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Existence of solutions for two dimensional generalized nonlinear functional integral equations via Petryshyn's fixed point theorem

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Abstract In this paper, we provide the conditions for the existence of solutions to two-dimensional generalized functional integral equations in the space $C([0,c] \times [0,d])$. The proof makes use of Petryshyn's fixed point theorem and the concept of measure of non-compactness. Our results encompass a wide range of functional integral equations encountered in nonlinear analysis. Additionally, we present several examples of equations to demonstrate the applicability of our findings to a diverse set of integral equations.

1 Introduction

Nonlinear functional integral equations (NFIEs) have broad class of applications in the modeling, physics, engineering, traffic, neutron transport, control theory, population dynamics, optimization, and theory of gases (see [5, 6, 17, 20, 23, 30]). It also plays a significant role in studying various problems related to functional differential equations. Darbo's and Petryshyn's theorems also play an important role in analyzing NFIEs. Moreover, there have been many rich efforts to apply M.N.C for the existence result of several NFIEs (see [8, 9, 11, 21, 22, 24, 37, 39, 41]). The idea of M.N.C goes back to the work of Kuratowski[31] which are functions that measure the degree of noncompactness of sets in Banach space. In 1955, Darbo presented a fixed point theorem using this notion. In 1971, W. V. Petryshyn [36] introduced several fixed point theorems for condensing mapping. This conception is directly to M.N.C. The idea of using the Petryshyn's fixed point theorem in order to investigate the existence of solution of nonlinear functional integral equations for the first time has been introduced by Kazemi and Ezzati [24].
The advantage of the Petryshyn's fixed point theorem, compared to other methods like Darbo and Schauder fixed point theorem , is that it does not require verifying that the operator involved maps a closed convex subset onto itself. As a result, this theorem is renowned for its simplicity and ease of application, making it a valuable tool in the proof of existence of solutions for integral equations. Recently many researchers used Petryshyn's fixed point theorem to discuss the existence of solution of nonlinear functional integral equations in Banach spaces as well as Banach Algebra (for instance see [1, 7, 11, 13, 14, 15, 16, 19, 25, 26, 27, 28, 29, 32, 37, 40] and references therein).

Motivated by the above literature, in this work, we study the existence result of the following NFIE

$$\begin{aligned}
&u(\kappa,t) = \\
&\Bigg(p(\kappa,t,u(\kappa,t),u(\alpha(\kappa,t))) + F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy, \\
&\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Bigg)\Bigg) \\
&\times\Bigg(q(\kappa,t,u(\kappa,t),u(\theta(\kappa,t))) + G\Bigg(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^\kappa W(\kappa,t,y,u(\beta(y,t)))dy, \\
&\int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\Bigg)\Bigg), \qquad (1.1)
\end{aligned}$$

where $(\kappa,t)\in I_0=[0,c]\times[0,d]$.
Das et al. [8] studied the existence of solutions for 2D FIE

$$u(\kappa,t) = p(\kappa,t) + F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(z,v))dvdz\Bigg) \qquad (1.2)$$

for $(\kappa,t)\in[0,1]\times[0,1]$.

Kazemi et al. [24] studied the following non-linear 2D NFIE

$$u(\kappa,t) = F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(z,v))dvdz\Bigg) \quad (1.3)$$

for $(\kappa,t)\in I_0$.

Kazemi et al. [24] discussed the existence result for the following non-linear 2D NFIE

$$u(\kappa,t) = p(\kappa,t,u(\kappa,t))+F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(z,v))dvdz\Bigg) \quad (1.4)$$

for $(\kappa,t)\in I_0$.

Deep et al. [10] studied the existence of solutions for 2D NFIE

$$\begin{aligned}
u(\kappa,t) &= \Bigg(p(\kappa,t,u(\kappa,t)) + F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(z,v))dvdz\Bigg)\Bigg) \\
&\quad\times\Bigg(q(\kappa,t,u(\kappa,t)) + G\Bigg(\kappa,t,u(\kappa,t),\int_0^c\int_0^d V(\kappa,t,z,v,u(z,v))dvdz\Bigg)\Bigg) \qquad (1.5)
\end{aligned}$$

for $(\kappa,t)\in I_0$.

Mishra et al. [32] studied the existence of solutions for 2D NFIE

$$u(\kappa,t) = F\Bigg(\kappa,t,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(z,v))dvdz\Bigg)\times G\Bigg(\kappa,t,\int_0^c\int_0^d V(\kappa,t,z,v,u(z,v))dvdz\Bigg).$$

for $(\kappa,t)\in I_0$.

Srivastava et al. [41] studied the existence of solutions for 2D NFIE

$$\begin{aligned} u(\kappa,t) &= \left(p(\kappa,t,u(\kappa,t)) + F\left(\kappa,t,\int_0^{\kappa}\int_0^{t} H(\kappa,t,z,v,u(z,v))dvdz, u(\kappa,t)\right)\right) \\ &\quad \times G\left(\kappa,t,\int_0^{c}\int_0^{d} V(\kappa,t,z,v,u(z,v))dvdz, u(\kappa,t)\right). \end{aligned} \tag{1.6}$$

for $(\kappa,t)\in[0,c]\times[0,d]$.

Further, a famous 2D NFIE of Hammerstein type [35] has the form

$$u(\kappa,t) = p(\kappa,t) + \int_0^{\kappa}\int_0^{t} p_1(\kappa,t,z,v)p_2(z,v,u(z,v))dvdz.$$

It is an interest that Eq. (1.1) is very general in nature and covers Eqs. (1.2–1.6) as particular problems, which are very valuable in real-world applications, for more details, we refer (see [5, 6, 17]). The main purpose of this study is to use Petryshyn's theorem (preferably Darbo's theorem) to examine the solvability of Eq. (1.1). Now we mention the main reason why we study Eq. (1.1) and what we achieve. The first advantage is that the conditions given in multiple papers will be investigated and the second reason is that it unifies the related work in this area. The third condition is the bounded condition shows that the "sublinear condition" that has been considered in literature will not play a significant role here.
The paper is designed into four sections involving the introduction. In Section 2, we present some preliminaries and represent the concept of M.N.C. Section 3 is dedicated to state and prove an existence theorem for equations connecting condensing operators using Petryshyn's fixed point theorem. In Section 4, we contribute some examples that prove the application of this kind of nonlinear NFIEs.

