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AN ANALYTICAL OBSERVATION FOR SET-THEORETICAL SOLUTIONS OF YANG-BAXTER EQUATION ON BCC-ALGEBRAS

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Abstract In this paper, we explore set-theoretical solutions to the Yang-Baxter equation within the framework of BCC-algebras. We systematically analyze various mappings to determine their adherence to the braid condition. First we produce basic results by identifying several mappings, including the identity map, constant mappings, and combinations involving elements and constants, as solutions to the Yang-Baxter equation. We then introduce the left and right α-extension mappings and demonstrate that they satisfy the braid condition when idempotent. Our results are further supported by examining the validity of right extension mappings under similar conditions. We conclude with generalized findings that confirm the braid condition for combinations of left and right extensions and scenarios where idempotent extensions are equal or interchanged. This study significantly enhances the understanding of BCC-algebras and their role in solving the Yang-Baxter equation.

1 Introduction

The Yang-Baxter equation, initially introduced by the Nobel laureate C.N. Yang in the realm of theoretical physics [1], and independently by Baxter in the field of statistical mechanics [2, 3], has garnered significant attention across various disciplines. This equation plays a pivotal role in knot theory, link invariants, quantum computing, braided categories, quantum groups, integrable systems, and quantum mechanics. In pure mathematics, the quest to find set-theoretical solutions within algebraic structures has been particularly influential. Notable contributions include the examination of algebraic structures arising from Yang-Baxter systems by Berceanu et al. [4], the construction of new set-theoretical solutions in MV-algebras by Oner, Senturk et al. [5], and the exploration of classical solutions for simple Lie algebras by Belavin and Drinfeld [6]. Further, Senturk et al. studies on set-theoretical solutions for the Yang-Baxter equation in triangle algebras [7], Massuyeau and Nichita's knot invariants from Yang-Baxter operators [8], Gateva-Ivanova's work on braces and symmetric groups [9], Wang and Ma's framework for singular solutions [10], and Nichita and Parashar's studies on spectral-parameter dependent operators [11], have significantly advanced the field. Similarly, Rota-Baxter operators are studied on complex-semi-simple algebras [12].

In recent years, there has been a growing focus on t-norm based logic systems and their corresponding pseudo-logic systems, driven by both theoretical and practical motivations. The algebraic investigations of these systems often precede their logical counterparts, as evidenced in BCK, BCI and BE-algebras, which are inspired by implicational logic. These algebras, along with their corresponding logics, often have a strong connection that allows for the translation of well-formed formulas and theorems into algebraic terms and theorems.[13, 14, 15, 16]

To address specific problems in BCK-algebras, Y. Komori introduced BCC-algebras [17], which are also known as BIK$^+$-algebras [18] or BZ-algebras [19] due to their close relationship with BIK$^+$ logic. Various generalizations of BCC-algebras have been studied extensively, particularly by mathematicians from China, Japan, and Korea. These algebras share a common set of identities, with one of the most significant being $(xy)z = (xz)y$. This identity is prevalent in pre-logics [20], Hilbert algebras, implication algebras [21], and MV-algebras [22]. Although BCK-algebras satisfy this identity, it does not hold in BCC-algebras unless specific conditions are met. The class of all bounded commutative BCC-algebras is equivalent to the class of all MV-algebras [23], which justifies the exploration of such BCC-algebras and their generalizations, particularly those satisfying this identity under certain conditions [24].

In this manuscript, we have explored the set-theoretical solutions to the Yang-Baxter equation within the framework of BCC-algebras. By developing and analyzing various mappings, including the identity map, constant mappings, and specific combinations of elements and constants, we have demonstrated their adherence to the braid condition, confirming their validity as solutions to the Yang-Baxter equation. Further, we introduced and examined the left and right α-extension mappings, proving their role as viable solutions through rigorous algebraic analysis, and showed that both left and right extension mappings, when defined with idempotent properties, serve as robust solutions. These findings were generalized through theorems, highlighting their applicability and reinforcing the versatility of BCC-algebras in addressing complex algebraic equations. Concrete examples illustrated the practical implementation of these theoretical results, reinforcing their significance with tangible applications. Our findings underscore the potential of BCC-algebras in broader mathematical contexts. Future research may delve deeper into the implications of these solutions, exploring their potential applications in fields such as cryptography, quantum computing, and other areas where the Yang-Baxter equation plays a critical role.

2 Preliminaries

This section encompasses essential definitions, propositions and some concepts pertaining to BCC-algebras and the Yang-Baxter equation, laying the groundwork for subsequent sections.

Definition 2.1. [25] A BCC-algebra is defined as a non-empty set X equipped with a binary internal operation "$\cdot$" and a distinguished element "0" that satisfy the following axioms:

(BCC_1) For all $x, y, z \in X$, $((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$;

(BCC_2) For all $x \in X$, $x \cdot x = 0$;

(BCC_3) For all $x \in X$, $x \cdot 0 = 0$;

(BCC_4) For all $x \in X$, $0 \cdot x = x$; and

(BCC_5) For all $x, y \in X$, if $x \cdot y = 0$ and $y \cdot x = 0$, then $x = y$.

Proposition 2.2. *In a BCC-algebra B, the following properties hold for any $x, y, z \in B$:*

(i) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(ii) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(iii) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(iv) $x \cdot (y \cdot x) = 0$,

(v) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(vi) $x \cdot (y \cdot y) = 0$.

Now, we explain Yang-Baxter equation on BCC-algebras.

