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STRUCTURE OF STRONG DISTRIBUTIVE LATTICE OF GROUP-SEMIRINGS

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Abstract By a group-semiring, we mean a semiring $(G, +, \cdot)$ such that $(G, +)$ is a left zero semigroup and $(G, \cdot)$ is a group. In this paper, we consider certain unions of group-semirings; specifically collection of group-semirings indexed over a distributive lattice. The concepts of semilattice of groups and strong semilattice of groups available in semigroups are carried over to the distributive lattice of semirings and strong distributive lattice (SDL) of semirings. It is shown that, for SDL of group-semirings, the group-semirings are mutually isomorphic, and hence a structure of such semirings is identified as the direct product of the distributive lattice and a group-semiring. Moreover, a characterization for SDL of group-semirings, simplifying the available characterization of SDL of semirings, was also obtained.

1 Introduction and preliminaries

A semiring $(S, +, \cdot)$ is a non-empty set S together with two binary operations $+$ and $\cdot$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and for any $a, b, c \in S$,

$$(a + b)c = ac + bc \quad \text{and} \quad a(b + c) = ab + ac.$$

Let D be a distributive lattice and $\{S_\alpha, \alpha \in D\}$ be a collection of disjoint semirings indexed over D. A semiring $S = \bigcup\{S_\alpha, \alpha \in D\}$ is regarded as a *distributive lattice of semirings*, if $S_\alpha + S_\beta \subseteq S_{\alpha+\beta}$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. If there exists two families of homomorphisms $\{\phi_{\alpha,\beta} : S\alpha \to S_\beta | \alpha, \beta \in D \text{ and } \beta \leq \alpha\}$ and $\{\psi_{\alpha,\beta} : S_\beta \to S_\alpha | \alpha, \beta \in D \text{ and } \beta \leq \alpha \in D\}$ such that the sum and product in S are given as follows. For $a \in S_\alpha$ and $b \in S_\beta$,

$$a + b = a\psi_{\alpha,\alpha+\beta} + b\psi_{\beta,\alpha+\beta} \quad \text{and} \quad ab = a\phi_{\alpha,\alpha\beta} b\phi_{\beta,\alpha\beta}.$$

Then we say that $S = \bigcup\{S_\alpha, \alpha \in D\}$ is a *strong distributive lattice of semirings* (SDL of semirings). Bendelt and Petrich [1] ensures the existence of such a system, whenever the collection of homomorphisms satisfies certain additional conditions. Ghosh [4] refined the conditions given by Bendelt and Petrich and proved that only one collection of homomorphisms is sufficient to establish such semirings. Instead of assuming conditions on the classes of homomorphisms, in this work, we put some conditions on sub-semirings.

A group $(G, \cdot)$ becomes a semiring if we define the addition by $x + y = x$ (similarly $x + y = y$) for all $x, y \in G$, we call it a *group-semiring*. In the following discussion, we restrict our attention to SDL of semirings $S = \bigcup\{G_\alpha, \alpha \in D\}$, such that each G_α is a group-semiring. In this case we can see that only one set of homomorphisms is required to describe the addition and multiplication in S and each homomorphism is an isomorphism. Thus $\{G_\alpha\}$ is a collection of mutually isomorphic groups, and hence S becomes a direct product of the distributive lattice D and a group G_α.

1.1 Union of Groups

A semigroup S is referred as a semilattice union of groups if there is a semilattice Λ and groups $\{G_\alpha : \alpha \in \Lambda\}$ such that $S = \cup_{\alpha\in\Lambda} G_\alpha$ and

$$G_\alpha G_\beta \subseteq G_{\alpha\beta}$$

for all $\alpha, \beta \in \Lambda$ [6]. It is termed as a strong semilattice of groups if there exists a family of homomorphisms $\{\phi_{\alpha,\beta} : \beta \leq \alpha \in \Lambda\}$ such that the following holds.

(i) $\phi_{\alpha,\alpha}$ is the identity map on G_α,

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$, whenever $\gamma \leq \beta \leq \alpha$, and

(iii) $xy = x\phi_{\alpha,\alpha\beta} y\phi_{\beta,\alpha\beta}$ for $x \in G_\alpha$ and $y \in G_\beta$.

