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Quotient Inclines of a Regular Incline

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Abstract Fundamental properties of the equivalence classes with respect to an ideal I of an incline R are presented in this paper. The set R/I of equivalence classes of a regular incline R with respect to an ideal I which is a regular incline is discussed, under certain operations and I is the least element of R/I. Further it is shown that R/I is an integral incline if I is a prime ideal of R and vice versa.

1 Introduction

Inclines are semirings with inclusively idempotent property and products are less than (or) equal to either factor. The idea of incline was invented by Cao and later it was refined by Cao et al. in [2]. Roush and Kim made a survey on incline in [3]. Incline algebra is a generalization of both Boolean Algebra and Fuzzy Algebra, which has poset and semiring structures. In [1], S.S. Ahn, Y.B. Jun and H.S. Kim, considered the equivalence relation in R, with respect to I and established that the equivalence classes form an incline, named quotient incline which elates the structure of incline algebras.

Here, the results on equivalence relation in R, with respect to I were postulated and conceived that the results found in [1] are valid only for an incline R, whose elements are idempotent. In section 2, the basic definitions, notations and required results of an incline R are given. Section 3 is dedicated to some basic properties of the equivalence classes with respect to an ideal I of an incline R and it is proved that the set of all equivalence classes of R with respect to I denoted as R/I is a regular incline, under suitable operations, where I is an ideal of a regular incline R. Further, we have proved that if R/I is an integral incline then I is a prime ideal of a regular incline R and vice versa.

2 Preliminaries

Here we recall some basic definitions, notations and results in an incline.

Definition 2.1. [7] A left ideal of a semiring R is a non-empty subset I of R such that for $a, b \in \mathsf{I}$ and $r \in \mathsf{R}$

(i) $a + b \in \mathsf{I}$

(ii) $ra \in \mathsf{I}$ (i.e.) $\mathsf{R}a \subseteq \mathsf{I}$.

Definition 2.2. [7] A right ideal of a semiring R is a non-empty subset I of R such that for $a, b \in \mathsf{I}$ and $r \in \mathsf{R}$

(i) $a + b \in \mathsf{I}$

(ii) $ar \in \mathsf{I}$ (i.e.) $a\mathsf{R} \subseteq \mathsf{I}$.

By an ideal of R, we mean both a left and right ideals of R. We call this ideal, 'ideal as a semiring structure'.

Definition 2.3. [1] An incline is a non-empty set R with binary operations addition and multiplication denoted as $+, .$ defined on $\mathsf{R} \times \mathsf{R} \mapsto \mathsf{R}$ such that for $x, y, z \in \mathsf{R}$

$$x + y = y + x \qquad x + (y + z) = (x + y) + z$$
$$x(y + z) = xy + xz \qquad (y + z)x = yx + zx$$
$$x(yz) = (xy)z \qquad x + xy = x$$
$$x + x = x \qquad y + xy = y$$

An incline R is commutative if $xy = yx$ for $x, y \in \mathsf{R}$.

Definition 2.4. [1] $(\mathsf{R}, \leq)$ is an incline with order relation '$\leq$' defined as $x \leq y$ if and only if $x + y = y$ for $x, y \in \mathsf{R}$. If $x \leq y$ then y dominate x.

Property 2.5. [1] For x, y in an incline R, $x + y \geq x$ and $x + y \geq y$

Property 2.6. [1] For x, y in an incline R, $xy \leq x$ and $xy \leq y$

Definition 2.7. [1] A subincline of an incline R is a non-empty subset I of R which is closed under the incline operations addition and multiplication.

Definition 2.8. [1] A subincline I is an ideal of an incline R if $x \in \mathsf{I}$ and $y \leq x$ then $y \in \mathsf{I}$

We name this ideal, 'ideal as a subincline'. In earlier work [6], we have proved that an ideal I as a subincline implies that I is an ideal as a semiring structure but not conversely.

Definition 2.9. [1] Let R be an incline. A proper ideal P of R is prime if for $x, y \in \mathsf{R}, xy \in \mathsf{P}$ signify either $x \in \mathsf{P}$ or $y \in \mathsf{P}$.

Definition 2.10. [1] For $x, y \in \mathsf{R}$, a relation $\sim$ on R is defined by $x \sim y$ with respect to an ideal I of R if and only if there exist $i_1, i_2 \in \mathsf{I}$ such that $x + i_1 = y + i_2$.

