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QUANTITATIVE ESTIMATORS OF EXTENDED BETA TYPE SZÁSZ-SCHURER OPERATORS

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Abstract In this article, we introduce generalized beta extension of Szász-Schurer Integral type operators and study their approximation properties. First, we calculate the some estimates for these operators. Further, we study the uniform convergence and order of approximation in terms of Korovkin type theorem and modulus of continuity for the space of univariate continuous functions and bivariate continuous functions in their sections. In continuation, local and global approximation properties are studied in terms of first and second order modulus of smoothness, Peetre's K-functional and weight functions in various functional spaces.

1 Introduction

Szász in 1950 [1], introduced a sequence of operators over the infinite length of interval, i.e., $[0, \infty)$ as:

$$P_m(h;u) = \sum_{i=0}^{\infty} h\left(\frac{i}{m}\right) p_{m,i}(u), \tag{1.1}$$

where $u \in [0,\infty),\ h \in C[0,\infty)$ and $p_{m,i}(u) = e^{-mu}\frac{(mu)^i}{i!}$.

In 1962, Schurer [2] constructed a new sequence of Bernstein operators [3] which is denoted as $B_{m+p} : C[0, 1+p] \to C[0, 1+p]$ and defined by:

$$B_{m+p}(g;\mu) = \sum_{j=0}^{m+p} g\left(\frac{j}{m}\right) \begin{pmatrix} m+p \\ j \end{pmatrix} \mu^j (1-\mu)^{m+p-j}, \mu \in [0, 1+p], \tag{1.2}$$

where $p \in \mathbb{N} \cup \{0\}$ and $g \in C[0, 1+p]$. But these sequences of operations given in (1.2) are restricted to $C[0, 1+p]$. The classical Szász-Schurer operators are linear positive operators and approximate the continuous functions over the positive semi axises. Several mathematicians have constructed various generalizations of Szász Schurer operators given by (1.1), e.g., Mursaleen et al. [4, 5, 6], Alotaibi et al. [7], Wafi et al. [8], Raiz et al. [9], Acu et al. [10], Mursaleen et al. ([11]-[12]), Aslan ([13]-[16]), Rao et al. ([17]-[19]), Khan et al. ([20]-[21]), Mohiuddine et al. [22], Nasiruzzaman et al. ([23, 24, 25, 26]), and Kajla et al. [28], etc. We also refer to reader for a deep historical background ([29]-[30]).
Motivated with the above development, we define Szász Schurer Durryemer type operators $S^{\nu}_{m+p} : L_B[0,\infty) \to L_B[0,\infty)$, with generalized beta function as ($L_B[0,\infty)$ denotes the space of bounded and Lebesgue measurable functions):

$$S^{\nu}_{m+p}(h;u) = \sum_{i=0}^{\infty} p_{m+p,i}(u) C^{\nu}_{m+p,i}(t)h(t), \tag{1.3}$$

where $C^{\nu}_{m+p,i}(t) = \int_0^1 D^{\nu}_{m+p,i}(t)dt$ and $D^{\nu}_{m+p,i}(t)$ is given by the formula

$$D^{\nu}_{m+p,i}(t) = \frac{t^{i\nu-1}(1-t)^{(m+p-i)\nu-1}}{B(i\nu,(m+p-i)\nu)}.$$

Now, we calculate some estimates for operators defined in equation (1.3).
Let $e_k(t) = t^k$, $k = 0,1,2,3,4$. Then, in the following Lemmas we give the some moments and estimates for the operators given by (1.3).

2 Basic estimates

Lemma 2.1. *For the operators $S^{\nu}_{m+p}(.;.)$ given by (1.3), the following identities are obtained*

$$\begin{aligned}
S^{\nu}_{m+p}(1;u) &= 1,\\
S^{\nu}_{m+p}(t;u) &= u,\\
S^{\nu}_{m+p}(t^2;u) &= \left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2 + \left(\frac{\nu+1}{(m+p)\nu+1}\right)u,
\end{aligned}$$

$$\begin{aligned}
S^{\nu}_{m+p}(t^3;u) &= \left(\frac{(m+p)^2\nu^2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^3\\
&+ \left(\frac{3(m+p)\nu(\nu+1)}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^2\\
&+ \left(\frac{\nu^2+3\nu+2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u,\\
S^{\nu}_{m+p}(t^4;u) &= \left(\frac{(m+p)^3\nu^3}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^4\\
&+ \left(\frac{6(m+p)^2\nu^2(\nu+1)}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^3\\
&+ \left(\frac{7(m+p)\nu^3+18(m+p)\nu^2+11(m+p)\nu}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^2\\
&+ \left(\frac{\nu^3+6\nu^2+11\nu+6}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u.
\end{aligned}$$

Proof. If $k = 0$, then

$$\begin{aligned}
(i)\ S^{\nu}_{m+p}(1;u) &= \sum_{i=0}^{\infty} p_{m+p,i}(u)\int_0^1 D^{\nu}_{m+p,i}(t)dt\\
&= \sum_{i=0}^{\infty} \frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)} \times B(i\nu,(m+p-i)\nu)\\
&= \sum_{i=0}^{\infty} p_{m+p,i}(u)\\
&= 1.
\end{aligned}$$

If $k=1$, then

$$\begin{aligned}(ii)\ S^{\nu}_{m+p}(t;u) &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\int_0^1 t^{i\nu}(1-t)^{(m+p-i)\nu-1}dt\\ &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\times B\left(i\nu+1,(m+p-i)\nu\right)\\ &= \sum_{i=0}^{\infty}\left(\frac{i}{m+p}\right)p_{m+p,i}(u)\\ &= u.\end{aligned}$$

If $k=2$, then

$$\begin{aligned}(iii)\ S^{\nu}_{m+p}(t^2;u) &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\int_0^1 t^{i\nu+1}(1-t)^{(m+p-i)\nu-1}dt\\ &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\times B\left(i\nu+2,(m+p-i)\nu\right)\\ &= \left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)\sum_{i=0}^{\infty}\left(\frac{i^2}{(m+p)^2}\right)p_{m+p,i}(u)\\ &+ \left(\frac{1}{(m+p)\nu+1}\right)\sum_{i=0}^{\infty}\left(\frac{i}{m+p}\right)p_{m+p,i}(u)\\ &= \left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{(m+p)\nu}{\nu+1}\right)u.\end{aligned}$$