2 Auxiliary facts and notations

In this study, let X be a real Banach space and B_r denote closed ball center at 0 with radius r and $\partial B_r = \{u \in X : \|u\| = r\}$ for the sphere in X around 0 with radius $r > 0$. M.N.C are rich tools in non-linear analysis, for existence in the fixed point theory and operator equations in X.

Definition 2.1. [31] If$N \subset X$, then Kuratowski M.N.C of N,

$$\hat{\beta}(N) = \inf\left\{\rho > 0 : N = \bigcup_{i=1}^{n} N_i \text{with } \text{diam} N_i \le \rho,\ i = 1, 2, ..., n\right\}.$$

Definition 2.2. [18] The Hausdroff M.N.C

$$\mu(N) = \inf\{\rho > 0 : \exists \text{ a finite } \rho\text{-net for } N \text{ in } X\}, \tag{2.1}$$

where from a finite ρ-net for N in X that means like a set $\{u_1, u_2, ..., u_n\} \subset X$ such that the ball $B_\rho(X, u_1), B_\rho(X, u_2), ..., B_\rho(X, u_n)$ over N. These MNC are mutually equivalent in the sense that

$$\mu(N) \le \hat{\beta}(N) \le 2\mu(N),$$

for a bounded set $N \subset X$.

Theorem 2.3. *Let $N, \hat{N} \subset X$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(N) = 0$ *if and only if N is relatively compact;*

(ii) $N \subseteq \hat{N} \implies \mu(N) \le \mu(\hat{N})$*;*

(iii) $\mu(\overline{N}) = \mu(\mathit{Conv}N) = \mu(N)$*;*

(iv) $\mu(N \cup \hat{N}) = \max\{\mu(N), \mu(\hat{N})\}$*;*

(v) $\mu(\lambda N) = |\lambda|\mu(N)$,

(vi) $\mu(N + \hat{N}) \leq \mu(N) + \mu(\hat{N})$.

Here, $C(I_0) = C([0, c] \times [0, d])$ consisting of all real valued continuous function with the usual norm

$$\|u\| = \max\{|u(\kappa, t)| : (\kappa, t) \in I_0\}.$$

The space $C(I_0)$ is also the structure of Banach algebra. Fix a set $N \in C(I_0)$ and $u \in N$, given $\rho > 0$, the modulus of continuity of u defined as

$$\omega(u, \rho) = \sup\{|u(\kappa, t) - u(\hat{\kappa}, \hat{t})| : \kappa, \hat{\kappa} \in [0, c], t, \hat{t} \in [0, d], |\kappa - \hat{\kappa}| \leq \rho, |t - \hat{t}| \leq \rho\}.$$

Further,

$$\omega(N, \rho) = \sup\{\omega(u, \rho) : u \in N\},\ \omega_0(N) = \lim_{\rho \to 0} \omega(N, \rho).$$

Theorem 2.4. *[24] The Hausdorff MNC is similar to*

$$\mu(N) = \lim_{\rho \to 0} \sup_{u \in N} \omega(u, \rho) \tag{2.2}$$

for all bounded sets $N \subset C(I_0)$.

Definition 2.5. [33] Assume $T : X \to X$ be a continuous mapping of X. T is called k set contraction, if $\forall\ D \subset X$ with D bounded, $T(D)$ is bounded and $\hat{\beta}(TD) \leq k\hat{\beta}(D), k \in (0, 1)$. If $\hat{\beta}(TD) < \hat{\beta}(D),\ \forall\ \hat{\beta}(D) > 0$, then T is called densifying or condensing mapping. A k-set contraction is condensing but converse not be true.

Theorem 2.6. *[36] Suppose that $T : B_r \to X$ is a condensing mapping, which fulfills the boundary condition if $T(u) = ku$, for some $u \in \partial B_r$, then $k \leq 1$. Then the set of fixed points of T in B_r is non-empty.*

Theorem 2.7. *[3] Let X is a Banach space. If the operators L and S each fulfil the condensing map on a bounded set D of X with constant k and $\hat{k}$, respectively, then the operator $\nabla = L.S$ fulfil the condensing map on D with the constant $\|L(D)\|\hat{k} + \|S(D)\|k$, Particularly, if $\|L(D)\|\hat{k} + \|S(D)\|k < 1$, then ∇ is a contraction with respect to the measure μ and it has at least one fixed point in the set D.*

3 An existence theorem for NFIEs model

We study the Eq. (1.1) with the following assumptions;

(1) $p, q \in C(I_0 \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), F, G \in C(I_1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), P, W \in C(I_0 \times I_c \times \mathbb{R}, \mathbb{R}), H, V \in C(I_2 \times \mathbb{R}, \mathbb{R})$,
where

$$I_0 = [0, c] \times [0, d],\ I_1 = \{(\kappa, t, u) : 0 \leq \kappa \leq c, 0 \leq t \leq b, u \in \mathbb{R}\},$$
$$I_2 = \{((\kappa, t), (z, v)) \in I_0^2 : 0 \leq z \leq \kappa \leq c, 0 \leq v \leq t \leq d\}, I_c = [0, c],$$
$$and \alpha, \beta, \theta, \eta : I_0 \to I_0,\ \gamma, \tau, \delta : I_0 \to I_0.$$

(2) There exist non-negative constants $k,\ \hat{k} < \frac{1}{3}$, such that

$$|p(\kappa, t, u_1, u_2) - p(\kappa, t, v_1, v_2)| \leq k\left(|u_1 - v_1| + |u_2 - v_2|\right),$$
$$|q(\kappa, t, u_1, u_2) - q(\kappa, t, v_1, v_2)| \leq \hat{k}\left(|u_1 - v_1| + |u_2 - v_2|\right),$$
$$|F(\kappa, t, u_1, u_2, u_3) - F(\kappa, t, v_1, v_2, v_3)| \leq k\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$
$$|G(\kappa, t, u_1, u_2, u_3) - G(\kappa, t, u_1, u_2, u_3)| \leq \hat{k}\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$

for all $(\kappa, t) \in I_0$ and $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$.