In [1], it is proven that the relation $\sim$ is an equivalence relation on R

Definition 2.11. [1] An element 0_{R} in an incline is the zero element of R if $x + 0_{\mathsf{R}} = 0_{\mathsf{R}} + x = x$ and $x.0_{\mathsf{R}} = 0_{\mathsf{R}}.x = 0_{\mathsf{R}}$ for $x \in \mathsf{R}$.

Definition 2.12. [8] An element 0 in an incline R is the least element of R if $0 \leq x$ for $x \in \mathsf{R}$.

Definition 2.13. [8] An element 1 in an incline R is the greatest element of R if $x \leq 1$ for $x \in \mathsf{R}$.

Remark 2.14. From Definitions 2.4, it follows that the zero element of R coincides with the least element of R. In general, the greatest element 1 of R need not be the multiplicative identity of R. (that is $1.x = x.1 = x$ for $x \in \mathsf{R}$) (Refer Example 3.5)

Lemma 2.15. *[5] An incline* R *is regular if and only if* $x^2 = x$ *for all* $x \in \mathsf{R}$.

Notation 2.16. For any $x \in \mathsf{R}$, $[x]_{\mathsf{I}} = \{y \in \mathsf{R} : x \sim y$, with respect to $\mathsf{I}\}$ denotes the equivalence class of x, with respect to an ideal I of R.

3 Quotient Inclines of a regular Incline

In this section, we prove that if I is an ideal as a subincline of R, then each pair of elements of I are related and I is their equivalence class. We have exhibited that the set of equivalence classes with respect to an ideal I of a regular incline R, forms a regular incline under certain operations. Further we have proved that R/I is an integral incline if and only if I is a prime ideal of a regular incline R. Throughout this section, R is an incline with zero element 0 and I is an ideal as a subincline of R. For $x, y \in \mathsf{R}, x \sim y$, means that x is related to y with respect to the ideal I as in Definition 2.10

Lemma 3.1. *In an incline* R*, for* $x, y \in \mathsf{R}, x \sim y$ *if and only if* $[x]_{\mathsf{I}} = [y]_{\mathsf{I}}$.

Proof. For $x, y \in \mathsf{R}$, let $x \sim y$, with respect to I. Let u be an arbitrary element of $[x]_{\mathsf{I}}$.

$$\Rightarrow u \sim x \text{ together with } x \sim y \text{ and by transitivity, we get } u \sim y$$
$$\Rightarrow u \in [y]_{\mathsf{I}}$$

Thus, $[x]_{\mathsf{I}} \subseteq [y]_{\mathsf{I}}$. In the same manner, we can show that $[y]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}}$. Hence, $[x]_{\mathsf{I}} = [y]_{\mathsf{I}}$.
Conversely, let $[x]_{\mathsf{I}} = [y]_{\mathsf{I}}$, then $x \sim y$ follows by reflexivity, $x \in [x]_{\mathsf{I}} = [y]_{\mathsf{I}}$. □

Lemma 3.2. *In an incline* R*, for* $x, y \in \mathsf{R}$*,* $x \nsim y$ *if and only if* $[x]_{\mathsf{I}}$ *and* $[y]_{\mathsf{I}}$ *are disjoint.*

Proof. If $x, y \in \mathsf{R}$ and $x \nsim y$ with respect to I then $[x]_{\mathsf{I}}$ and $[y]_{\mathsf{I}}$ are disjoint.
For if, $[x]_{\mathsf{I}}$ and $[y]_{\mathsf{I}}$ are not disjoint then there exists $z \in \mathsf{R}$ such that $z \in [x]_{\mathsf{I}}$ and $z \in [y]_{\mathsf{I}}$.

$$\Rightarrow z \sim x \text{ and } z \sim y$$
$$\Rightarrow x \sim z \text{ and } z \sim y \text{ (by Symmetry)}$$
$$\Rightarrow x \sim y \text{ (by Transitivity), which is a contradiction to hypothesis}$$

Therefore, $[x]_{\mathsf{I}}$ and $[y]_{\mathsf{I}}$ are disjoint.
Conversely, if $[x]_{\mathsf{I}}$ and $[y]_{\mathsf{I}}$ are disjoint, then $x \nsim y$, with respect to I. For, if $x \sim y$ then by Lemma 3.1, $[x]_{\mathsf{I}} = [y]_{\mathsf{I}}$ which is not possible. □