If $k=3$, then

$$\begin{aligned}(iv)\ S^{\nu}_{m+p}(t^3;u) &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\int_0^1 t^{i\nu+2}(1-t)^{(m+p-i)\nu-1}dt\\ &= \sum_{i=0}^{\infty}\frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)}\times B\left(i\nu+3,(m+p-i)\nu\right)\\ &= \left(\frac{(m+p)^2\nu^2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)\sum_{i=0}^{\infty}\left(\frac{i^3}{(m+p)^3}\right)p_{m+p,i}(u)\\ &+ \left(\frac{3(m+p)\nu}{(m+p)^2\nu^2}(m+p)^2\nu^2+3(m+p)\nu+2\right)\\ &\quad \left(\sum_{i=0}^{\infty}\left(\frac{i^2}{(m+p)^2}\right)p_{m+p,i}(u)\right)\\ &+ \left(\frac{2}{(m+p)^2\nu^2}(m+p)^2\nu^2+3(m+p)\nu+2\right)\\ &\quad \left(\sum_{i=0}^{\infty}\left(\frac{i}{m+p}\right)p_{m+p,i}(u)\right)\\ &= \left(\frac{(m+p)^2\nu^2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^3\\ &+ \left(\frac{3(m+p)^2\nu^2+3(m+p)^2\nu^2}{(m+p)((m+p)^2\nu^2+3(m+p)\nu+2)}\right)u^2\\ &+ \left(\frac{3(m+p)^2\nu^2+3(m+p)^2\nu^2}{(m+p)^2((m+p)^2\nu^2+3(m+p)\nu+2)}\right)u.\end{aligned}$$

If $k = 4$, then

$$
\begin{aligned}
(v)\ S^{\nu}_{m+p}(t^4;u) &= \sum_{i=0}^{\infty} \frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)} \int_0^1 t^{i\nu+3}(1-t)^{(m+p-i)\nu-1}dt \\
&= \sum_{i=0}^{\infty} \frac{p_{m+p,i}(u)}{B(i\nu,(m+p-i)\nu)} \times B\left(i\nu+4,(m+p-i)\nu\right) \\
&= \left(\frac{(m+p)^3\nu^3}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right) \\
&\quad \left(\sum_{i=0}^{\infty}\left(\frac{i^4}{(m+p)^4}\right)p_{m+p,i}(u)\right) \\
&+ \left(\frac{6(m+p)^2\nu^2}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right) \\
&\quad \left(\sum_{i=0}^{\infty}\left(\frac{i^3}{(m+p)^3}\right)p_{m+p,i}(u)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{11(m+p)\nu}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)\sum_{i=0}^{\infty}\left(\frac{i^2}{(m+p)^2}\right)p_{m+p,i}(u) \\
&+ \left(\frac{6}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)\sum_{i=0}^{\infty}\left(\frac{i}{m+p}\right)p_{m+p,i}(u) \\
&= \left(\frac{(m+p)^3\nu^3}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^4 \\
&+ \left(\frac{6(m+p)^2\nu^3+6(m+p)^2\nu^2}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^3 \\
&+ \left(\frac{7(m+p)\nu^3+18(m+p)\nu^2+11(m+p)\nu}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u^2 \\
&+ \left(\frac{\nu^3+6\nu^2+11\nu+6}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)u.
\end{aligned}
$$

□

Lemma 2.2. *The central moments of beta Szász-Schurer operators using Lemma 2.1 are easily calculated as follows:*

$$
\begin{aligned}
S^{\nu}_{m+p}((t-u)^2;u) &= \left(\frac{(m+p)\nu}{(m+p)\nu+1}-1\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u = A^{\nu}_{m+p},\\
S^{\nu}_{m+p}((t-u)^4;u) &= \left(\frac{(m+p)^3\nu^3}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right. \quad (2.1)\\
&- \left.\frac{4(m+p)^2\nu^2}{(m+p)^2\nu^2+3(m+p)\nu+2}+\frac{6(m+p)\nu}{(m+p)\nu+1}-3\right)u^4\\
&+ \left(\frac{6(m+p)^2\nu^3(\nu+1)}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right. \quad (2.2)\\
&- \left.\frac{12(m+p)\nu(\nu+1)}{(m+p)^2\nu^2+3(m+p)\nu+2}+\frac{6(\nu+1)}{(m+p)\nu+1}\right)u^3\\
&+ \left(\frac{7(m+p)\nu^3+18(m+p)\nu^2+11(m+p)\nu}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right. \quad (2.3)\\
&- \left.\frac{4\nu^2+12\nu+8}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^2\\
&+ \left(\frac{\nu^3+6\nu^2+11\nu+6}{(m+p)^3\nu^3+6(m+p)^2\nu^2+11(m+p)\nu+6}\right)\\
&= B^{\nu}_{m+p}. \quad (2.4)
\end{aligned}
$$

3 Rate of convergence

Definition 3.1. The modulus smoothness for a uniformly continuous function τ is presented as:

$$\omega(\tau;\eta) = \sup_{|t-u|\le\eta}\{|\tau(t)-\tau(u)|, t,u\in[0,\infty)\},$$

for $\tau \in C[0,\infty)$.