(3) There exists a $r > 0$ such that the resulting bounded condition is fulfilled

$$\sup\left\{\left|p(\kappa,t):(\kappa,t)\in I_0\right| + \left|F(\kappa,t,u_1,u_2,u_3):(\kappa,t)\in I_0, -r\le u_1\le r\ -cN_1\le u_2\le cN_1,\right.\right.$$
$$\left.\left. -cdN_2\le u_3\le cdN_2\right|\right\}\times\sup\left\{\left|q(\kappa,t):(\kappa,t)\in I_0\right| + \left|G(\kappa,t,u_1,u_2,u_3):(\kappa,t)\in I_0\right.\right.$$
$$\left.\left.,-r\le u_1,u_2\le r, -cd\hat{N}_2\le u_3\le cd\hat{N}_2\right|\right\}.$$

such that $\sup\{|p+F|\}\hat{k}+\sup\{|q+G|\}k<\frac{1}{3}$ where

$$N_1 = \sup\{|P(\kappa,t,y,u(\beta(y,t)))| : \textit{for all } (\kappa,t)\in I_0, y\in I_c, \textit{and } u\in[-r,r]\},$$
$$\hat{N}_1 = \sup\{|W(\kappa,t,y,u(\beta(y,t)))| : \textit{for all } (\kappa,t)\in I_0, y\in I_c, \textit{and } u\in[-r,r]\},$$
$$N_2 = \sup\{|H(\kappa,t,z,v,u(\gamma(z,v)))| : \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\},$$
$$\hat{N}_2 = \sup\{|V(\kappa,t,z,v,u(\delta(z,v)))| : \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\}.$$

Theorem 3.1. *By assumptions* $(1)-(3)$, *Eq.(1.1) has at least one solution in* $X = C(I_0)$.

Proof. Define the operators $L, S : B_r \to X$ as

$$\begin{aligned}(Lu)(\kappa,t) &= \Bigg(p(\kappa,t,u(\kappa,t),u(\alpha(\kappa,t))) + F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\\ &\quad \int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Bigg)\Bigg),\\ (Su)(\kappa,t) &= \Bigg(q(\kappa,t,u(\kappa,t),u(\theta(\kappa,t))) + G\Bigg(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^\kappa W(\kappa,t,y,u(\tau(y,t)))dy,\\ &\quad \int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\Bigg)\Bigg),\end{aligned}$$

for $(\kappa,t)\in I_0$.
Further, define T on the X by setting

$$Tu = (Lu)(Su).$$

Step 1. Now, we show that T is continuous on B_r. For this, take $\rho > 0$ and any $u, w \in B_r$ such that $|u-w|<\rho$. Then,

$$
\begin{aligned}
& |(Lu)(\kappa,t)-(Lw)(\kappa,t)| \\
= & \Big|p(\kappa,t,u(\kappa,t),u(\alpha(\kappa,t))) \\
& +F\Big(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big) \\
& -p(\kappa,t,w(\kappa,t),w(\alpha(\kappa,t))) \\
& -F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,w(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,w(\gamma(z,v)))dvdz\Big)\Big| \\
\leq & k_1|u(\kappa,t)-w(\kappa,t)|+k_2|u(\alpha(\kappa,t))-w(\alpha(\kappa,t))| \\
& +\Big|F\Big(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big) \\
& -F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big)\Big| \\
& +\Big|F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big) \\
& -F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,w(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big)\Big| \\
& +\Big|F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,w(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Big) \\
& -F\Big(\kappa,t,w(\kappa,t),\int_0^\kappa P(\kappa,t,y,w(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,w(\gamma(z,v)))dvdz\Big)\Big| \\
& \leq k|u(\kappa,t)-w(\kappa,t)|+k|u(\alpha(\kappa,t))-w(\alpha(\kappa,t))| \\
& +k|u(\kappa,t)-w(\kappa,t)|+k\int_0^\kappa |P(\kappa,t,y,u(\beta(y,t)))-P(\kappa,t,y,w(\beta(y,t)))|dy \\
& +k\int_0^\kappa\int_0^t |H(\kappa,t,z,v,u(\gamma(z,v)))-H(\kappa,t,z,v,w(\gamma(z,v)))|dvdz \\
\leq & 3k\|u-w\|+kc\omega(P,\rho)+kcd\omega(H,\rho),
\end{aligned}
$$

and similarly, we have

$$
\begin{aligned}
& |(Su)(\kappa,t)-(Sw)(\kappa,t)| = \Big|q(\kappa,t,u(\kappa,t),u(\theta(\kappa,t))) \\
& + G\Bigg(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\Bigg) \\
& - q(\kappa,t,w(\kappa,t),w(\theta(\kappa,t))) \\
& - G\Bigg(\kappa,t,w(\kappa,t),w(\eta(\kappa,t)),\int_0^\kappa W(\kappa,t,y,w(\beta(y,t)))dy,\int_0^c\int_0^d V(\kappa,t,z,v,w(\delta(z,v)))dvdz\Bigg)\Bigg| \\
& \leq 3\hat{k}\|u-w\|+\hat{k}c\omega(W,\rho)+\hat{k}cd\omega(V,\rho),
\end{aligned}
$$