Proposition 3.3. *Let* x *be any element of* R*, then* $x \in \mathsf{I}$ *if and only if* $[x]_{\mathsf{I}} = \mathsf{I}$*. In particular,* $[0]_{\mathsf{I}} = \mathsf{I}$ *where* 0 *is the zero element of* R*.*

Proof. If $x \in \mathsf{I}$ then we have to show that $[x]_{\mathsf{I}} = \mathsf{I}$. For any $y \in [x]_{\mathsf{I}}$, we have

$$y \sim x \Rightarrow y + i_1 = x + i_2 \text{ for some } i_1, i_2 \in \mathsf{I}, \text{ (by Definition 2.10)}$$
$$\Rightarrow y + i_1 = i_3 \text{ where } i_3 = x + i_2 \in \mathsf{I}, \text{ (by Definition 2.8)}$$
$$\Rightarrow y \leq i_3, \text{ (by Incline Property 2.5)}$$
$$\Rightarrow y \in \mathsf{I}, \text{ (by Definition 2.8)}$$

Hence $[x]_{\mathsf{I}} \subseteq \mathsf{I}$. Since $x \in \mathsf{I}$ for any arbitrary element $y \in \mathsf{I}$ and $y \neq x$, we have

$$y + x = x + y \text{ for some } x, y \in \mathsf{I}$$
$$\Rightarrow y \sim x, \text{ (by Definition 2.10)}$$
$$\Rightarrow y \in [x]_{\mathsf{I}}$$

Thus, $\mathsf{I} \subseteq [x]_{\mathsf{I}}$. Therefore, $[x]_{\mathsf{I}} = \mathsf{I}$.
Conversely, let $[x]_{\mathsf{I}} = \mathsf{I}$. Suppose $x \notin \mathsf{I}$ then $x \notin [x]_{\mathsf{I}}$. That is, $x \nsim x$, with respect to I, which is a contradiction to the reflexivity of the equivalence relation on R, with respect to I. Thus, $x \in \mathsf{I}$.
From Definition 2.12, $0 \leq x$ for $x \in \mathsf{I}$ and by Definition 2.8, $0 \in \mathsf{I}$. Hence $[0]_{\mathsf{I}} = \mathsf{I}$. □

Remark 3.4. In Proposition 3.3, the fact that I should be an ideal as a subincline is essential. This is illustrated in the following example.

Example 3.5. Let $\mathsf{R} = \{0, a, b, c, d, 1\}$ be an incline. Define $\bullet : \mathsf{R} \bullet \mathsf{R} \mapsto \mathsf{R}$ by $x \bullet y = d$ for $x, y \in \{b, c, d, 1\}$ and 0 otherwise. Then by Lemma 2.15, $(\mathsf{R}, +, \bullet)$ is a commutative, non-regular incline, since 0 and 1 are the only idempotent elements of R.

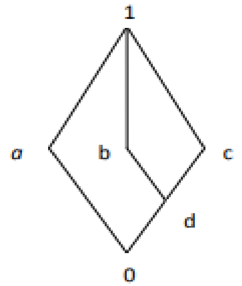

Figure 1. Illustration on Essential of being subincline

Here 1 is the greatest element of R but 1 is not multiplicative identity of R. Let us consider $\mathsf{I} = \{0, b, d, 1\}$. I is an ideal as a semiring structure but I is not an ideal as a subincline. For instance $c \in \mathsf{R}$ and $c \leq 1$ but $c \notin \mathsf{I}$. Here $[0]_{\mathsf{I}} = [a]_{\mathsf{I}} = [b]_{\mathsf{I}} = [c]_{\mathsf{I}} = [d]_{\mathsf{I}} = [1]_{\mathsf{I}} = \mathsf{R}$. Thus, $[x]_{\mathsf{I}} \neq \mathsf{I}$ for $x \in \mathsf{I}$.

Corollary 3.6. *In an incline* R*, for* $x, y \in \mathsf{R}$*, if* $x \sim y$ *then* $x \in \mathsf{I}$ *if and only if* $y \in \mathsf{I}$.