Let F_{BCC} be a field where the tensor product is defined, and consider W_{BCC} as an F_{BCC}-vector space. We represent the mapping on $W_{BCC} \otimes W_{BCC}$ by δ. The twist map for this structure is given by $\delta(w_1 \otimes w_2) = w_2 \otimes w_1$, while the identity map on F_{BCC} is indicated by $I : W_{BCC} \to W_{BCC}$. For an F_{BCC}-linear map $\Im : W_{BCC} \otimes W_{BCC} \to W_{BCC} \otimes W_{BCC}$, we define $\Im^{12} = \Im \otimes I$, $\Im^{13} = (I \otimes \delta)(\Im \otimes I)(\delta \otimes I)$, and $\Im^{23} = I \otimes \Im$.

Definition 2.3. [26] A Yang–Baxter operator is defined as an invertible F_{BCC}-linear map $\Im : W_{BCC} \otimes W_{BCC} \to W_{BCC} \otimes W_{BCC}$ that adheres to the braid condition, commonly known as the Yang–Baxter equation:

$$\Im^{12} \circ \Im^{23} \circ \Im^{12} = \Im^{23} \circ \Im^{12} \circ \Im^{23}. \tag{2.1}$$

If $\Im$ satisfies Equation (2.1), then the compositions $\Im \circ \mu$ and $\mu \circ \Im$ fulfill the quantum Yang–Baxter equation (QYBE):

$$\Im^{12} \circ \Im^{13} \circ \Im^{23} = \Im^{23} \circ \Im^{13} \circ \Im^{12}. \tag{2.2}$$

To determine set-theoretical solutions of the Yang–Baxter equation within BCC-algebras, the following definition is pivotal.

Definition 2.4. [26] Consider a set B, and let $\Im : B \times B \to B \times B$ be a mapping given by $\Im(b_1, b_2) = (\overline{b_1}, \overline{b_2})$. The map $\Im$ is a set-theoretical solution of the Yang–Baxter equation if it satisfies:

$$\Im^{12} \circ \Im^{23} \circ \Im^{12} = \Im^{23} \circ \Im^{12} \circ \Im^{23}, \tag{2.3}$$

which is also equivalent to:

$$\Im^{12} \circ \Im^{13} \circ \Im^{23} = \Im^{23} \circ \Im^{13} \circ \Im^{12}. \tag{2.4}$$

where

$$\begin{aligned}
&\Im^{12} : B^3 \to B^3, \qquad && \Im^{12}(b_1, b_2, b_3) = (\overline{b_1}, \overline{b_2}, b_3),\\
&\Im^{23} : B^3 \to B^3, \qquad && \Im^{23}(b_1, b_2, b_3) = (b_1, \overline{b_2}, \overline{b_3}),\\
&\Im^{13} : B^3 \to B^3, \qquad && \Im^{13}(b_1, b_2, b_3) = (\overline{b_1}, b_2, \overline{b_3}).
\end{aligned}$$

3 Set-Theoretical Solutions of Yang-Baxter Equation on BCC-algebras

In this section, we explored various set-theoretical solutions to the Yang-Baxter equation within the framework of BCC-algebras. We began by identifying several specific mappings, including the identity map and constant mappings, that satisfy the Yang-Baxter equation, as outlined in Lemma 3.1. We then extended our analysis to mappings involving associative binary operations, demonstrating their adherence to the braid condition as shown in Lemma 3.2. We introduced the concepts of left and right α-extension mappings, $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\alpha$, and established their validity as solutions through detailed proofs in Lemmas 3.4 to 3.8. The section culminates with Theorems 3.9 to 3.11, which provide generalized results for mappings involving these extensions, ensuring their compliance with the braid condition. Finally, Corollary 3.12 further reinforces the applicability of these mappings by confirming that certain combinations of idempotent extensions also fulfill the braid condition, thereby validating their role as set-theoretical solutions to the Yang-Baxter equation in BCC-algebras.

Lemma 3.1. *Let B be a BCC-algebra. The following mappings are set-theoretical solutions of the Yang-Baxter equation for any element $\alpha \in B$:*

(i) *If $\Im$ is the identity map,*

(ii) *If $\Im(b_1, b_2) = (0, 0)$,*

(iii) *If $\Im(b_1, b_2) = (\alpha, \alpha)$,*

(iv) *If $\Im(b_1, b_2) = (b_1, \alpha)$,*

(v) *If $\Im(b_1, b_2) = (\alpha, b_2)$,*

(vi) *If $\Im(b_1, b_2) = (b_1, 0)$,*

(vii) *If $\Im(b_1, b_2) = (0, b_2)$.*

Proof. (v) : Consider the mappings $\Im^{12}$ and $\Im^{23}$ defined as follows:

$$\Im^{12} : B^3 \to B^3, \qquad \Im^{12}(b_1, b_2, b_3) = (\alpha, b_2, b_3),$$
$$\Im^{23} : B^3 \to B^3, \qquad \Im^{23}(b_1, b_2, b_3) = (b_1, \alpha, b_3).$$

We need to verify that the equation

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3)$$

holds for all $(b_1, b_2, b_3) \in B^3$. We compute as follows:

$$\begin{aligned}
(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\
&= \Im^{12}(\Im^{23}(\alpha, b_2, b_3)) \\
&= \Im^{12}(\alpha, \alpha, b_3) \\
&= (\alpha, \alpha, b_3) \\
&= \Im^{23}(\alpha, \alpha, b_3) \\
&= \Im^{12}(\Im^{23}(b_1, \alpha, b_3)) \\
&= \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\
&= (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3).
\end{aligned}$$