where

$$
\begin{aligned}
\omega(P,\rho) &= \sup\{|P(\kappa,t,\nu,u)-P(\kappa,t,\nu,w)| : (\kappa,t)\in I_0, \nu\in I_c, u,w\in[-r,r], |u-w|\leq\rho\}, \\
\omega(W,\rho) &= \sup\{|W(\kappa,t,\nu,u)-W(\kappa,t,\nu,w)| : (\kappa,t)\in I_0, \nu\in I_c, u,w\in[-r,r], |u-w|\leq\rho\}, \\
\omega(H,\rho) &= \sup\{|H(\kappa,t,z,v,u)-H(\kappa,t,z,v,w)| : (\kappa,t,z,v)\in I_2, u,w\in[-r,r], |u-w|\leq\rho\}, \\
\omega(V,\rho) &= \sup\{|V(\kappa,t,z,v,u)-V(\kappa,t,z,v,w)| : (\kappa,t,z,v)\in I_2, u,w\in[-r,r], |u-w|\leq\rho\}.
\end{aligned}
$$

Since, $P = P(\kappa, t, y, u)$, $W = W(\kappa, t, y, u)$, $H = H(\kappa, t, z, v, u)$ and $V = V(\kappa, t, z, v, u)$ are uniformly continuous on $I_0 \times I_c \times [-r, r]$, $I_0 \times I_c \times [-r, r]$, $I_2 \times [-r, r]$ and $I_2 \times [-r, r]$, respectively, we infer that $\omega(P, \rho)$, $\omega(W, \rho)$, $\omega(H, \rho)$ and $\omega(V, \rho) \to 0$ as $\rho \to 0$. From above fact prove that L and S is continuous on B_r. Hence, T is also continuous on B_r.

Step 2. We prove that the operators L and S satisfies the condensing condition with μ. For this, take a fixed arbitrary $\rho > 0$ and $u \in N$, where N is bounded subset of $X, (\kappa_1, t_1), (\kappa_2, t_2) \in I_0$ with $\kappa_1 \le \kappa_2, t_1 \le t_2$ and $\kappa_2 - \kappa_1 \le \rho, t_2 - t_1 \le \rho$ we get

$$
\begin{aligned}
& |(Lu)(\kappa_2, t_2) - (Lu)(\kappa_1, t_1)| \\
= \; & \bigg| p(\kappa_2, t_2, u(\kappa_2, t_2), u(\alpha(\kappa_2, t_2))) + F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_2} P(\kappa_2, t_2, y, u(\beta(y, t_2)))dy, \\
& \int_0^{\kappa_2}\int_0^{t_2} H(\kappa_2, t_2, z, v, u(\gamma(z, v)))dvdz\Big) - p(\kappa_1, t_1, u(\kappa_1, t_1), u(\alpha(\kappa_1, t_1))) \\
& -F\Big(\kappa_1, t_1, u(\kappa_1, t_1), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big)\bigg| \\
\le \; & |p(\kappa_2, t_2, u(\kappa_2, t_2), u(\alpha(\kappa_2, t_2))) - p(\kappa_2, t_2, u(\kappa_1, t_1), u(\alpha(\kappa_1, t_1)))| \\
& +|p(\kappa_2, t_2, u(\kappa_1, t_1), u(\alpha(\kappa_1, t_1))) - p(\kappa_1, t_1, u(\kappa_1, t_1), u(\alpha(\kappa_1, t_1)))| \\
& +\bigg|F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_2} P(\kappa_2, t_2, y, u(\beta(y, t_2)))dy, \int_0^{\kappa_2}\int_0^{t_2} H(\kappa_2, t_2, z, v, u(\gamma(z, v)))dvdz\Big) \\
& -F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_2} P(\kappa_2, t_2, y, u(\beta(y, t_2)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big)\bigg| \\
& +\bigg|F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_2} P(\kappa_2, t_2, y, u(\beta(y, t_2)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big) \\
& -F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big)\bigg| \\
& +\bigg|F\Big(\kappa_2, t_2, u(\kappa_2, t_2), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big) \\
& -F\Big(\kappa_2, t_2, u(\kappa_1, t_1), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big)\bigg| \\
& +\bigg|F\Big(\kappa_2, t_2, u(\kappa_1, t_1), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \\
& \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big) - F\Big(\kappa_1, t_1, u(\kappa_1, t_1), \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy, \\
& \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\Big)\bigg| \\
& \le k|u(\kappa_2, t_2) - u(\kappa_1, t_1)| + k|u(\alpha(\kappa_2, t_2)) - u(\alpha(\kappa_1, t_1))| + \omega_p(I_0, \rho) \\
& +k\left|\int_0^{\kappa_2}\int_0^{t_2} H(\kappa_2, t_2, z, v, u(\gamma(z, v)))dvdz - \int_0^{\kappa_1}\int_0^{t_1} H(\kappa_1, t_1, z, v, u(\gamma(z, v)))dvdz\right| \\
& +k\left|\int_0^{\kappa_2} P(\kappa_2, t_2, y, u(\beta(y, t_2)))dy - \int_0^{\kappa_1} P(\kappa_1, t_1, y, u(\beta(y, t_1)))dy\right| \\
& +k|u(\kappa_2, t_2) - u(\kappa_1, t_1)| + \omega_F(I_0, \rho) \\
& \le k\omega(u, \rho) + k_2\omega(u, \omega(\alpha, \rho)) + k\omega(u, \rho) + \omega_p(I_0, \rho) + \omega_F(I_0, \rho)
\end{aligned}
$$