Proof. Since $x \sim y$, by Lemma 3.1, $[x]_{\mathsf{I}} = [y]_{\mathsf{I}}$. Then the result follows from Proposition 3.3. □

Remark 3.7. We observe that in an incline R, each pair of elements in I are related with respect to the ideal I. Further, the equivalence classes of R with respect ot an ideal I of R give rise to partitioning of R.

Lemma 3.8. *In a regular incline* R*, for* $x, y \in \mathsf{R}$*, if* $x \sim y$ *with respect to an ideal* I *of* R*, then*

(i) $x + y \sim x$ *and* $x + y \sim y$ *with respect to* I.

(ii) $xy \sim x$ *and* $xy \sim y$ *with respect to* I.

Proof. Let R be a regular incline and I be an ideal of R. For $x, y \in \mathsf{R}$

(i) Since $x \sim y$ by symmetry, $y \sim x$, with respect to I.

$\Rightarrow x + i_1 = y + i_2$ and $y + i_3 = x + i_4$ for some $i_1, i_2, i_3, i_4 \in \mathsf{I}$ (by Definition 2.10)

$\Rightarrow x + i_1 + x = y + i_2 + x$ and $y + i_3 + y = x + i_4 + y$

$\Rightarrow x + x + i_1 = y + x + i_2$ and $y + y + i_3 = x + y + i_4$ (by commutativity)

$\Rightarrow x + i_1 = y + x + i_2$ and $y + i_3 = x + y + i_4$ (by Definition 2.3)

$\Rightarrow x \sim y + x$ and $y \sim x + y$

$\Rightarrow x \sim x + y$ and $y \sim x + y$ with respect to I (by commutativity)

Thus (i) holds.

(ii) Since $x \sim y$, as in the proof of (i), we have

$$x + i_1 = y + i_2 \text{ and } y + i_3 = x + i_4 \text{ for some } i_1, i_2, i_3, i_4 \in \mathsf{I}$$

On Pre multiplication by 'x' of the first equation and post multiplication by 'y' of the second equation yield,

$$x(x + i_1) = x(y + i_2) \text{ and } (y + i_3)y = (x + i_4)y$$
$$\Rightarrow xx + xi_1 = xy + xi_2 \text{ and } yy + i_3y = xy + i_4y$$
$$\Rightarrow x + xi_1 = xy + xi_2 \text{ and } y + i_3y = xy + i_4y \text{ (by Lemma 2.15)}$$
$$\Rightarrow x + i_5 = xy + i_6 \text{ and } y + i_7 = xy + i_8 \text{ (by Definition 2.8)}$$
$$\Rightarrow x \sim xy \text{ and } y \sim xy \text{ with respect to I}$$

Thus (ii) holds.

□

Lemma 3.9. *In an incline* R*, for* $x, y \in \mathsf{R}$*, if* $x \nsim y$ *with respect to an ideal* I *of* R*, then*

(i) either $x + y \nsim x$ *or* $x + y \sim y$ *with respect to* I.

(ii) either $xy \nsim x$ *or* $xy \nsim y$ *with respect to* I.

Proof. For $x, y \in \mathsf{R}$, if $x \nsim y$, then we have to show that either $x + y \nsim x$ or $x + y \nsim y$, with respect to I.
Suppose, $x + y \sim x$ and $x + y \sim y$, with respect to I. Then by symmetry, $x \sim x + y$ and $y \sim x + y$. And by Transitivity, $x \sim y$ which is a contradiction to the hypothesis. Thus, $x + y \nsim x$ or $x + y \nsim y$, with respect to I.
In the same manner we can prove (ii). □

Theorem 3.10. *In a regular incline* R*, if* I *is an ideal of* R *then the set of equivalence classes with respect to* I *denoted by* R/I *is an incline under the operations*

(i) $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$

(ii) $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} = [xy]_{\mathsf{I}}$ *for any* $x, y \in \mathsf{R}$.