Thus, the mapping $\Im(b_1, b_2) = (\alpha, b_2)$ satisfies the Yang-Baxter equation as a set-theoretical solution on BCC-algebras. The proofs for the other mappings follow a similar process to that shown for (v). □

Lemma 3.2. *Let $(B; \cdot)$ be a BCC-algebra. The mappings defined by*

(i) $\Im(b_1, b_2) = (b_1 \cdot b_2, 0)$

(ii) $\Im(b_1, b_2) = (0, b_2 \cdot b_1)$

satisfy the braid condition in this structure, where the operation $\cdot$ is associative, i.e., $b_1 \cdot (b_2 \cdot b_3) = (b_1 \cdot b_2) \cdot b_3$ for all $b_1, b_2, b_3 \in B$. Consequently, these mappings are set-theoretical solutions of Yang-Baxter equation in BCC-algebras.

Proof. (i) : Consider the mappings $\Im^{12}$ and $\Im^{23}$ defined as follows:

$$\Im^{12} : B^3 \to B^3, \qquad \Im^{12}(b_1, b_2, b_3) = (b_1 \cdot b_2, 0, b_3),$$
$$\Im^{23} : B^3 \to B^3, \qquad \Im^{23}(b_1, b_2, b_3) = (b_1, b_2 \cdot b_3, 0).$$

We need to verify that the equation

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3)$$

holds for all $(b_1, b_2, b_3) \in B^3$. We compute as follows:
By Definition 2.1 (BCC_3), we have:

$$\begin{aligned}
(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\
&= \Im^{12}(\Im^{23}(b_1 \cdot b_2, 0, b_3)) \\
&= \Im^{12}(b_1 \cdot b_2, 0 \cdot b_3, 0) \\
&= \Im^{12}(b_1 \cdot b_2, b_3, 0) \\
&= ((b_1 \cdot b_2) \cdot b_3, 0, 0).
\end{aligned} \tag{3.1}$$

Similarly, we get

$$\begin{aligned}
(\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\
&= \Im^{23}(\Im^{12}(b_1, b_2 \cdot b_3, 0)) \\
&= \Im^{23}(b_1 \cdot (b_2 \cdot b_3), 0, 0) \\
&= (b_1 \cdot (b_2 \cdot b_3), 0, 0).
\end{aligned} \tag{3.2}$$

The Equation (3.1) and the Equation (3.4) are equal to each other since we have $b_1 \cdot (b_2 \cdot b_3) = (b_1 \cdot b_2) \cdot b_3$ for all $b_1, b_2, b_3 \in B$. It follows that:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3)$$

for all $(b_1, b_2, b_3) \in B^3$. Therefore, the mapping $\Im(b_1, b_2) = (b_1 \cdot b_2, 0)$ satisfies the Yang-Baxter equation as a set-theoretical solution in BCC-algebras.
(ii) : It is clearly obtained by using a similar method as in the proof of Lemma 3.2 (i). □

Definition 3.3. Let $(B; \cdot)$ be a BCC-algebra. We define the following mappings for any $\alpha \in B$:

$$\begin{array}{rcl} \mathfrak{L}_\alpha : B & \to & B \\ b & \mapsto & \mathfrak{L}_\alpha(b) := \alpha \cdot b, \end{array}$$

where $\mathfrak{L}_\alpha$ is called the *left α-extension mapping* in the BCC-algebra.

Similarly, the mapping

$$\begin{array}{rcl} \mathfrak{R}_\alpha : B & \to & B \\ b & \mapsto & \mathfrak{R}_\alpha(b) := b \cdot \alpha \end{array}$$

is called the *right α-extension mapping* in the BCC-algebra.

Lemma 3.4. *Let $(B; \cdot)$ be a BCC-algebra. Consider the mapping defined by*

$$\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_1)),$$

where $\mathfrak{L}_\alpha(b) = \alpha \cdot b$ denotes the left α-extension mapping. This mapping satisfies the braid condition within this structure. Consequently, it serves as a set-theoretical solution to the Yang-Baxter equation in BCC-algebras.

Proof. Let us examine the mappings $\Im^{12}$ and $\Im^{23}$, which are defined as follows:

$$\begin{array}{ll} \Im^{12} : B^3 \to B^3, & \Im^{12}(b_1, b_2, b_3) = (\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_1), b_3), \\ \Im^{23} : B^3 \to B^3, & \Im^{23}(b_1, b_2, b_3) = (b_1, \mathfrak{L}_\alpha(b_3), \mathfrak{L}_\alpha(b_2)). \end{array}$$

Our goal is to verify whether the following equality holds:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3).$$

for every element $(b_1, b_2, b_3) \in B^3$. We proceed with the calculations as follows:

First, we compute:

$$\begin{array}{rcl} (\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) & = & \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\ & = & \Im^{12}(\Im^{23}(\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_1), b_3)) \\ & = & \Im^{12}(\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_3), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1))) \\ & = & (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2)), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1))). \end{array} \tag{3.3}$$

Then, we obtain:

$$\begin{array}{rcl} (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3) & = & \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\ & = & \Im^{23}(\Im^{12}(b_1, \mathfrak{L}_\alpha(b_3), \mathfrak{L}_\alpha(b_2))) \\ & = & \Im^{23}(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)), \mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(b_2)) \\ & = & (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2)), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1))). \end{array} \tag{3.4}$$