$$
\begin{aligned}
&+k\int_0^{\kappa_1}\int_0^{t_1}|H(\kappa_2,t_2,z,v,u(\gamma(z,v)))-H(\kappa_1,t_1,z,v,u(\gamma(z,v)))|dvdz\\
&+k\int_{\kappa_1}^{\kappa_2}\int_0^{t_1}|H(\kappa_2,t_2,z,v,u(\gamma(z,v)))|dvdz+k\int_0^{\kappa_1}\int_{t_1}^{t_2}|H(\kappa_2,t_2,z,v,u(\gamma(z,v)))|dvdz\\
&+\int_{\kappa_1}^{\kappa_2}\int_{t_1}^{t_2}|H(\kappa_2,t_2,z,v,u(\gamma(z,v)))|dvdz\\
&+k\int_0^{\kappa_1}|P(\kappa_2,t_2,y,u(\beta(y,t_2)))dy-P(\kappa_1,t_1,y,u(\beta(y,t_1)))|dy\\
&+k\int_{\kappa_1}^{\kappa_2}|P(\kappa_2,t_2,y,u(\beta(y,t_2)))|dy\\
&\le k\omega(u,\rho)+k\omega(u,\omega(\alpha,\rho))+k\omega(u,\rho)+\omega_p(I_0,\rho)+\omega_F(I_0,\rho)\\
&+kcd\omega_H(I_0,\rho)+k\rho dN_2+k\rho cN_2+\rho^2kN_2+kc\omega_P(I_0,\rho)+k\rho N_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|(Su)(\kappa_2,t_2)-(Su)(\kappa_1,t_1)|=\\
&\bigg|q(\kappa_2,t_2,u(\kappa_2,t_2),u(\theta(\kappa_2,t_2)))+G\bigg(\kappa_2,t_2,u(\kappa_2,t_2),u(\eta(\kappa_2,t_2)),\int_0^{\kappa_2}W(\kappa_2,t_2,y,u(\beta(y,t_2)))dy,\\
&\qquad\int_0^c\int_0^d V(\kappa_2,t_2,z,v,u(\delta(z,v)))dvdz\bigg)\\
&-q(\kappa_1,t_1,u(\kappa_1,t_1),u(\theta(\kappa_1,t_1)))-G\bigg(\kappa_1,t_1,u(\kappa_1,t_1),u(\eta(\kappa_1,t_1)),\int_0^{\kappa_1}W(\kappa_1,t_1,y,u(\beta(y,t_1)))dy,\\
&\qquad\int_0^c\int_0^d V(\kappa_1,t_1,z,v,u(\delta(z,v)))dvdz\bigg)\bigg|\\
&\le \hat{k}\omega(u,\rho)+\hat{k}\omega(u,\omega(\eta,\rho))+\hat{k}\omega(u,\rho)+\omega_q(I_0,\rho)+\omega_G(I_0,\rho)\\
&\quad+\hat{k}cd\omega_V(I_0,\rho)+\hat{k}\rho d\hat{N}_2+\hat{k}\rho c\hat{N}_2+\rho^2\hat{k}\hat{N}_2+\hat{k}c\omega_W(I_0,\rho)+\hat{k}\rho\hat{N}_1 \quad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_p(I_0,\rho)&=\sup\{|p(\kappa,t,u_1,u_2)-p(\hat{\kappa},\hat{t},u_1,u_2)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t)\in I_0,(\hat{\kappa},\hat{t})\in I_0,\\
&\qquad u_1,u_2\in[-r,r]\},\\
\omega_q(I_0,\rho)&=\sup\{|q(\kappa,t,u_1,u_2)-q(\hat{\kappa},\hat{t},u_1,u_2)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t)\in I_0,(\hat{\kappa},\hat{t})\in I_0,\\
&\qquad u_1,u_2\in[-r,r]\},\\
\omega_H(I_0,\rho)&=\sup\{|H(\kappa,t,z,v,u)-H(\hat{\kappa},\hat{t},z,v,u)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t,z,v)\in I_2,u\in[-r,r]\},\\
\omega_V(I_0,\rho)&=\sup\{|V(\kappa,t,z,v,u)-V(\hat{\kappa},\hat{t},z,v,u)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t,z,v)\in I_2,u\in[-r,r]\},\\
\omega_P(I_0,\rho)&=\sup\{|P(\kappa,t,y,u)-P(\hat{\kappa},\hat{t},y,u)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t)\in I_0,y\in I_c,u\in[-r,r]\},\\
\omega_W(I_0,\rho)&=\sup\{|W(\kappa,t,y,u)-W(\hat{\kappa},\hat{t},y,u)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,(\kappa,t)\in I_0,y\in I_c,u\in[-r,r]\},\\
\omega_F(I_0,\rho)&=\sup\{|F(\kappa,t,u_1,u_2,u_3)-F(\hat{\kappa},\hat{t},u_1,u_2,u_3)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,u_2\in[-cN_1,cN_1],\\
&\qquad u_3\in[-cdN_2,cdN_2],u_1\in[-r,r]\},\\
\omega_G(I_0,\rho)&=\sup\{|G(\kappa,t,u_1,u_2,u_3)-G(\hat{\kappa},\hat{t},u_1,u_2,u_3)|:|\kappa-\hat{\kappa}|,|t-\hat{t}|\le\rho,u_3\in[-cd\hat{N}_2,cd\hat{N}_2],\\
&\qquad u_1,u_2\in[-r,r]\}. \quad (3.2)
\end{aligned}
$$

From above relation, we get

$$\begin{aligned}|(Lu)(\kappa_2,t_2)-(Lu)(\kappa_1,t_1)| &\le k\omega(u,\rho)+k\omega(u,\omega(\alpha,\rho))+k\omega(u,\rho)+\omega_p(I_0,\rho)\\ &\quad +\omega_F(I_0,\rho)+kcd\omega_H(I_0,\rho)+k\rho dN_2+k\rho cN_2\\ &\quad +\rho^2kN_2+kc\omega_P(I_0,\rho)+k\rho N_1,\\ |(Su)(\kappa_2,t_2)-(Su)(\kappa_1,t_1)| &\le \hat{k}\omega(u,\rho)+\hat{k}\omega(u,\omega(\eta,\rho))+\hat{k}\omega(u,\rho)+\omega_q(I_0,\rho)\\ &\quad +\omega_G(I_0,\rho)+\hat{k}cd\omega_V(I_0,\rho)+\hat{k}\rho d\hat{N}_2+\hat{k}\rho c\hat{N}_2\\ &\quad +\rho^2\hat{k}\hat{N}_2+\hat{k}c\omega_W(I_0,\rho)+\hat{k}\rho\hat{N}_1.\end{aligned}$$