Proof. Let R be a regular incline and I be an ideal of R. For any $x, y \in \mathsf{R}$, we prove that $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$ and $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} = [xy]_{\mathsf{I}}$ are well defined for the following cases.
Case (i) $x \sim y$, with respect to I
Case (ii) $x \nsim y$, with respect to I
First we prove that $[x]_{\mathsf{I}} + [x]_{\mathsf{I}} = [x]_{\mathsf{I}}$ and $[x]_{\mathsf{I}}.[x]_{\mathsf{I}} = [x]_{\mathsf{I}}$ for any $x \in \mathsf{R}$.

$$\begin{aligned} \text{Let } u, v \in [x]_{\mathsf{I}} &\Rightarrow u \sim x \text{ and } v \sim x \\ &\Rightarrow u + v \sim x + x = x \text{ (by Definition 2.3)} \\ &\Rightarrow [x]_{\mathsf{I}} + [x]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}} \end{aligned}$$

By Definition 2.3, since $u = u + u$, we have $[x]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}} + [x]_{\mathsf{I}}$. Thus, $[x]_{\mathsf{I}} + [x]_{\mathsf{I}} = [x]_{\mathsf{I}}$. $\rightarrow$ **(3.1)**

$$\begin{aligned} \text{Now, Let } u, v \in [x]_{\mathsf{I}} &\Rightarrow u \sim x \text{ and } v \sim x \\ &\Rightarrow u.v \sim x.x = x \text{ (by Lemma 2.15)} \\ &\Rightarrow [x]_{\mathsf{I}}.[x]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}} \end{aligned}$$

Since R is regular, by Lemma 2.15, $u = u.u$, we have $[x]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}}.[x]_{\mathsf{I}}$.
Thus, $[x]_{\mathsf{I}}.[x]_{\mathsf{I}} = [x]_{\mathsf{I}}.\rightarrow$ **(3.2)**.

Proof of case (i):

For $x, y \in \mathsf{R}$, since $x \sim y$, by Lemma 3.8, we have $x + y \sim x$, $x + y \sim y$, $xy \sim x$ and $xy \sim y$.
And by Lemma 3.1, we have $[x+y]_{\mathsf{I}} = [x]_{\mathsf{I}} = [y]_{\mathsf{I}} = [xy]_{\mathsf{I}} \rightarrow$ **(3.3)**.

$$\begin{aligned} \text{Therefore, } [x+y]_{\mathsf{I}} &= [x]_{\mathsf{I}} \text{ (by Equation 3.3)} \\ &= [x]_{\mathsf{I}} + [x]_{\mathsf{I}} \text{ (by Equation 3.1)} \\ &= [x]_{\mathsf{I}} + [y]_{\mathsf{I}} \text{ (by Equation 3.3)} \end{aligned}$$

Thus, $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$.

$$\begin{aligned} \text{Similarly, } [xy]_{\mathsf{I}} &= [x]_{\mathsf{I}} \text{ (by Equation 3.3)} \\ &= [x]_{\mathsf{I}}.[x]_{\mathsf{I}} \text{ (by Equation 3.2)} \\ &= [x]_{\mathsf{I}}.[y]_{\mathsf{I}} \text{ (by Equation 3.3)} \end{aligned}$$

Thus, $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} = [xy]_{\mathsf{I}}$.

Proof of Case (ii):

To prove $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$. Let z be an arbitrary element of $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} \Rightarrow z = p + q$ for some $p \in [x]_{\mathsf{I}}$ and $q \in [y]_{\mathsf{I}}$

$$\begin{aligned} \text{Then, } p \sim x \text{ and } q \sim y &\Rightarrow p + q \sim x + y \\ &\Rightarrow z \sim x + y, \text{ with respect to } \mathsf{I} \\ &\Rightarrow z \in [x+y]_{\mathsf{I}} \end{aligned}$$

Thus, $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} \subseteq [x+y]_{\mathsf{I}} \rightarrow$ **(3.4)**
Since $x \nsim y$, with respect to I, by Lemma 3.9, we have either $x + y \nsim x$ or $x + y \nsim y$.
Suppose $x + y \nsim y$, then $x + y \sim x$.

$$\begin{aligned} x + y \sim x &\Rightarrow xy + yy \sim xy \\ &\Rightarrow xy + y \sim xy \text{ (by Lemma 2.15)} \\ &\Rightarrow y \sim xy \text{ (by Definition 2.3)} \end{aligned}$$

Therefore, by Lemma 3.1, $[x+y]_{\mathsf{I}} = [x]_{\mathsf{I}}$ and $[xy]_{\mathsf{I}} = [y]_{\mathsf{I}} \rightarrow$ **(3.5)**
In the same manner, if $x+y \nsim x$ then $x+y \sim y$ and we get $[x+y]_{\mathsf{I}} = [y]_{\mathsf{I}}$ and $[xy]_{\mathsf{I}} = [x]_{\mathsf{I}}$
Let z be an arbitrary element of $[x+y]_{\mathsf{I}}$.