For a uniformly continuous function τ in $C[0,\infty)$ and $\eta > 0$, one has

$$|\tau(t)-\tau(u)| \le \left(1+\frac{(1-t)^2}{\eta^2}\right)\omega(\tau;\eta). \quad (3.1)$$

Theorem 3.2. *For $S^{\nu}_{m+p}(.;.)$ the operators introduced by (1.3) and for each $\tau \in C[0,\infty)\cap E$, $S^{\nu}_{m+p}(\tau;u) \longrightarrow \tau(u)$ on each compact subset of $[0,\infty)$, where $E = \Big\{\tau : u \ge 0, \frac{\tau(u)}{1+u^2}$ is convergent as $u \longrightarrow \infty\Big\}$.*

Proof. In view of Korovkin-type property (iv) of Theorem 4.1.4 in [31], it is sufficient to show that $S^{\nu}_{m+p}(e_k;u) \longrightarrow e_k$, for $k = 0,1,2$. Using Lemma 2.1, it is obvious $S^{\nu}_{m+p}(e_0;u) \longrightarrow e_0(u)$ as $n \longrightarrow \infty$ and for $k = 1$

$$\lim_{m+p} S^{\nu}_{m+p}(e_1;u) = lim_{m+p\longrightarrow\infty}\left(\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u\right) = e_1(u).$$

Similarly, we can prove for $k = 2$, $S^{\nu}_{m+p}(e_2;u) \longrightarrow e_2(u)$, which completes the proof of Theorem 3.2. □

Theorem 3.3. *(See[32]) Let $L : C[c,d] \longrightarrow B[c,d]$ be the positive linear operator and let γ_u be the function defined by*

$$\beta_u(y) = |y-u|, (u,y) \in [c,d]\times[c,d].$$

If $\tau \in C_B([c,d])$, for any $u \in [c,d]$ and $\delta > 0$, the operator L verifies:

$$|(L\tau)(u)-\tau(u)| \le |\tau(u)||(Le_0)(u) - L|(Le_0)(u) + \lambda^{-1}\sqrt{(Le_0)(u)(L\gamma_u^2(u))}\omega_\tau(\lambda).$$

Theorem 3.4. *Let* $\tau \in C_B[0,\infty)$. *Then, for the operator* $S^{\nu}_{m+p}(.;.)$ *presented by* (1.3) *one has*

$$|S^{\nu}_{m+p}(\tau;u)-\tau(u)| \leq 2\omega(\tau;\lambda),\ \textit{where}\ \lambda=\sqrt{S^{\nu}_{m+p}(A^{\nu}_{m+p};u)}.$$

Proof. In terms of Lemma 2.1, 2.2 and Theorem 3.2, one has

$$\left|S^{\nu}_{m+p}(\tau;u)-\tau(u)\right| \leq \left\{1+\lambda^{-1}\sqrt{S^{\nu}_{m+p}(A^{\nu}_{m+p};u)}\right\}\omega(\tau;\lambda),$$

which prove the Theorem 3.4 choosing $\lambda=\sqrt{S^{\nu}_{m+p}(A^{\nu}_{m+p};u)}$. □

4 Local approximations

The local approximation results in $C_B[0,\infty)$, which is the space of real valued continuous and bounded functions equipped with norm, $||h|| = sup_{0\leq u<\infty}|h(u)|$. For any $h \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as:

$$K_2(h;\delta)=inf\left\{||h-f||+\delta||f^{''}|| : f\in C^2_B[0,\infty)\right\},$$

where $C^2_B[0,\infty)=\left\{f\in C_B[0;\infty) : f^{'},f^{''}\in C_B[0,\infty)\right\}$. By DeVore and Lorentz ([33] $p.177$, Theorem 2.4), there is fixed real constant $C>0$. As a result, it exists

$$K_2g(h,f)\leq C_B(h,\sqrt{\delta}). \tag{4.1}$$

The modulus of smoothness of second order is denoted by $\omega_2(.;.)$ and is defined as:

$$\omega_2(h;\sqrt{\delta})=sup_{0<h\leq\sqrt{\delta}}sup_{u\in[0,\infty)}|h(u+2h)-2h(u+h)+h(u)|.$$

Now, for $h\in C_B[0,\infty), u\geq 0$, the auxiliary operator is taken into consideration $\widehat{S}^{\nu}_{m+p}(.;.)$ as follows:

$$\widehat{S}^{\nu}_{m+p}(h;u)=S^{\nu}_{m+p}(h;u)+h(u)-h\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u. \tag{4.2}$$

Lemma 4.1. *Let* $h\in C^2_B[0,\infty)$. *Then, for all* $u\geq 0$, *one has*

$$|\widehat{S}^{\nu}_{m+p}(h;u)-h(u)|\leq \xi_{m+p}(u)||h^{''}||,$$

where

$$\begin{aligned}\xi_{m+p}(u) &= \left(\frac{(m+p)^2\nu^2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^3+\left(\frac{3(m+p)\nu(\nu+1)}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u^2\\ &+ \left(\frac{\nu^2+3\nu+2}{(m+p)^2\nu^2+3(m+p)\nu+2}\right)u.\end{aligned}$$

Proof. For the auxiliary operators are defined in the Definition (4.2), we have

$$\widehat{S}^{\nu}_{m+p}(1;u)=1,\ \widehat{S}^{\nu}_{m+p}(\eta_u;u)=0 \text{ and } |\widehat{S}^{\nu}_{m+p}(h;u)|\leq 3||h||. \tag{4.3}$$

By Taylor's expansion and $h\in C^2_B[0,\infty)$, we have

$$h(t)=h(v)+(t-v)h^{'}(v)+\int_v^t(t-v)g^{''}(v)dv. \tag{4.4}$$

Operating (4.2) both the side in above equation, we have

$$\widehat{S}^{\nu}_{m+p}(h;u)-h(u)=f^{'}(u)\widehat{S}^{\nu}_{m+p}(t-u;u)+\widehat{S}^{\nu}_{m+p}\left(\int_u^1(t-v)g^{''}(v)dv\right).$$

From (4.2) and (4.3), we have

$$\widehat{S}^{\nu}_{m+p}(h;u)-h(u)=\widehat{S}^{\nu}_{m+p}\left(\int_u^1 (t-v)(g)^{''}(v)dv;u\right)$$

$$=S^{\nu}_{m+p}\left(\int_u^1 (t-u)g^{''}(v)dv;u\right)$$

$$-\int_u^{\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u}\left(\frac{(m+p)\nu u^2}{(m+p)\nu+1}+\frac{(\nu+1)u}{(m+p)\nu+1}-v\right)g^{''}(v)dv. \tag{4.5}$$