Since the results from Equation (3.3) and the Equation (3.4) match, it follows that the mapping $\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_1))$ satisfies the Yang-Baxter equation. Hence, it represents a valid set-theoretical solution within BCC-algebras. □

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0
3	0	1	2	0	0	0	0
4	0	1	2	3	0	0	0
5	0	1	2	3	4	0	6
6	0	1	2	3	5	5	0

Table 1. Definition of the operation $\cdot$ on B

Example 3.5. Consider the set $B = \{0, 1, 2, 3, 4, 5, 6\}$. The binary operation $\cdot$ on B is defined as depicted in Table 1.
Then, the structure $(B; \cdot)$ forms a BCC-algebra. To verify the mapping $\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_1))$ where $\mathfrak{L}_\alpha(b) = \alpha \cdot b$ and for $\alpha = 5$, we perform the following calculations:
• For $b_1 = 0$:

$$\begin{aligned}
\Im(0,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(0)) = (0,0)\\
\Im(0,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(0)) = (1,0)\\
\Im(0,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(0)) = (2,0)\\
\Im(0,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(0)) = (3,0)\\
\Im(0,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(0)) = (4,0)\\
\Im(0,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(0)) = (0,0)\\
\Im(0,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(0)) = (6,0)
\end{aligned}$$

• For $b_1 = 1$:

$$\begin{aligned}
\Im(1,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(1)) = (0,1)\\
\Im(1,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(1)) = (1,1)\\
\Im(1,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(1)) = (2,1)\\
\Im(1,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(1)) = (3,1)\\
\Im(1,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(1)) = (4,1)\\
\Im(1,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(1)) = (0,1)\\
\Im(1,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(1)) = (6,1)
\end{aligned}$$

• For $b_1 = 2$:

$$\begin{aligned}
\Im(2,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(2)) = (0,2)\\
\Im(2,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(2)) = (1,2)\\
\Im(2,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(2)) = (2,2)\\
\Im(2,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(2)) = (3,2)\\
\Im(2,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(2)) = (4,2)\\
\Im(2,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(2)) = (0,2)\\
\Im(2,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(2)) = (6,2)
\end{aligned}$$

• For $b_1 = 3$:

$$\begin{aligned}
\Im(3,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(3)) = (0,3)\\
\Im(3,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(3)) = (1,3)\\
\Im(3,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(3)) = (2,3)\\
\Im(3,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(3)) = (3,3)\\
\Im(3,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(3)) = (4,3)\\
\Im(3,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(3)) = (0,3)\\
\Im(3,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(3)) = (6,3)
\end{aligned}$$

• For $b_1 = 4$:

$$\begin{aligned}
\Im(4,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(4)) = (0,4)\\
\Im(4,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(4)) = (1,4)\\
\Im(4,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(4)) = (2,4)\\
\Im(4,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(4)) = (3,4)\\
\Im(4,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(4)) = (4,4)\\
\Im(4,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(4)) = (0,4)\\
\Im(4,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(4)) = (6,4)
\end{aligned}$$

• For $b_1 = 5$:

$$\begin{aligned}
\Im(5,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(5)) = (0,0)\\
\Im(5,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(5)) = (1,0)\\
\Im(5,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(5)) = (2,0)\\
\Im(5,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(5)) = (3,0)\\
\Im(5,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(5)) = (4,0)\\
\Im(5,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(5)) = (0,0)\\
\Im(5,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(5)) = (6,0)
\end{aligned}$$

• For $b_1 = 6$:

$$\begin{aligned}
\Im(6,0) &= (\mathfrak{L}_5(0), \mathfrak{L}_5(6)) = (0,6)\\
\Im(6,1) &= (\mathfrak{L}_5(1), \mathfrak{L}_5(6)) = (1,6)\\
\Im(6,2) &= (\mathfrak{L}_5(2), \mathfrak{L}_5(6)) = (2,6)\\
\Im(6,3) &= (\mathfrak{L}_5(3), \mathfrak{L}_5(6)) = (3,6)\\
\Im(6,4) &= (\mathfrak{L}_5(4), \mathfrak{L}_5(6)) = (4,6)\\
\Im(6,5) &= (\mathfrak{L}_5(5), \mathfrak{L}_5(6)) = (0,6)\\
\Im(6,6) &= (\mathfrak{L}_5(6), \mathfrak{L}_5(6)) = (6,6)
\end{aligned}$$

Let's verify the braid condition for any selected example to demonstrate. We chose $b_1 = 0, b_2 = 1, b_3 = 2$ and $\alpha = 5$, then we get

$$
\begin{aligned}
(\Im^{12}\circ\Im^{23}\circ\Im^{12})(0,1,2) &= \Im^{12}(\Im^{23}(\Im^{12}(0,1,2))) \\
&= \Im^{12}(\Im^{23}(\mathfrak{L}_5(1),\mathfrak{L}_5(0),2)) \\
&= \Im^{12}(\Im^{23}(1,0,2)) \\
&= \Im^{12}(1,\mathfrak{L}_5(2),\mathfrak{L}_5(0)) \\
&= \Im^{12}(1,2,0) \\
&= (\mathfrak{L}_5(2),\mathfrak{L}_5(1),0) \\
&= (2,1,0),
\end{aligned}
\tag{3.5}
$$

and also,

$$
\begin{aligned}
(\Im^{23}\circ\Im^{12}\circ\Im^{23})(0,1,2) &= \Im^{23}(\Im^{12}(\Im^{23}(0,1,2))) \\
&= \Im^{23}(\Im^{12}(0,\mathfrak{L}_5(2),\mathfrak{L}_5(1))) \\
&= \Im^{23}(\Im^{12}(0,2,1)) \\
&= \Im^{23}(\mathfrak{L}_5(2),\mathfrak{L}_5(0),1) \\
&= \Im^{23}(2,0,1) \\
&= (2,\mathfrak{L}_5(1),\mathfrak{L}_5(0)) \\
&= (2,1,0).
\end{aligned}
\tag{3.6}
$$