Taking limit as $\rho\to 0$, we get

$$\mu(LN)\le 3k\mu(N) \tag{3.3}$$

and

$$\mu(SN)\le 3\hat{k}\mu(N). \tag{3.4}$$

From Eqs. (3.3),(3.4) and Theorem 2.7, we obtain T is a condensing map. Now, assume $u\in\partial B_r$ and if $Tu=ku$ then, we get $\|Tu\|=k\|u\|=kr$ and by assumption (3), we have

$$\begin{aligned}\|Tu(\kappa,t)\| &= \Bigg(p(\kappa,t,u(\kappa,t),u(\alpha(\kappa,t)))+F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\\ &\quad \int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\Bigg)\Bigg)\\ &\quad \times\Bigg(q(\kappa,t,u(\kappa,t),u(\theta(\kappa,t)))+G\Bigg(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^\kappa W(\kappa,t,y,u(\tau(y,t)))dy,\\ &\quad \int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\Bigg)\Bigg)\\ &\le r,\end{aligned}$$

for all $(\kappa,t)\in I_0$, hence $\|Tu\|\le r$, i.e $k\le 1$. □

In following, the desired results obtained from Theorem 3.1.

Corollary 3.2. *Assume that*

(1) $p\in C(I_0\times\mathbb{R}\times\mathbb{R},\mathbb{R}), F,G\in C(I_1\times\mathbb{R}\times\mathbb{R},\mathbb{R}), P\in C(I_0\times I_c\times\mathbb{R},\mathbb{R}), H,V\in C(I_2\times\mathbb{R},\mathbb{R})$, *where*

$$I_0=[0,c]\times[0,d],\ I_1=\{(\kappa,t,u):0\le\kappa\le c,0\le t\le d,u\in\mathbb{R}\},$$

$$I_2=\{(\kappa,t,z,v)\in I_0^2:0\le z\le\kappa\le c,0\le v\le t\le d\},$$

$$\alpha,\theta,\eta:I_0\to I_0,\ \gamma,\delta:I_0\to I_0.$$

(2) *There exist non-negative constants* $k<\frac{1}{3}$, $\hat{k}<\frac{1}{2}$ *such that*

$$\begin{aligned}|p(\kappa,t,u_1,u_2)-p(\kappa,t,v_1,v_2)| &\le k\left(|u_1-v_1|+|u_2-v_2|\right),\\ |F(\kappa,t,u_1,u_2,u_3)-F(\kappa,t,v_1,v_2,v_3)| &\le k\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),\\ |G(\kappa,t,u_1,u_2,u_3)-G(\kappa,t,u_1,u_2,u_3)| &\le \hat{k}\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),\end{aligned}$$

for all $(\kappa,t)\in I_0$.

(3) *There exists a* $r\ge 0$ *such that the resulting bounded condition is fulfilled*

$$\sup\left(\{|(p+F)\times G|\}\right)\le r,$$

and $\sup\{|p+F|\}\hat{k}+\sup\{|G|\}k<1$ *where,*

$$\begin{aligned}
\sup p &= \sup\{|p(\kappa,t,u_1,u_2|: \textit{for all } (\kappa,t)\in I_0, -r\le u_1,u_2\le r\},\\
\sup F &= \sup\{|F(\kappa,t,u_1,u_2,u_3)|: \textit{for all } (\kappa,t)\in I_0, -r\le u_1,\le r\ -cN_1\le u_2\le cN_1,\\
&\quad\ \textit{and } -cdN_2\le u_3\le cdN_2\},\\
N_1 &= \sup\{|P(\kappa,t,y,u(\beta(y,t)))|: \textit{for all } (\kappa,t)\in I_0, y\in I_c, \textit{and } u\in[-r,r]\},\\
N_2 &= \sup\{|H(\kappa,t,z,v,u(\gamma(z,v)))|: \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\},\\
\hat{N}_2 &= \sup\{|V(\kappa,t,z,v,u(\delta(z,v)))|: \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\},\\
\sup G &= \sup\{|G(\kappa,t,u_1,u_2,u_3)|: \textit{for all } (\kappa,t)\in I_0, -r\le u_1,u_2\le r\\
&\quad \textit{and } -cd\hat{N}_2\le u_3\le cd\hat{N}_2\}.
\end{aligned}$$

Then

$$\begin{aligned}
u(\kappa,t) &= \Bigg(p(\kappa,t,u(\kappa,t),u(\alpha(\kappa,t)))\\
&\quad +F\Bigg(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz)\Bigg)\Bigg)\\
&\quad \times G\Bigg(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz)\Bigg) \qquad (3.5)
\end{aligned}$$

has at least one solution in $I_0=[0,c]\times[0,d]$.