Since,

$$\left|\int_u^1 (t-v)g^{''}(v)dv\right|\leq (t-u)^2||g^{''}||. \tag{4.6}$$

Then, we get

$$\left|\int_u^{\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u}\left(\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u-v\right)g^{''}(v)dv\right|$$

$$\leq\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}-u\right)^2||g^{''}||. \tag{4.7}$$

Applying (4.6) and (4.7) in (4.5), we obtain

$$\left|S^{\nu}_{m+p}(h;u)-h(u)\right|$$

$$\leq\left\{S^{\nu}_{m+p}((t-u)^2;u)+\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)u\right\}||h^{''}||$$

$$=\xi_{m+p}(u)||h^{''}||,$$

which completes the proof of Lemma 4.1. □

Theorem 4.2. *Let $h\in C_B^2[0,\infty)$. Then, there exist a constant $C>0$ such that*

$$|S^{\nu}_{m+p}(h;u)-h(u)|\leq C\omega_2(h;\sqrt{\xi_{m+p}})+\omega(h;S^{\nu}_{m+p}(\xi_{m+p};u)),$$

where $\xi_{m+p}(u)$ is defined by the Lemma 4.1.

Proof. For $g\in C_B^2[0,\infty)$, $h\in C_B[0,\infty)$ and in view of the definition of $\widehat{S}^{\nu}_{m+p}(.;.)$, one has

$$|S^{\nu}_{m+p}(h;u)-h(u)|\leq|\widehat{S}^{\nu}_{m+p}(h-g;u)|+|(h-g)(u)|+|\widehat{S}^{\nu}_{m+p}(g;u)-g(u)|$$
$$+\left|h\left(\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)\right)-h(u)\right|.$$

With the aid of Lemma 4.1 and relation in (4.3), we get

$$\begin{aligned}\left|S^{\nu}_{m+p}(h;u)-h(u)\right| &\leq 4||h-g||+|S^{\nu}_{m+p}(h;u)-h(u)|\\ &+\left|h\left(\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2+\left(\frac{\nu+1}{(m+p)\nu+1}\right)\right)-h(u)\right|\\ &\leq 4||h-g||+\xi_{m+p}(u)||g^{''}||+\omega(h;S^{\nu}_{m+p}(\xi_u;u)).\end{aligned}$$

From definition of Peetre's K-functional

$$\left|S_{m+p}^{\nu}(h;u)-h(u)\right| \leq C\omega_2\left(h;\sqrt{\xi_{m+p}(u)}\right)+\omega(h;S_{m+p}^{\nu}(\xi_u;u)),$$

which completes the proof of Theorem 4.2.
Let $\rho_1>0$ and $\rho_2>0$, are two fixed real values. Then, we recall Lipschitz-type space here [34] as:

$$Lip_M^{\rho_1\rho_2}(\gamma) := \left\{h \in C_B[0,\infty) : |h(t)-h(u)| \leq M\frac{|t-u|^{\gamma}}{(t+\rho_1 u+\rho_2 u^2)^{\gamma/2}} : u,t \in (0,\infty)\right\},$$

$M>0$ is a constant and $0<\gamma\leq 1$. □

Theorem 4.3. *Let $h \in Lip_M^{\rho_1,\rho_2}(\gamma)$ and $u \in (0,\infty)$. Then, for the operators defined by (1.3), one has*

$$\left|S_{m+p}^{\nu}(h;u)-h(u)\right| \leq M\left(\frac{\eta_{m+p}(u)}{\rho_1 u+\rho_2 u^2}\right)^{\frac{\gamma}{2}}, \tag{4.8}$$

where $\gamma \in (0,1)$ and $\eta_{m+p}(u)=S_{m+p}^{\nu}(\xi_u^2;u)$.

Proof. For $\gamma=1$ and $u\in[0,\infty)$, one has

$$\left|S_{m+p}^{\nu}(h;u)-h(u)\right| \leq S_{m+p}^{\nu}(|h(t)-h(u);u) \leq MS_{m+p}^{\nu}\left(\frac{|t-u|}{(t+\rho_1 u+\rho_2 u^2)^{1/2}};u\right).$$

It is obvious that

$$\frac{1}{t+\rho_1 u+\rho_2 u^2}<\frac{1}{(\rho_1 u+\rho_2 u^2)},$$

for all $u\in[0;\infty)$, we have

$$\begin{aligned}\left|S_{m+p}^{\nu}(h;u)-h(u)\right| &\leq \frac{M}{(\rho_1 u+\rho_2 u^2)^{1/2}}\left(S_{m+p}^{\nu}(t-u)^2;u\right)^{1/2}\\ &\leq M\left(\frac{\eta_{m+p}(u)}{\rho_1 u+\rho_2 u^2}\right)^{1/2}.\end{aligned}$$

Using Hölder's inequality, the Theorem 4.3 now holds for $\gamma=1$ and $\gamma\in(0,\infty)$. with $q_1=2/\gamma$ and $q_2=2/2-\gamma$, we have

$$\begin{aligned}\left|S_{m+p}^{\nu}(h;u)-h(u)\right| &\leq \left(S_{m+p}^{\nu}(|h(t)-h(u)|^{\gamma/2};u)\right)^{\gamma/2}\\ &\leq MS_{m+p}^{\nu}\left(\frac{|t-u|^2}{t+\rho_1 u+\rho_2 u^2};u\right)^{\gamma/2}.\end{aligned}$$

Since $\frac{1}{t+\rho_1 u+\rho_2 u^2}<\frac{1}{\rho_1 u+\rho_2 u^2}$ for all $u\in(0,\infty)$, we have

$$|S_{m+p}^{\nu}(h;u)-h(u)| \leq M\left(\frac{S_{m+p}^{\nu}(|t-u|^2;u)}{\rho_1 u+\rho_2 u^2}\right)^{\gamma/2} \leq M\left(\frac{\eta_{m+p}(u)}{\rho_1 u+\rho_2 u^2}\right)^2.$$