$$\Rightarrow z \sim x+y \text{ with respect to } \mathsf{I}$$
$$\Rightarrow z + i_1 = x + y + i_2 \text{ for some } i_1, i_2 \in \mathsf{I} \text{ (by Definition 2.10)}$$
$$\Rightarrow (z+i_1)y = (x+y+i_2)y$$
$$\Rightarrow zy + i_1 y = xy + yy + i_2 y$$
$$\Rightarrow zy + i_3 = xy + y + i_4 \text{ (by Lemma 2.15 and Definition 2.8)}$$
$$\Rightarrow zy + i_3 = y + i_4 \text{ (by Definition 2.3)}$$
$$\Rightarrow zy \sim y$$
$$\Rightarrow zy \in [y]_{\mathsf{I}}$$
$$\text{Here, } z = z + zy \text{ (by Definition 2.3)}$$
$$\Rightarrow z \in [x]_{\mathsf{I}} + [y]_{\mathsf{I}} \text{ where, } z \in [x+y]_{\mathsf{I}} = [x]_{\mathsf{I}} \text{ (by Equation } \mathbf{3.5} \text{ and } zy \in [y]_{\mathsf{I}})$$

Hence, $[x+y]_{\mathsf{I}} \subseteq [x]_{\mathsf{I}} + [y]_{\mathsf{I}} \rightarrow$ **(3.6)**
From **(3.4)** and **(3.6)** we get, $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$
To prove $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} = [xy]_{\mathsf{I}}$.
Let z be an arbitrary element of $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} \Rightarrow z = pq$ for some $p \in [x]_{\mathsf{I}}$ and $q \in [y]_{\mathsf{I}}$

$$\text{Then, } p \sim x \text{ and } q \sim y \Rightarrow xp \sim x \text{ and } qy \sim y \text{ (by Lemma 2.15)}$$
$$\Rightarrow xpqy \sim xy$$
$$\text{Now, } z = pq \Rightarrow xzy = xpqy$$
$$\Rightarrow xzy \in [x]_{\mathsf{I}}.[y]_{\mathsf{I}} \text{ where } xp \in [x]_{\mathsf{I}} \text{ and } pq \in [y]_{\mathsf{I}}$$

Also, $xzy = xpqy \in [xy]_{\mathsf{I}} \Rightarrow xzy \in [xy]_{\mathsf{I}} \rightarrow$ **(3.7)**
On the other hand, let z be an arbitrary element of $[xy]_{\mathsf{I}}$.

$$\Rightarrow z \sim xy \text{ with respect to } \mathsf{I}$$
$$\Rightarrow z + i_1 = xy + i_2 \text{ for some } i_1, i_2 \in \mathsf{I} \text{ (by Definition 2.10)}$$
$$\Rightarrow xzy + xi_1 y = xxyy + xi_2 y$$
$$\Rightarrow xzy + i_3 = xy + i_4 \text{ (by Lemma 2.15 and Definition 2.8)}$$
$$\Rightarrow xzy \sim xy$$
$$\Rightarrow xzy \in [xy]_{\mathsf{I}}$$

We have, $z \sim xy \Rightarrow zy \sim xy$, (by Lemma 2.15) and hence $zy \in [xy]_{\mathsf{I}}$
Now, $xzy = (x)(zy) \Rightarrow xzy \in [x]_{\mathsf{I}}.[y]_{\mathsf{I}}$ where $x \in [x]_{\mathsf{I}}$ and $zy \in [xy]_{\mathsf{I}} = [y]_{\mathsf{I}}$ (by **(3.5)**) $\rightarrow$ **(3.8)**
From **(3.7)** and **(3.8)** we get, $[x]_{\mathsf{I}}.[y]_{\mathsf{I}} = [xy]_{\mathsf{I}}$
Therefore, the operations on R/I are well defined. Now, it can be verified that $(\mathsf{R}/\mathsf{I}, +, .)$ is an incline under the operations (i) and (ii) and is called the quotient incline with respect to I. □

Remark 3.11. In the above Theorem 3.10, the regularity of R is essential. This is illustrated in the following examples.