Proof. The proof is relevant to the Theorem 3.1 and leave parts. □

Corollary 3.3. *Let*

(1) $F,G\in C(I_1\times\mathbb{R}\times\mathbb{R},\mathbb{R}), P\in C(I_0\times I_c\times\mathbb{R},\mathbb{R}), H,V\in C(I_2\times\mathbb{R},\mathbb{R})$, *where*

$$I_0=[0,c]\times[0,d],\ I_1=\{(\kappa,t,u):0\le\kappa\le c, 0\le t\le d, u\in\mathbb{R}\},$$
$$I_2=\{(\kappa,t,z,v)\in I_0^2: 0\le z\le\kappa\le c, 0\le v\le t\le d\},$$
$$\eta: I_0\to I_0,\ \gamma,\delta: I_0\to I_0.$$

(2) *There exist non-negative constants* $k,\hat{k}<\frac{1}{2}$ *such that*

$$\begin{aligned}
|F(\kappa,t,u_1,u_2,u_3)-F(\kappa,t,v_1,v_2,v_3)| &\le k\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),\\
|G(\kappa,t,u_1,u_2,u_3)-G(\kappa,t,u_1,u_2,u_3)| &\le \hat{k}\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),
\end{aligned}$$

for all $(\kappa,t)\in I_0$..

(3) *There exists a* $r\ge 0$ *such that the resulting bounded condition is fulfilled*

$$\sup\Big(\{|F\times G|\}\Big)\le r,$$

and $\hat{k}\sup G+k\sup F<1$ *where,*

$$\begin{aligned}
\sup F &= \sup\{|F(\kappa,t,u_1,u_2,u_3)|: \textit{for all } (\kappa,t)\in I_0, -r\le u_1,\le r\ -cN_1\le u_2\le cN_1,\\
&\quad\ \textit{and } -cdN_2\le u_3\le cdN_2\},\\
N_1 &= \sup\{|P(\kappa,t,\nu,u(\beta(\nu,s)))|: \textit{for all } (\kappa,t)\in I_0, \nu\in I_c, \textit{and } u\in[-r,r]\},\\
N_2 &= \sup\{|H(\kappa,t,z,v,u(\gamma(z,v)))|: \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\},\\
\hat{N}_2 &= \sup\{|V(\kappa,t,z,v,u(\delta(z,v)))|: \textit{for all } (\kappa,t,z,v)\in I_2, \textit{and } u\in[-r,r]\},\\
\sup G &= \sup\{|G(\kappa,t,u_1,u_2,u_3)|: \textit{for all } (\kappa,t)\in I_0, -r\le u_1,u_2\le r\\
&\quad \textit{and } -cd\hat{N}_2\le u_3\le cd\hat{N}_2\}.
\end{aligned}$$

Then

$$u(\kappa,t) = F\left(\kappa,t,u(\kappa,t),\int_0^\kappa P(\kappa,t,y,u(\beta(y,t)))dy,\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\right)$$
$$\times G\left(\kappa,t,u(\kappa,t),u(\eta(\kappa,t)),\int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\right) \quad (3.6)$$

has at least one solution in $C(I_0), I_0 = [0,c]\times[0,d]$.

Proof. The proof is relevant to the Theorem 3.1 and leave this parts. □

Corollary 3.4. *If* $p(\kappa,t,u_1,u_2) = 0,\ F(\kappa,t,u_1,u_2,u_3) = F(\kappa,t,u_1,u_3)$ *and* $G(\kappa,t,u_1,u_2,u_3) = G(\kappa,t,u_1,u_3)$. *Then Eq. (1.1) can be converted into following FIE*

$$u(\kappa,t) = F\left(\kappa,t,u(\kappa,t),\int_0^\kappa\int_0^t H(\kappa,t,z,v,u(\gamma(z,v)))dvdz\right)$$
$$\times G\left(\kappa,t,u(\kappa,t),\int_0^c\int_0^d V(\kappa,t,z,v,u(\delta(z,v)))dvdz\right) \quad (3.7)$$

has at least one solution in $C(I_0), I_0 = [0,c]\times[0,d]$.

Proof. The proof is relevant to the Theorem 3.1 and leave this parts. □

4 Application via illustrative examples

In this part, we give some examples of NFIEs to explain the advantage of our results.

Example 4.1.

$$u(\kappa,t) = p(\kappa,t) + \int_0^\kappa\int_0^t p_1(\kappa,t,z,v)p_2(z,v,u(z,v))dvdz,$$

for $\gamma(\kappa,t) = (\kappa,t), p(\kappa,t,u_1,u_2) = p(\kappa,t), q(\kappa,t,u_1,u_2) = 1, F(\kappa,t,u_1,u_2,u_3) = u_3, G(\kappa,t,u_1,u_2,u_3) = 0$ and $H(\kappa,t,z,v,u(z,v)) = p_1(\kappa,t,z,v)p_2(z,v,u(z,v))$, which may be viewed like a two independent variables generalization of the famous Hammerstein type integral equation [35]

$$u(\kappa,t) = p(\kappa,t) + \int_0^1\int_0^1 H(v,z,v,u(z,v))dvdz,$$

which is the famous two-dimensional Fredholm integral equation analyzed by various authors in history [35].

Example 4.2. Putting $p(\kappa,t,u_1,u_2) = p(\kappa,t), F(\kappa,t,u_1,u_2,u_3) = \hat{k}(u_2+u_3), q(\kappa,t,u_1,u_2) = 1$ and $G(\kappa,t,u_1,u_2,u_3) = 0$, then Eq. (1.1) convert to the following equation

$$u(\kappa,t) = p(\kappa,t) + \int_0^\kappa \hat{k}P(\kappa,t,y,u(\beta(y,t)))dy + \int_0^\kappa\int_0^t \hat{k}H(\kappa,t,z,v,u(\gamma(z,v)))dvdz. \quad (4.1)$$

The Eq. (4.1) is studied by many authors, one can see [6, 35].