This completes the proof of Theorem 4.3.
Now, we recall r^{th} term order Lipschitz-type maximal function suggested by Lenze [35] as:

$$\widetilde{\omega}(h;u)=\sup_{\substack{t\neq u}},_{t\in(0,\infty)}\frac{|h(v)-f(u)|}{|v-u|^r}, u\in[0;\infty), \tag{4.9}$$

and $r\in(0,1]$. □

Theorem 4.4. *Let $h \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for all $u \in [0,\infty)$, one has*

$$\left|S^{\nu}_{m+p}(h;u) - h(u)\right| \leq \widetilde{\omega}_r(h;u)(\eta_{m+p}(u))^{\gamma/2}.$$

Proof. We know that

$$\left|S^{\nu}_{m+p}(h;u) - h(u)\right| \leq S^{\nu}_{m+p}\left(|h(v) - h(u)|;u\right).$$

From equation (4.9), one has

$$\left|S^{\nu}_{m+p}(h;u) - h(u)\right| \leq \widetilde{\omega_r}\left((h;u)(S^{\nu}_{m+p}|v-u|^r;u)\right).$$

By Hölder's inequality with $q_1 = 2/r$ and $q_2 = 2/2 - r$, we have

$$\left|S^{\nu}_{m+p}(h;u) - h(u)\right| \leq \widetilde{\omega_r}(h;u)\left(S^{\nu}_{m+p}|v-u|^2;u\right)^{r/2},$$

we arrive at our the desired result. □

5 Weighted approximation

To establish the next result, we recall some notation from [36]. Assume that $B_{1+u^2}[0,\infty) = \left\{h(u) : |h(u)| \leq M_h(1+u^2)\right\}$, is weighted functional space, M_h is a constant that is determined by h and in $B_{1+u^2}[0,\infty)$, $u \in [0,\infty), C_{1+u^2}[0,\infty)$ is the space continuous functions with the norm

$$||h(u)||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|h(u)|}{1+u^2},$$

and

$$C^k_{1+u^2}[0,\infty) = \left\{h \in C_{1+u^2}[0,\infty) : \lim_{|u|\longrightarrow\infty} \frac{h(u)}{1+u^2} = K\right\},$$

K is a constant that depends on h.

The modulus of continuity for the function h with $a > 0$ and a closed interval $[0,a]$ is as follows:

$$\omega_a(h;\delta) = \sup_{|v-u|\leq\delta}\ \sup_{u,v\in[0,a]} |h(v) - h(u)|. \tag{5.1}$$

Here, we observe that for $h \in C_{1+u^2}[0,\infty)$, the modulus of continuity tends to zero.

Theorem 5.1. *For $h \in C_{1+u^2}[0,\infty)$ and its modulus of continuity $\omega_{b+1}(h;\delta)$ defined on $[0,b+1] \in [0,\infty)$, we have*

$$||S^{\nu}_{m+p}(h;u) - h(u)||_{C[0,b]} \leq 6M_h(1+b)\delta_{(m+p)}(b) + 2\omega_{b+1}\left(h;\sqrt{\delta_{(m+p)}(b)}\right),$$

where

$$\delta_{m+p}(b) = S^{\nu}_{m+p}(\phi^2_b;b).$$

Proof. From ([37] p.378) for $u \in [a,b]$ and $v \in [0,\infty)$, we have

$$|h(v) - h(u)| \leq 6M_h(1+b^2)(v-u)^2 + \left(1 + \frac{|v-u|}{\delta}\right)\omega_{b+1}(h;\delta).$$

Applying both side $S^{\nu}_{m+p}(.;.)$, one has

$$|S^{\nu}_{m+p}h(v) - h(u)| \leq 6M_h(1+b^2)|S^{\nu}_{m+p}(v-u)^2 + \left(1 + \frac{|S^{\nu}_{m+p}(1+\sqrt{|v-u|})}{\delta}\right)\omega_{b+1}(h;\delta).$$

In view of Lemma (2.2) and $u \in [a,b]$, we get

$$|S^{\nu}_{m+p}(h;u) - h(u)| \leq 6M_h(1+b)\delta_{(m+p)}(b) + \left(1 + \frac{\sqrt{\delta_{(m+p)}(b)}}{\delta}\right)\omega_{b+1}(h;\delta).$$

Choosing $\delta = \delta_{m+p}(b)$, we arrive at our desired result □

Theorem 5.2. *If the operators $S^{\nu}_{m+p}(.;.)$ given by (1.3) from $C^k_{1+u^2}[0;\infty)$ to $B_{1+u^2}[0;\infty)$ satisfying the conditions*

$$\lim_{m+p\longrightarrow\infty} ||S^{\nu}_{m+p}(e_j;.) - e_j||_{1+u^2} = 0,$$

for $j = 0, 1, 2$ Then, for each $h \in C^k_{1+u^2}[0,\infty)$, one has

$$\lim_{m+p\longrightarrow\infty} ||S^{\nu}_{m+p}(h;.) - h||_{1+u^2} = 0.$$

Proof. To prove this Theorem, it is enough to show that

$$\lim_{m+p\longrightarrow\infty} ||S^{\nu}_{m+p}(e_j;.) - e_j||_{1+u^2} = 0, j = 0, 1, 2.$$

From Lemma 2.2, we have for $j = 0$

$$||S^{\nu}_{m+p}(e_0;.) - e_0||_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|S^{\nu}_{m+p}(e_0;.) - 1|}{1+u^2} = 0.$$

For $j = 1$ it is obvious.
For $j = 2$

$$||S^{\nu}_{m+p}(e_2;.) - e_2||_{1+u^2} = \sup_{u\in[0;\infty)} \frac{\left|\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right)u^2 + \left(\frac{\nu+1}{(m+p)\nu+1}\right)u - u^2\right|}{1+u^2},$$

$$= \sup_{u\in[0,\infty)} \frac{u^2}{1+u^2}\left(\frac{(m+p)\nu}{(m+p)\nu+1}\right) + \left(\frac{\nu+1}{(m+p)\nu+1}\right) sup_{u\in[0,\infty)} \frac{u}{1+u^2}.$$

This implies that $||S^{\nu}_{m+p}(e_2;.) - e_2||_{1+u^2} \longrightarrow 0$ as $m+p \longrightarrow \infty$. Hence, we complete the proof of Theorem 5.2. □