Since the Equation (3.5) and the Equation (3.6) are identical, the mapping $\Im(b_1,b_2)=(\mathfrak{L}_\alpha(b_2),\mathfrak{L}_\alpha(b_1))$ verifies the braid condition for $b_1=0, b_2=1, b_3=2$ and $\alpha=5$. As a similar technique, braid condition can be implemented for all elements on this example.

Lemma 3.6. *Let $(B;\cdot)$ be a BCC-algebra. Consider the mapping defined by*

$$\Im(b_1,b_2)=(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(b_2)),$$

where $\mathfrak{L}_\alpha(b)=\alpha\cdot b$ represents the left extension mapping with $\mathfrak{L}_\alpha$ being an idempotent operation. This mapping satisfies the braid condition for this algebraic structure. Thus, it provides a valid set-theoretical solution to the Yang-Baxter equation within the framework of BCC-algebras.

Proof. Consider the mappings $\Im^{12}$ and $\Im^{23}$ defined as follows:

$$
\begin{aligned}
&\Im^{12}: B^3\to B^3, \qquad \Im^{12}(b_1,b_2,b_3)=(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(b_2),b_3),\\
&\Im^{23}: B^3\to B^3, \qquad \Im^{23}(b_1,b_2,b_3)=(b_1,\mathfrak{L}_\alpha(b_2),\mathfrak{L}_\alpha(b_3)).
\end{aligned}
$$

To show that these mappings satisfy the braid condition, we need to verify:

$$(\Im^{12}\circ\Im^{23}\circ\Im^{12})(b_1,b_2,b_3)=(\Im^{23}\circ\Im^{12}\circ\Im^{23})(b_1,b_2,b_3)$$

for all $(b_1,b_2,b_3)\in B^3$. We will compute each side step-by-step.

Firstly, we calculate:

$$
\begin{aligned}
(\Im^{12}\circ\Im^{23}\circ\Im^{12})(b_1,b_2,b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1,b_2,b_3))) \\
&= \Im^{12}(\Im^{23}(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(b_2),b_3)) \\
&= \Im^{12}(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2)),\mathfrak{L}_\alpha(b_3)) \\
&= (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)),\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2))),\mathfrak{L}_\alpha(b_3)).
\end{aligned}
\tag{3.7}
$$

Also, we compute:

$$
\begin{aligned}
(\Im^{23}\circ\Im^{12}\circ\Im^{23})(b_1,b_2,b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1,b_2,b_3))) \\
&= \Im^{23}(\Im^{12}(b_1,\mathfrak{L}_\alpha(b_2),\mathfrak{L}_\alpha(b_3))) \\
&= \Im^{23}(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2)),\mathfrak{L}_\alpha(b_3)) \\
&= (\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2))),\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3))).
\end{aligned}
\tag{3.8}
$$

We need to show that:

$$(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2))), \mathfrak{L}_\alpha(b_3)) = (\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_2))), \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3))).$$

Given that $\mathfrak{L}_\alpha$ is an idempotent mapping, we have $\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b)) = \mathfrak{L}_\alpha(b)$ for all $b \in B$. Thus:

$$\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)) = \mathfrak{L}_\alpha(b_1) \quad \text{and} \quad \mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)) = \mathfrak{L}_\alpha(b_3).$$

Therefore, the computed expressions from the Equation (3.7) and the Equation (3.8) are identical. Hence, the mapping $\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(b_2))$ satisfies the Yang-Baxter equation, proving it is a valid set-theoretical solution within BCC-algebras. □

Lemma 3.7. *Let $(B;\cdot)$ be a BCC-algebra. Define the mapping as follows:*

$$\Im(b_1, b_2) = (\mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_1)),$$

where $\mathfrak{R}_\alpha(b) = b\cdot\alpha$ represents the right α-extension mapping. This mapping adheres to the braid condition within this algebraic structure. As a result, it constitutes a set-theoretical solution to the Yang-Baxter equation in BCC-algebras.

Proof. Let us consider the mappings $\Im^{12}$ and $\Im^{23}$, which are defined as follows:

$$\begin{aligned} &\Im^{12} : B^3 \to B^3, && \Im^{12}(b_1, b_2, b_3) = (\mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_1), b_3), \\ &\Im^{23} : B^3 \to B^3, && \Im^{23}(b_1, b_2, b_3) = (b_1, \mathfrak{R}_\alpha(b_3), \mathfrak{R}_\alpha(b_2)). \end{aligned}$$

We need to check whether the following equation holds:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3)$$

for all $(b_1, b_2, b_3) \in B^3$. We proceed with the calculations as follows:

First of all, we compute:

$$\begin{aligned} (\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\ &= \Im^{12}(\Im^{23}(\mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_1), b_3)) \\ &= \Im^{12}(\mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_3), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))) \\ &= (\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2)), \mathfrak{R}_\alpha(b_1)). \end{aligned} \tag{3.9}$$