Example 3.12. Let us consider the incline in Example 3.5. Let $\mathsf{I} = \{0, d\}$ be an ideal of R. Here, $[0]_{\mathsf{I}} = [d]_{\mathsf{I}} = \{0, d\} = \mathsf{I}$, $[a]_{\mathsf{I}} = [1]_{\mathsf{I}} = \{a, 1\}$, $[b]_{\mathsf{I}} = \{b\}$ and $[c]_{\mathsf{I}} = \{c\}$.
Now, let us consider $b, c \in \mathsf{R}$ where $b \nsim c$, with respect to I.
Here $[b]_{\mathsf{I}} + [c]_{\mathsf{I}} = b + c = \{1\}$ and $[b+c]_{\mathsf{I}} = [1]_{\mathsf{I}} = \{a, 1\}$. Thus, $[b]_{\mathsf{I}} + [c]_{\mathsf{I}} \neq [b+c]_{\mathsf{I}}$.
Here, $[b]_{\mathsf{I}}.[c]_{\mathsf{I}} = b.c = \{d\}$ and $[b.c]_{\mathsf{I}} = [d]_{\mathsf{I}} = \{0, d\}$.
Thus, $[b]_{\mathsf{I}}.[c]_{\mathsf{I}} \neq [b.c]_{\mathsf{I}}$

Example 3.13. Let us consider the incline in Example 3.5. Let $\mathsf{I} = \{0, b, d\}$ be an ideal of R. Here, $[0]_{\mathsf{I}} = [b]_{\mathsf{I}} = [d]_{\mathsf{I}} = \mathsf{I}$ and $[a]_{\mathsf{I}} = [c]_{\mathsf{I}} = [1]_{\mathsf{I}} = \{a, c, 1\}$.

Now, let us consider $b, d \in \mathsf{R}$ where $b \sim d$, with respect to I. Here, $[b]_{\mathsf{I}}.[d]_{\mathsf{I}} = \{0, d\}$ and $[b.d]_{\mathsf{I}} = [d]_{\mathsf{I}} = \{0, b, d\}$. Thus, $[b]_{\mathsf{I}}.[d]_{\mathsf{I}} \neq [b.d]_{\mathsf{I}}$.
However, $[x]_{\mathsf{I}} + [y]_{\mathsf{I}} = [x+y]_{\mathsf{I}}$ for $x, y \in \mathsf{R}$.

Theorem 3.14. *If* R *is a regular incline with zero element 0 and greatest element 1 and for any ideal* I *of* R*,* R/I *is also a regular incline with least element* I *and greatest element* $[1]_{\mathsf{I}}$.

Proof. Let R be a regular incline and I be an ideal of R, By Theorem 3.10, R/I is an quotient incline. Now, we prove that $[0]_{\mathsf{I}}$ is the least element and $[1]_{\mathsf{I}}$ is the greatest element of R/I. Since I is the greatest element of R, by Definition 2.13, we have $x \leq 1$ for $x \in \mathsf{R}$.

$$\Rightarrow x + 1 = 1 \text{ (by Definition 2.4)}$$
$$\Rightarrow [x+1]_{\mathsf{I}} = [1]_{\mathsf{I}}$$
$$\Rightarrow [x]_{\mathsf{I}} + [1]_{\mathsf{I}} = [1]_{\mathsf{I}} \text{ (by incline operation (i) in R/I)}$$
$$\Rightarrow [x]_{\mathsf{I}} \leq [1]_{\mathsf{I}} \text{ for } [x]_{\mathsf{I}} \in \mathsf{R}/\mathsf{I} \text{ (by incline property 2.5)}$$

Similarly, we can prove that $[0]_{\mathsf{I}} = \mathsf{I}$ (by Proposition 3.3) is the least element of R/I. Further, for $[x]_{\mathsf{I}} \in \mathsf{R}/\mathsf{I}$.

$$[x]_{\mathsf{I}}.[x]_{\mathsf{I}} = [x.x]_{\mathsf{I}} \text{ (by Incline operation (ii) in R/I)}$$
$$= [x]_{\mathsf{I}} \text{ (by Lemma 2.15)}$$

Therefore, $[x]_{\mathsf{I}}.[x]_{\mathsf{I}} = [x]_{\mathsf{I}}$ for $x \in \mathsf{R}$. Thus, R/I is a regular quotient incline. □

Remark 3.15. By the above Theorem 3.14, we observe that $[1]_{\mathsf{I}} \neq \mathsf{R}$ and Theorem 2.3 of $[1]$ fails. The regularity condition on R is essential. Since for ideal I in example 3.13, $[0]_{\mathsf{I}} = \mathsf{I}$ but $[0]_{\mathsf{I}}$ is not the zero element of R/I. Now, we shall recall the definitions of integral pair and integral incline in an incline R.