Theorem 5.3. *Let $h \in C^k_{1+u^2}[0,\infty)$ and $\gamma > 0$ Then, we have*

$$\lim_{m+p\longrightarrow\infty} sup_{u\in[0,\infty)} \frac{|S^{\nu}_{m+p}(h;u) - h(u)|}{(1+u^2)^{1+\gamma}} = 0.$$

Proof. For fixed number $u_0 > 0$, one possesses

$$\begin{aligned} sup_{u\in(0,\infty)} \frac{|S^{\nu}_{m+p}(h;u) - h(u)|}{(1+u^2)^{1+\gamma}} &\leq sup_{u\leq u_0} \frac{|S^{\nu}_{m+p}(h;u) - h(u)|}{(1+u^2)^{1+\gamma}} \\ &+ sup_{u\geq u_0} \frac{|S^{\nu}_{m+p}(h;u) - h(u)|}{(1+u^2)^{1+\gamma}} \\ &\leq ||S^{\nu}_{m+p}(h.;.)| - h||_{C[0,u_0]} \\ &+ ||h||_{1+u^2} sup_{u\geq u_0} \frac{|S^{\nu}_{m+p}(1+v^2;u)|}{(1+u^2)^{1+\gamma}} \\ &+ sup_{u\geq u_0} \frac{|h(u)|}{(1+u)^{1+u}} \\ &= T_1 + T_2 + T_3. \end{aligned} \tag{5.2}$$

Since
$|h(a)| \leq ||h||_{1+u^2}(1+u^2)$, we have

$$T_3 = sup_{u\geq u_0} \frac{|h(u)|}{(1+u^2)^{1+\gamma}} \leq sup_{u\geq u_0} \frac{||h||_{1+u^2}}{(1+u^2)^{1+\gamma}} \leq \frac{||h||_{1+u^2}}{(1+u^2)^{\gamma}}.$$

Let $\epsilon > 0$ be random real number. Then, from (3.2), there exist $m_1 \in \mathbb{N}$, such that

$$T_2 > \frac{1}{(1+u^2)^{\gamma}} ||h||_{1+u^2} \left(1 + u^2 + \frac{\epsilon}{3||h||_{1+u^2}}\right),$$

for all

$$m_1 \geq m + p \leq \frac{||h||_{1+u^2}}{(1+u^2)^{\gamma}} + \frac{\epsilon}{3},$$

for all $m_1 \geq m + p$. This implies that ,$T_2 + T_3 < 2\frac{||h||_{1+u^2}}{1+u^2} + \frac{\epsilon}{3}$.

For a large enough value of u_0, we get $\frac{||h||_{1+u^2}}{1+u^2} + \frac{\epsilon}{6}$,

$$T_2 + T_3 < \frac{2\epsilon}{3} \text{ for all } m_1 \geq m + p. \tag{5.3}$$

By Theorem 5.2, there exist $m_2 > m + p$ such that

$$T_1 = ||S^{\nu}_{m+p}(h) - h||_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } \ m_2 \geq m. \tag{5.4}$$

Let $m_3 = max\{m_1 + p, m_2 + p\}$. Then, joining (5.2), (5.3) and (5.4) we get

$$sup_{u\in[0,\infty)} \frac{|S^{\nu}_{m+p}(h;u) - h(u)|}{(1+u^2)^{\gamma}} < \epsilon.$$

The proof of the above theorem (5.2) is complete. □

6 Bivariate case of extended Beta type Szász-Schurer operator $S^{\nu}_{m+p}(h;u)$

Take $T^2 = \{(u_1, u_2) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}$ and $C(T^2)$ is the class of all continuous function on T^2 equipped with norm $||f||_{C(T^2)} = \sup_{(u_1,u_2)\in T^2} |f(u_1, u_2)|$. Then, for all $g \in C(T^2)$ and $m_1 + p, m_2 + p \in \mathbb{N}$, we introduce a bivariate sequence as:

$$S^{\nu}_{m_1+p,m_2+p}(h;u_1,u_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{m_1+p,m_2+p,j,k}(t_1,t_2)p_{m_1+p,m_2+p,j,k}(u_1,u_2)h(t_1,t_2) \tag{6.1}$$

where

$$\begin{aligned} C^{\nu}_{m_1+p,m_2+p,j,k}(t_1,t_2) &= C^{\nu}_{m_1+p,j}(t_1)C^{\nu}_{m_2+p,k}(t_2), \\ with\ C^{\nu}_{m_i+p,j}(t_i) &= \int_0^1 D^{\nu}_{m_i+p,j}(t_i)dt_i,\ for\ i = 1,2 \end{aligned}$$

and

$$\begin{aligned} p_{m_1+p,m_2+p,j,k}(u_1,u_2) &= p_{m_1+p,j}(u_1)p_{m_2+p,k}(u_2), \\ with\ p_{m_i+p,j}(u_i) &= e^{-(m_i+p)u_i}\frac{((m_i+p)u_i)^j}{j!},\ for\ i = 1,2. \end{aligned}$$

Lemma 6.1. *Let $e_{j,k} = u_1^j u_2^k$. Then, for the operator (6.1), we get*

$$
\begin{aligned}
S^{\nu}_{m_1+p,m_2+p}(e_{0,0};u_1,u_2) &= 1,\\
S^{\nu}_{m_1+p,m_2+p}(e_{1,0};u_1,u_2) &= u_1,\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,1};u_1,u_2) &= u_2,\\
S^{\nu}_{m_1+p,m_2+p}(e_{2,0};u_1,u_2) &= \left(\frac{(m_1+p)\nu}{(m_1+p)\nu+1}\right)u_1^2+\left(\frac{\nu+1}{(m_1+p)\nu+1}\right)u_1,\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,2};u_1,u_2) &= \left(\frac{(m_2+p)\nu}{(m_2+p)\nu+1}\right)u_2^2+\left(\frac{\nu+1}{(m_2+p)\nu+1}\right)u_2,\\
S^{\nu}_{m_1+p,m_2+p}(e_{3,0};u_1,u_2) &= \left(\frac{(m_1+p)^2\nu^2}{(m_1+p)^2\nu^2+3(m_1+p)\nu+2}\right)u_1^3\\
&+ \left(\frac{3(m_1+p)\nu(\nu+1)}{(m_1+p)^2\nu^2+3(m_1+p)\nu+2}\right)u_1^2\\
&+ \left(\frac{\nu^2+3\nu+2}{(m_1+p)^2\nu^2+3(m_1+p)\nu+2}\right)u_1,\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,3};u_1,u_2) &= \left(\frac{(m_2+p)^2\nu^2}{(m_2+p)^2\nu^2+3(m_2+p)\nu+2}\right)u_2^3\\
&+ \left(\frac{3(m_2+p)\nu(\nu+1)}{(m_2+p)^2\nu^2+3(m_2+p)\nu+2}\right)u_2^2\\
&+ \left(\frac{\nu^2+3\nu+2}{(m_2+p)^2\nu^2+3(m_2+p)\nu+2}\right)u_2.
\end{aligned}
$$