Then, we get:

$$\begin{aligned} (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\ &= \Im^{23}(\Im^{12}(b_1, \mathfrak{R}_\alpha(b_3), \mathfrak{R}_\alpha(b_2))) \\ &= \Im^{23}(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)), \mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(b_2)) \\ &= (\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2)), \mathfrak{R}_\alpha(b_1)). \end{aligned} \tag{3.10}$$

Since the results from Equation (3.9) and the Equation (3.10) match, it follows that the mapping $\Im(b_1, b_2) = (\mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_1))$ satisfies the Yang-Baxter equation. Therefore, it serves as a valid set-theoretical solution within BCC-algebras. □

Lemma 3.8. *Let $(B;\cdot)$ be a BCC-algebra. Define the mapping by*

$$\Im(b_1, b_2) = (\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(b_2)),$$

where $\mathfrak{R}_\alpha(b) = b\cdot\alpha$ is the right extension mapping, and $\mathfrak{L}_\alpha$ is idempotent. This mapping adheres to the braid condition within this algebraic structure, making it a valid set-theoretical solution to the Yang-Baxter equation in the context of BCC-algebras.

Proof. Let us consider the mappings $\Im^{12}$ and $\Im^{23}$, defined as follows:

$$\Im^{12} : B^3 \to B^3, \qquad \Im^{12}(b_1, b_2, b_3) = (\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(b_2), b_3),$$
$$\Im^{23} : B^3 \to B^3, \qquad \Im^{23}(b_1, b_2, b_3) = (b_1, \mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_3)).$$

To prove that these mappings satisfy the braid condition, we need to verify the following equality:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3),$$

for all $(b_1, b_2, b_3) \in B^3$. We will compute each side step-by-step.

For the left hand side of the braid condition, we calculate:

$$\begin{aligned}(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\ &= \Im^{12}(\Im^{23}(\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(b_2), b_3)) \\ &= \Im^{12}(\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2)), \mathfrak{R}_\alpha(b_3)) \\ &= (\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1)), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2))), \mathfrak{R}_\alpha(b_3)).\end{aligned} \tag{3.11}$$

For the right hand side of the braid condition, we compute:

$$\begin{aligned}(\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\ &= \Im^{23}(\Im^{12}(b_1, \mathfrak{R}_\alpha(b_2), \mathfrak{R}_\alpha(b_3))) \\ &= \Im^{23}(\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2)), \mathfrak{R}_\alpha(b_3)) \\ &= (\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2))), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3))).\end{aligned} \tag{3.12}$$

We need to show that:

$$(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1)), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2))), \mathfrak{R}_\alpha(b_3)) = (\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_2))), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3))).$$

Given that $\mathfrak{R}_\alpha$ is an idempotent mapping, we have $\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b)) = \mathfrak{R}_\alpha(b)$ for all $b \in B$. Thus:

$$\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1)) = \mathfrak{R}_\alpha(b_1) \quad \text{and} \quad \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)) = \mathfrak{R}_\alpha(b_3).$$

Therefore, the expressions obtained in Equation (3.11) and Equation (3.12) are identical. Consequently, the mapping $\Im(b_1, b_2) = (\mathfrak{R}_\alpha(b_1), \mathfrak{R}_\alpha(b_2))$ satisfies the Yang-Baxter equation, proving that it is a valid set-theoretical solution within BCC-algebras. □

Theorem 3.9. *Let $(B; \cdot)$ be a BCC-algebra. Define the mapping as*

$$\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_2), \mathfrak{R}_\alpha(b_1)),$$

where $\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)) = 0$ for every $b_2 \in B$. This mapping satisfies the braid condition within this algebraic structure, thereby providing a valid set-theoretical solution to the Yang-Baxter equation in BCC-algebras.

Proof. Consider the mappings $\Im^{12}$ and $\Im^{23}$, defined as:

$$\Im^{12} : B^3 \to B^3, \qquad \Im^{12}(b_1, b_2, b_3) = (\mathfrak{L}_\alpha(b_2), \mathfrak{R}_\alpha(b_1), b_3),$$
$$\Im^{23} : B^3 \to B^3, \qquad \Im^{23}(b_1, b_2, b_3) = (b_1, \mathfrak{L}_\alpha(b_3), \mathfrak{R}_\alpha(b_2)).$$

To demonstrate that these mappings meet the braid condition, we need to verify the equality:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3),$$

for all $(b_1, b_2, b_3) \in B^3$. Let's compute each side step-by-step.

Starting with the left-hand side:

$$\begin{aligned}(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\ &= \Im^{12}(\Im^{23}(\mathfrak{L}_\alpha(b_2), \mathfrak{R}_\alpha(b_1), b_3)) \\ &= \Im^{12}(\mathfrak{L}_\alpha(b_2), \mathfrak{L}_\alpha(b_3), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))) \\ &= (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)), \mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2)), \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))).\end{aligned} \tag{3.13}$$

Now, for the right-hand side:

$$\begin{aligned}(\Im^{23}\circ\Im^{12}\circ\Im^{23})(b_1,b_2,b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1,b_2,b_3))) \\ &= \Im^{23}(\Im^{12}(b_1,\mathfrak{L}_\alpha(b_3),\mathfrak{R}_\alpha(b_2))) \\ &= \Im^{23}(\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_3)),\mathfrak{R}_\alpha(b_1),\mathfrak{R}_\alpha(b_2)) \\ &= (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)),\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)),\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))).\end{aligned} \tag{3.14}$$