Example 4.3. Let the following NFIEs

$$u(\kappa,t) = \left(\frac{\kappa^2}{8(1+\kappa^2t^2)}e^{-\kappa t} + \frac{1}{8+\kappa^2+t^2}\left(\frac{u(\kappa,t)}{1+|u(\kappa,t)|}\right) + \frac{1}{8}\int_0^\kappa\left(\frac{t^2+y^2}{9} + \sqrt[3]{u(y,t)} + u(y,t)\right)dy\right.$$
$$\left.+\frac{1}{9}\int_0^\kappa\int_0^t e^{\frac{-\kappa t+zv}{2}}|u(\kappa,t)|dvdz\right)$$
$$\times\left(\frac{1}{9}\sin\frac{\kappa+t}{3} + \frac{\sin u(\kappa,t)}{5\left(\left(\sqrt{\kappa+t}+1\right)\right)} + \frac{1}{\pi^2}\int_0^\kappa\int_0^t \frac{vz\sin(v+u(v,z))+3\kappa t\ln(1+u(\kappa,t))}{v^2\kappa^2+t^2z^2+6}dvdz\right) \quad (4.2)$$

for $(\kappa, t) \in I = [0,1] \times [0,1]$.
Here,

$$p(\kappa, t, u_1, u_2) = \tfrac{\kappa^2}{8(1+\kappa^2 t^2)} e^{-\kappa t} + \tfrac{1}{8+\kappa^2+t^2} u_1,\ P(\kappa, t, y, u) = \tfrac{t^2+y^2}{9} + \sqrt[3]{u(y,t)} + u(y,t),$$

$$H(\kappa, t, z, v, u) = e^{\frac{-\kappa t + zv}{2}} |u(\kappa, t)|,\ F(\kappa, t, u_1, u_2, u_3) = \tfrac{1}{8} u_2 + \tfrac{1}{9} u_3,$$

$$q(\kappa, t, u_1, u_2) = \tfrac{1}{9} \sin \tfrac{\kappa + t}{3} + \tfrac{\sin u(\kappa,t)}{5(\sqrt{\kappa+t+1})},\ W(\kappa, t, y, u) = 0$$

$$V(\kappa, t, z, v, u) = \tfrac{vz \sin(v + u(v,z)) + 3\kappa t ln(1+u(\kappa,t))}{v^2\kappa^2 + t^2 z^2 + 6},\ G(\kappa, t, u_1, u_2, u_3) = \frac{1}{\pi^2} u_3.$$

Moreover, for all $(\kappa, t) \in I = [0,1] \times [0,1]$ we have

$$\begin{aligned}
|p(\kappa, t, u_1, u_2) - p(\kappa, t, \hat{u}_1, \hat{u}_2)| &\leq \frac{1}{8} |u_1 - \hat{u}_1|, \\
|q(\kappa, t, u_1, u_2) - q(\kappa, t, \hat{u}_1, \hat{u}_2)| &\leq \tfrac{1}{5} |u_1 - \hat{u}_1|, \\
|F(\kappa, t, u_1, u_2, u_3) - F(\kappa, t, \hat{u}_1, \hat{u}_2, \hat{u}_3)| &\leq \frac{1}{8} |u_2 - \hat{u}_2| + \frac{1}{9} |u_3 - \hat{u}_3|, \\
|G(\kappa, t, u_1, u_2, u_3) - G(\kappa, t, \hat{u}_1, \hat{u}_2, \hat{u}_3)| &\leq \frac{1}{\pi^2} |u_3 - \hat{u}_3|.
\end{aligned}$$

One can easily check the conditions (1) and (2) are hold. Also, $k = \frac{1}{8}, \hat{k} = \frac{1}{\pi^2}$. We show that (3) also fulfils. Let $|u(\kappa, t)| \leq r$, $r > 0$, then

$$\begin{aligned}
|u(\kappa, t)| = & \Bigg| \bigg(\tfrac{\kappa^2}{8(1+\kappa^2 t^2)} e^{-\kappa t} + \tfrac{1}{8+\kappa^2+t^2} \left(\tfrac{u(\kappa,t)}{1+|u(\kappa,t)|} \right) + \tfrac{1}{8} \int_0^\kappa \left(\tfrac{t^2+y^2}{9} + \sqrt[3]{u(y,t)} + u(y,t) \right) dy \\
& + \tfrac{1}{9} \int_0^\kappa \int_0^t e^{\frac{-\kappa t + zv}{2}} |u(\kappa, t)| dv dz \bigg) \\
& \times \left(\tfrac{1}{9} \sin \tfrac{\kappa+t}{3} + \tfrac{\sin u(\kappa,t)}{5\left(\left(\sqrt{\kappa+t}+1\right)\right)} + \tfrac{1}{\pi^2} \int_0^\kappa \int_0^t \tfrac{vz \sin(v+u(v,z)) + 3\kappa t ln(1+u(\kappa,t))}{v^2\kappa^2 + t^2 z^2 + 6} dv dz \right) \Bigg| \leq r.
\end{aligned}$$

Hence , the condition (3) in theorem 3.1 holds if,

$$\left(\tfrac{1}{4} + \tfrac{1}{8} \left(\tfrac{1}{9} + \sqrt[3]{r} + r \right) + \tfrac{1}{9} r \right) \times \left(\tfrac{1}{9} + \tfrac{1}{5} r + \tfrac{1}{\pi^2} (1 + 3r) \right) \leq r$$

It is easy to verify that the number $r \in [0.871229, 5.76441]$ satisfies the above inequality. Also, for $r \in [0.871229, 3.01147] \subset [0.871229, 5.76441]$, we have $\sup\{|p+F|\}\hat{k} + \sup\{|q+G|\}k < \frac{1}{3}$. hence, Theorem 3.1 implies that the Eq. (4.2) has at least one solution in $C(I)$.

5 Conclusion remarks

The theory of integral equations primarily focuses on the existence and uniqueness of solutions. Many researchers have contributed to this field by sharing their findings and methodologies. In line with this, the authors of this paper introduce a new approach utilizing measures of noncompactness and the Petryshyn's fixed point theorem for a nonlinear integral equation. This method offers several advantages compared to similar techniques, including fewer conditions and no requirement to verify the mapping of the operator onto a closed convex subset. The results of this research are diverse and remarkable, making it both intriguing and deserving of further investigation in future studies.

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