Proof. From (2.1) and linearity, property we get

$$
\begin{aligned}
S^{\nu}_{m_1+p,m_2+p}(e_{0,0};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{1,0};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_1;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,1};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_1;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{2,0};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_2;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,2};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_2;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{3,0};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_3;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2),\\
S^{\nu}_{m_1+p,m_2+p}(e_{0,3};u_1,u_2) &= S^{\nu}_{m_1+p,m_2+p}(e_0;u_1,u_2)S^{\nu}_{m_1+p,m_2+p}(e_3;u_1,u_2).
\end{aligned}
$$

□

7 Degree of Convergence

For any $g \in C(\mathcal{T}^2)$ and $\eta > 0$, the modulus of continuity of the second order is given by

$$\omega\left(g;\delta_{n_1},\eta_{n_2}\right) = \sup_{(u_1,u_2)\in T^2}\{\mid g(t,s)-g(u_1,u_2)\mid : (t,s),(u_1,u_2)\in\mathcal{T}^2\},$$

with $\mid t-u_1 \mid\leq \eta_{n_1},\ \mid s-u_2\mid\leq\eta_{n_2}$ defined by the partial modulus of continuity as:

$$\omega_1\left(g;\eta\right) = \sup_{0\leq u_2\leq\infty}\ \sup_{|x_1-x_2|\leq\eta}\{\mid g(x_1,u_2)-g(x_2,u_2)\mid\},$$

$$\omega_2\left(g;\eta\right) = \sup_{0\leq u_1\leq\infty}\ \sup_{|u_1-u_2|\leq\eta}\{\mid g(u_1,u_1)-g(u_1,u_2)\mid\}.$$

Theorem 7.1. *For any $g \in C(\mathcal{T}^2)$, we have*

$$\mid S^{\nu}_{m_1+p,m_2+p}(g;u_1,u_2)-g(u_1,u_2)\mid\leq 2\Big(\omega_1(g;\delta_{n_1,u_1})+\omega_2(g;\delta_{n_2,u_2})\Big).$$

Proof. In order to give the proof of Theorem 7.1, generally, we use the well-known Cauchy-Schwartz inequality. Thus, we see that

$$\begin{aligned} \mid S^{\nu}_{m_1+p,m_2+p}(g;u_1,u_2) &- g(u_1,u_2) \mid\leq S^{\nu}_{m_1+p,m_2+p}\left(\mid g(t,s)-g(u_1,u_2)\mid;u_1,u_2\right) \\ &\leq S^{\nu}_{m_1+p,m_2+p}\left(\mid g(t,s)-g(u_1,s)\mid;u_1,u_2\right) \\ &+ S^{\nu}_{m_1+p,m_2+p}\left(\mid g(u_1,s)-g(u_1,u_2)\mid;u_1,u_2\right) \\ &\leq S^{\nu}_{m_1+p,m_2+p}\left(\omega_1(g;\mid t-u_1\mid);u_1,u_2\right) \\ &+ S^{\nu}_{m_1+p,m_2+p}\left(\omega_2(g;\mid s-u_2\mid);u_1,u_2\right) \\ &\leq \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}S^{\nu}_{m_1+p,m_2+p}(\mid t-u_1\mid;u_1,u_2)\right) \\ &+ \omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}S^{\nu}_{m_1+p,m_2+p}(\mid s-u_2\mid;u_1,u_2)\right) \\ &\leq \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{S^{\nu}_{m_1+p,m_2+p}((t-u_1)^2;u_1,u_2)}\right) \\ &+ \omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{S^{\nu}_{m_1+p,m_2+p}((s-u_2)^2;u_1,u_2)}\right). \end{aligned}$$

If we choose $\delta^2_{n_1}=\delta^2_{n_1,u_1}=S^{\nu}_{m_1+p,m_2+p}((t-u_1)^2;u_1,u_2)$ and $\delta^2_{n_2}=\delta^2_{n_2,u_2}=S^{\nu}_{m_1+p,m_2+p}((s-u_2)^2;u_1,u_2)$, then we can simply achieve our objectives. □

Here, we analyse convergence in terms of the Lipschitz class for bivariate functions. For $M>0$ and $\tau,\tau\in[0,1]$, maximal Lipschitz function space on $E\times E\subset\mathcal{T}^2$ given by

$$\begin{aligned} \mathcal{L}_{\tau,\tau}(E\times E) &= \Big\{g:\sup(1+t)^{\tau}(1+s)^{\tau}\left(g_{\tau,\tau}(t,s)-g_{\tau,\tau}(u_1,u_2)\right) \\ &\leq M\frac{1}{(1+u_1)^{\tau}}\frac{1}{(1+u_2)^{\tau}}\Big\}, \end{aligned}$$

where g is continuous and bounded on $\mathcal{T}^2$, and

$$g_{\tau,\tau}(t,s)-g_{\tau,\tau}(u_1,u_2)=\frac{\mid g(t,s)-g(u_1,u_2)\mid}{\mid t-u_1\mid^{\tau}\mid s-u_2\mid^{\tau}};\quad (t,s),(u_1,u_2)\in\mathcal{T}^2. \tag{7.1}$$

Theorem 7.2. *Let $g\in\mathcal{L}_{\tau,\tau}(E\times E)$. Then, for any $\tau,\tau\in[0,1]$, there exists $M>0$ such that*

$$\begin{aligned} \mid S^{\nu}_{m_1+p,m_2+p}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq M\Bigg\{\left((d(u_1,E))^{\tau}+\left(\delta^2_{n_1,u_1}\right)^{\frac{\tau}{2}}\right) \\ &\times \left((d(u_2,E))^{\tau}+\left(\delta^2_{n_2,u_2}\right)^{\frac{\tau}{2}}\right) \\ &+ \left(d((u_1,E))^{\tau}\left(d(u_2,E)\right)^{\tau}\right)\Bigg\}, \end{aligned}$$

where δ_{n_1,u_1} and δ_{n_2,u_2} defined by Theorem 7.1.