Definition 3.16. [1] A pair of elements $x, y \in \mathsf{R}$ is an integral pair if $xy = 0$ and $x + y = 1$ where 0 and 1 are the least and greatest elements of R respectively.

Definition 3.17. [1] An incline R with least element 0 and greatest element 1 is an integral incline if it has no integral pair.

Theorem 3.18. *Let* R *be a regular incline with least element 0 and greatest element 1. An ideal* P *of* R *is prime if and only if the regular incline* R/P *is an integral incline.*

Proof. Let R be a regular incline and P be a prime ideal of R, we claim that $[1]_{\mathsf{P}} \neq \mathsf{P}$. For if, $[1]_{\mathsf{P}} = \mathsf{P}$, then by Proposition 3.3, $1 \in \mathsf{P}$. Since 1 is the greatest element of R, $x \leq 1$ for $x \in \mathsf{R}$ and by Definition 2.8, $x \in \mathsf{P}$ for $x \in \mathsf{R}$, hence $\mathsf{P} = \mathsf{R}$ which is a contradiction to P is a proper ideal. Therefore, $[1]_{\mathsf{P}} \neq \mathsf{P}$.

We claim that R/P is an integral incline. Suppose R/P is not an integral incline, then by Definition 3.17 for some $x, y \in \mathsf{R}$ there exists an integral pair in R/P that is, for $[x]_{\mathsf{P}} \neq [0]_{\mathsf{P}}$ and $[y]_{\mathsf{P}} \neq [0]_{\mathsf{P}}$ such that $[x]_{\mathsf{P}}.[y]_{\mathsf{P}} = [0]_{\mathsf{P}}$ and $[x]_{\mathsf{P}} + [y]_{\mathsf{P}} = [1]_{\mathsf{P}}$.
Let us consider

$$[x]_{\mathsf{P}}.[y]_{\mathsf{P}} = [0]_{\mathsf{P}}$$
$$\Rightarrow [x.y]_{\mathsf{P}} = [0]_{\mathsf{P}} \text{ (by incline operation (ii) in R/P)}$$
$$\Rightarrow [x.y]_{\mathsf{P}} = \mathsf{P} \text{ (by Proposition 3.3)}$$
$$\Rightarrow x.y \in \mathsf{P}$$
$$\Rightarrow x \in \mathsf{P} \text{ or } y \in \mathsf{P} \text{ (by Definition 2.9)}$$
$$\Rightarrow [x]_{\mathsf{P}} = \mathsf{P} \text{ or } [y]_{\mathsf{P}} = \mathsf{P} \text{ (by Proposition 3.3)}$$
$$\Rightarrow [x]_{\mathsf{P}} = [0]_{\mathsf{P}} \text{ or } [y]_{\mathsf{P}} = [0]_{\mathsf{P}} \text{ (by Proposition 3.3)}$$

This is a contradiction to our assumption. Therefore, R/P is an integral incline.
Conversely, let R/P be an integral incline. Then

$$[1]_P \neq [0]_P = P$$
$$\Rightarrow 1 \notin P$$
$$P \neq R$$

Thus, P is a proper ideal of R.
Suppose $xy \in P$ for some $x, y \in R$ then by Proposition 3.3.

$$[x.y]_P = P$$
$$\Rightarrow [x]_P.[y]_P = P = [0]_P \text{ (by incline operation (ii) in R/P)}$$
$$\Rightarrow [x]_P = P \text{ or } [y]_P = P \text{ (by Definition 3.17)}$$
$$\Rightarrow x \in P \text{ or } y \in P \text{ (by Proposition 3.3)}$$

Thus, P is a prime ideal of R. □

4 Conclusion

We have developed the results of equivalence relation in a regular incline R, with respect to an ideal I and proved that the results found in [1] are valid only for a regular incline R, whose elements are idempotent and illustrated it with suitable examples. This also helps to promote the concept of merging Incline Theories and this can be preferred to some other algebraic structures. The recent trends and development of Incline Algebra were discussed in [9, 10, 11].

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