We need to establish that:

$$(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)),\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2)),\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))) = (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_3)),\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)),\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_1))).$$

Using Proposition 2.2 (iv), we get $\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)) = 0$. Furthermore, we have $\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)) = 0$ for all $b_2 \in B$ from our assumption. Then we deduce that $\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2)) = \mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2))$. Therefore, the expressions from Equation (3.13) and Equation (3.14) are identical. Consequently, the mapping $\Im(b_1,b_2) = (\mathfrak{L}_\alpha(b_2),\mathfrak{R}_\alpha(b_2))$ satisfies the Yang-Baxter equation, establishing it as a valid set-theoretical solution within BCC-algebras. □

Theorem 3.10. *Let $(B;\cdot)$ be a BCC-algebra. Consider the mapping defined by:*

$$\Im(b_1,b_2) = (\mathfrak{L}_\alpha(b_1),\mathfrak{R}_\alpha(b_2)),$$

where $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\alpha$ are idempotent operations satisfying $\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2))) = 0$ for every $b_2 \in B$. This mapping meets the braid condition for this algebraic structure, thereby serving as a valid set-theoretical solution to the Yang-Baxter equation within the context of BCC-algebras.

Proof. We examine the mappings $\Im^{12}$ and $\Im^{23}$, which are given by:

$$\begin{aligned}\Im^{12} &: B^3 \to B^3, & \Im^{12}(b_1,b_2,b_3) &= (\mathfrak{L}_\alpha(b_1),\mathfrak{R}_\alpha(b_2),b_3), \\ \Im^{23} &: B^3 \to B^3, & \Im^{23}(b_1,b_2,b_3) &= (b_1,\mathfrak{L}_\alpha(b_2),\mathfrak{R}_\alpha(b_3)).\end{aligned}$$

To confirm that these mappings fulfill the braid condition, we must show that:

$$(\Im^{12}\circ\Im^{23}\circ\Im^{12})(b_1,b_2,b_3) = (\Im^{23}\circ\Im^{12}\circ\Im^{23})(b_1,b_2,b_3)$$

for any $(b_1,b_2,b_3) \in B^3$. We will compute both sides step by step.

First, we calculate the left-hand side:

$$\begin{aligned}(\Im^{12}\circ\Im^{23}\circ\Im^{12})(b_1,b_2,b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1,b_2,b_3))) \\ &= \Im^{12}(\Im^{23}(\mathfrak{L}_\alpha(b_1),\mathfrak{R}_\alpha(b_2),b_3)) \\ &= \Im^{12}(\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2)),\mathfrak{R}_\alpha(b_3)) \\ &= (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)),\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2))),\mathfrak{R}_\alpha(b_3)).\end{aligned} \tag{3.15}$$

Then, we calculate the right-hand side:

$$\begin{aligned}(\Im^{23}\circ\Im^{12}\circ\Im^{23})(b_1,b_2,b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1,b_2,b_3))) \\ &= \Im^{23}(\Im^{12}(b_1,\mathfrak{L}_\alpha(b_2),\mathfrak{R}_\alpha(b_3))) \\ &= \Im^{23}(\mathfrak{L}_\alpha(b_1),\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2)),\mathfrak{R}_\alpha(b_3)) \\ &= (\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2))),\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3))).\end{aligned} \tag{3.16}$$

We need to demonstrate that:

$$(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)),\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2))),\mathfrak{R}_\alpha(b_3)) = (\mathfrak{L}_\alpha(b_1),\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2))),\mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3))).$$

Since $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\alpha$ are idempotent, we get:

$$\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)) = \mathfrak{L}_\alpha(b_1) \quad \text{and} \quad \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)) = \mathfrak{R}_\alpha(b_3).$$

Moreover, we attain $\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2))) = 0$ for each $b_2 \in B$ via Proposition 2.2 (iv). By the assumption, we have $\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2))) = 0$ for every $b_2 \in B$. Then, we conclude:

$$\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(\mathfrak{R}_\alpha(b_2))) = \mathfrak{L}_\alpha(\mathfrak{R}_\alpha(\mathfrak{L}_\alpha(b_2)))$$

for each $b_2 \in B$. Therefore, the expressions in Equations (3.15) and (3.16) are identical. Consequently, the mapping $\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\alpha(b_2))$ adheres to the Yang-Baxter equation, establishing it as a valid set-theoretical solution in BCC-algebras. □

Theorem 3.11. *Consider a BCC-algebra $(B; \cdot)$. Suppose $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\beta$ are idempotent mappings such that $\mathfrak{L}_\alpha(b) = \mathfrak{R}_\beta(b)$ for every $b \in B$. Then, the mapping defined by*

$$\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\beta(b_2)),$$

satisfies the braid condition in this algebraic structure. Therefore, it provides a valid set-theoretical solution to the Yang-Baxter equation in the realm of BCC-algebras.

Proof. We examine the mappings $\Im^{12}$ and $\Im^{23}$, which are given by:

$$\begin{aligned} \Im^{12} &: B^3 \to B^3, \qquad \Im^{12}(b_1, b_2, b_3) = (\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\beta(b_2), b_3), \\ \Im^{23} &: B^3 \to B^3, \qquad \Im^{23}(b_1, b_2, b_3) = (b_1, \mathfrak{L}_\alpha(b_2), \mathfrak{R}_\beta(b_3)). \end{aligned}$$

To confirm that these mappings fulfill the braid condition, we must show that:

$$(\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) = (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3)$$

for any $(b_1, b_2, b_3) \in B^3$. We will compute both sides step by step.