Proof. Consider $\mid u_1-x_0\mid=d(u_1,E)$ and $\mid u_2-y_0\mid=d(u_2,E)$, for any $(u_1,u_2)\in\mathcal{T}^2$, and $(x_0,y_0)\in E\times E$.
Let $d(u_1,E)=\inf\{\mid u_1-u_2\mid: u_2\in E\}$. Then, we write here

$$\mid g(t,s)-g(u_1,u_2)\mid\leq M\mid g(t,s)-g(x_0,y_0)\mid+\mid g(x_0,y_0)-g(u_1,u_2)\mid. \tag{7.2}$$

Apply $S^{\nu}_{m_1+p,m_2+p}(.;.,.)$, we obtain

$$\begin{aligned} \mid S^{\nu}_{m_1+p,m_2+p}(g;u_1,u_2) &- g(u_1,u_2)\mid \\ &\leq S^{\nu}_{m_1+p,m_2+p}\left(\mid g(u_1,u_2)-g(x_0,y_0)\mid+\mid g(x_0,y_0)-g(u_1,u_2)\mid\right) \\ &\leq MS^{\nu}_{m_1+p,m_2+p}\left(\mid t-x_0\mid^{\tau}\mid s-y_0\mid^{\tau};u_1,u_2\right) \\ &+ M\mid u_1-x_0\mid^{\tau}\mid u_2-y_0\mid^{\tau}. \end{aligned}$$

For all $A, B \geq 0$ and $\tau \in [0,1]$, the inequality $(A+B)^\tau \leq A^\tau + B^\tau$, thus

$$\mid t - x_0 \mid^\tau \leq \mid t - u_1 \mid^\tau + \mid u_1 - x_0 \mid^\tau,$$

$$\mid s - y_0 \mid^\tau \leq \mid s - u_2 \mid^\tau + \mid u_2 - y_0 \mid^\tau .$$

Therefore,

$$\begin{aligned} &\mid S^\nu_{m_1+p,m_2+p}(g; u_1, u_2) - g(u_1, u_2) \mid \\ \leq\ & M S^\nu_{m_1+p,m_2+p}\left(\mid t - u_1 \mid^\tau \mid s - u_2 \mid^\tau; u_1, u_2\right) \\ +\ & M \mid u_1 - x_0 \mid^\tau S^\nu_{m_1+p,m_2+p}\left(\mid s - u_2 \mid^\tau; u_1, u_2\right) \\ +\ & M \mid u_2 - y_0 \mid^\tau S^\nu_{m_1+p,m_2+p}\left(\mid t - u_1 \mid^\tau; u_1, u_2\right) \\ +\ & 2M \mid u_1 - x_0 \mid^\tau \mid u_2 - y_0 \mid^\tau S^\nu_{m_1+p,m_2+p}\left(\mu_{0,0}; u_1, u_2\right). \end{aligned}$$

Apply Hölders inequality on $S^\nu_{m_1+p,m_2+p}(.;.,.)$, we get

$$\begin{aligned} S^\nu_{m_1+p,m_2+p}\left(\mid t - u_1 \mid^\tau \mid s - u_2 \mid^\tau; u_1, u_2\right) &= \mathcal{U}^{\lambda_1}_{n_1,k}\left(\mid t - u_1 \mid^\tau; u_1, u_2\right) \\ &\times \mathcal{V}^{\lambda_2}_{n_2,l}\left(\mid s - u_2 \mid^\tau; u_1, u_2\right) \\ &\leq \left(S^\nu_{m_1+p,m_2+p}(\mid t - u_1 \mid^2; u_1, u_2)\right)^{\frac{\tau}{2}} \\ &\times \left(S^\nu_{m_1+p,m_2+p}(\mu_{0,0}; u_1, u_2)\right)^{\frac{2-\tau}{2}} \\ &\times \left(S^\nu_{m_1+p,m_2+p}(\mid s - u_2 \mid^2; u_1, u_2)\right)^{\frac{\tau}{2}} \\ &\times \left(S^\nu_{m_1+p,m_2+p}(\mu_{0,0}; u_1, u_2)\right)^{\frac{2-\tau}{2}}. \end{aligned}$$

Thus, we can obtain

$$\begin{aligned} \mid S^\nu_{m_1+p,m_2+p}(g; u_1, u_2) - g(u_1, u_2) \mid &\leq M\left(\delta^2_{n_1,u_1}\right)^{\frac{\tau}{2}}\left(\delta^2_{n_2,u_2}\right)^{\frac{\tau}{2}} \\ &+ 2M\left(d(u_1, E)\right)^\tau \left(d(u_2, E)\right)^\tau \\ &+ M\left(d(u_1, E)\right)^\tau \left(\delta^2_{n_2,u_2}\right)^{\frac{\tau}{2}} \\ &+ L\left(d(u_2, E)\right)^\tau \left(\delta^2_{n_1,u_1}\right)^{\frac{\tau}{2}}. \end{aligned}$$

Which completes the proof. □

8 Conclusion

In this study, we introduce generalized beta Szász-Schurer operators and study their approximation properties. Further, we prove a Korovkin-type convergence theorem, the order of convergence concerning the usual modulus of continuity and as well as Peetre's K-functional and Lipschitz-type class of functions. Moreover, we introduced global approximation results and A-Statistical approximation properties of the constructed operators.

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