First, we calculate the left-hand side:

$$\begin{aligned} (\Im^{12} \circ \Im^{23} \circ \Im^{12})(b_1, b_2, b_3) &= \Im^{12}(\Im^{23}(\Im^{12}(b_1, b_2, b_3))) \\ &= \Im^{12}(\Im^{23}(\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\beta(b_2), b_3)) \\ &= \Im^{12}(\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2)), \mathfrak{R}_\beta(b_3)) \\ &= (\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)), \mathfrak{R}_\beta(\mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2))), \mathfrak{R}_\beta(b_3)). \end{aligned} \tag{3.17}$$

Then, we calculate the right-hand side:

$$\begin{aligned} (\Im^{23} \circ \Im^{12} \circ \Im^{23})(b_1, b_2, b_3) &= \Im^{23}(\Im^{12}(\Im^{23}(b_1, b_2, b_3))) \\ &= \Im^{23}(\Im^{12}(b_1, \mathfrak{L}_\alpha(b_2), \mathfrak{R}_\beta(b_3))) \\ &= \Im^{23}(\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2)), \mathfrak{R}_\beta(b_3)) \\ &= (\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(\mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2))), \mathfrak{R}_\beta(\mathfrak{R}_\beta(b_3))). \end{aligned} \tag{3.18}$$

We need to demonstrate that:

$$(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)), \mathfrak{R}_\beta(\mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2))), \mathfrak{R}_\beta(b_3)) = (\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(\mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2))), \mathfrak{R}_\beta(\mathfrak{R}_\beta(b_3))).$$

Since $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\alpha$ are idempotent, we get:

$$\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)) = \mathfrak{L}_\alpha(b_1) \quad \text{and} \quad \mathfrak{R}_\alpha(\mathfrak{R}_\alpha(b_3)) = \mathfrak{R}_\alpha(b_3).$$

Using our assumption, we attain:

$$\begin{aligned} \mathfrak{L}_\alpha(b_2) = \mathfrak{R}_\beta(b_2) \quad &\Rightarrow \quad \mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2)) = \mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2)) \\ &\Rightarrow \quad \mathfrak{L}_\alpha(\mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2))) = \mathfrak{R}_\beta(\mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2))) \end{aligned}$$

for each $b_2 \in B$. Then, we conclude:

$$(\mathfrak{L}_\alpha(\mathfrak{L}_\alpha(b_1)), \mathfrak{R}_\beta(\mathfrak{L}_\alpha(\mathfrak{R}_\beta(b_2))), \mathfrak{R}_\beta(b_3)) = (\mathfrak{L}_\alpha(b_1), \mathfrak{L}_\alpha(\mathfrak{R}_\beta(\mathfrak{L}_\alpha(b_2))), \mathfrak{R}_\beta(\mathfrak{R}_\beta(b_3))).$$

Therefore, the results from Equations (3.17) and (3.18) are equivalent. Hence, the mapping $\Im(b_1, b_2) = (\mathfrak{L}_\alpha(b_1), \mathfrak{R}_\beta(b_2))$ conforms to the Yang-Baxter equation, proving it to be a valid set-theoretical solution in BCC-algebras. □

Corollary 3.12. *Let $(B;\cdot)$ be a BCC-algebra. Assume that $\mathfrak{L}_\alpha$ and $\mathfrak{R}_\beta$ are idempotent mappings satisfying $\mathfrak{L}_\alpha(b) = \mathfrak{R}_\beta(b)$ for all $b \in B$. Then the mapping defined by*

$$\Im(b_1, b_2) = (\mathfrak{R}_\beta(b_1), \mathfrak{L}_\alpha(b_2)),$$

fulfills the braid condition within this algebraic structure. Consequently, it constitutes a valid set-theoretical solution to the Yang-Baxter equation within the context of BCC-algebras.

Proof. The result follows directly by applying the same approach used in the proof of Theorem 3.11. □

4 Conclusion

In this manuscript, we have explored the set-theoretical solutions to the Yang-Baxter equation within the framework of BCC-algebras. By developing and analyzing various mappings, we have demonstrated their adherence to the braid condition, thereby confirming their validity as solutions to the Yang-Baxter equation.

The study began by establishing fundamental lemmas, including mappings such as the identity map, constant mappings, and specific combinations of elements and constants, all shown to satisfy the braid condition. Further, we introduced and examined the left and right α-extension mappings, proving their role as viable solutions through rigorous algebraic analysis.

Significantly, we have shown that both left and right extension mappings, when defined with idempotent properties, serve as robust solutions. The results culminated in theorems that generalized these findings, highlighting their applicability and reinforcing the versatility of BCC-algebras in addressing complex algebraic equations.

Through the provision of concrete examples, the practical implementation of these theoretical results was illustrated, reinforcing the theoretical contributions with tangible applications. Our findings underscore the potential of BCC-algebras in broader mathematical contexts, especially in relation to the Yang-Baxter equation. Future research may delve deeper into the implications of these solutions, exploring their potential applications in fields such as cryptography, quantum computing, and other areas where the Yang-Baxter equation plays a critical role. In conclusion, this work significantly advances the understanding of BCC-algebras and their capacity to provide set-theoretical solutions to the Yang-Baxter equation, paving the way for further theoretical developments and practical applications in various scientific domains such as [27, 28].

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