By [2], a semigroup S is a semilattice of groups if and only if it is a strong semilattice of groups. Analogous to these constructions in semigroup theory, there are studies in semirings also [1, 4]. In this work, we are interested in the union of group-semirings, rather than the union of groups.

1.2 Distributive lattice of semirings

Analogous to the definition of semilattice of semigroups (or groups), we have the definition of the distributive lattice of semirings. Unless otherwise stated, D denotes a distributive lattice, a semiring in which both addition and multiplication are commutative and idempotent.

Definition 1.1. [1] Let $S = \bigcup\{S_\alpha, \alpha \in D\}$ be a distributive lattice of semirings indexed over D. Then S is said to be a *SDL of semirings* if there are two families of homomorphisms $\{\phi_{\alpha,\beta} : S_\alpha \to S_\beta : \beta \leq \alpha\}$ and $\{\psi_{\beta,\alpha} : S_\beta \to S_\alpha : \beta \leq \alpha\}$ such that

(i) $\phi_{\alpha,\alpha} : S_\alpha \to S_\alpha$ is identity map.

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ whenever $\gamma \leq \beta \leq \alpha$.

(iii) $\psi_{\alpha,\alpha} : S_\alpha \to S_\alpha$ is identity map.

(iv) $\psi_{\gamma,\beta}\psi_{\beta,\alpha} = \psi_{\gamma,\alpha}$ whenever $\gamma \leq \beta \leq \alpha$.

Further, for $x \in S_\alpha$ and $y \in S_\beta$

$$xy = x\phi_{\alpha,\alpha\beta} y\phi_{\beta,\alpha\beta} \tag{1.1a}$$

and

$$x + y = x\psi_{\alpha,\alpha+\beta} + y\psi_{\beta,\alpha+\beta}. \tag{1.1b}$$

For a given collection of semirings $\{S_\alpha, \alpha \in D\}$ and two families of homomorphisms $\{\phi_{\alpha,\beta} : \beta \leq \alpha\}$ and $\{\psi_{\beta,\alpha} : \beta \leq \alpha\}$ satisfying the conditions (i), (ii), (iii) and (iv), Bendelt and Petrich [1] proved that $S = \bigcup\{S_\alpha, \alpha \in D\}$ is an SDL of semirings under the operations defined by (1.1b) and (1.1a), whenever the following conditions hold.

B1) $S_\beta\psi_{\beta,\alpha}$ is an ideal of S_α for $\beta \leq \alpha$ and

B2) $\psi_{\alpha,\gamma}\phi_{\gamma,\beta} = \phi_{\beta,\alpha\beta}\psi_{\alpha\beta,\beta}$ whenever $\alpha + \beta \leq \gamma$.

In [4], Gosh introduced an equivalent similar construction which requires only one family of homomorphisms $\{\psi_{\beta,\alpha} : \beta \leq \alpha\}$, where each $\psi_{\beta,\alpha}$ is a monomorphism and $S_\alpha\psi_{\alpha,\gamma}S_\beta\psi_{\beta,\gamma} \subseteq S_{\alpha\beta}\psi_{\alpha\beta,\gamma}$ whenever $\alpha + \beta \leq \gamma$. The addition is defined by (1.1b) and the multiplication is given by

$$(xy)\psi_{\alpha\beta,\alpha+\beta} = x\psi_{\alpha,\alpha+\beta}\, y\psi_{\beta,\alpha+\beta}, \text{ for } x \in S_\alpha \text{ and } y \in S_\beta.$$

We follow the notations and terminologies of [5], [10] and [8, 7, 9] for semirings and [3] and [6] for semigroups.

2 Distributive Lattice of group-semirings

The term *group-semiring* is used to describe a semiring which is multiplicatively a group and additively a left (right) zero semigroup. Here we consider the left zero semigroup structure for the addition.

Proposition 2.1. Let $(G, \cdot)$ be a group. For $x, y \in G$ define

$$x + y = x.$$

Then $(G, +, \cdot)$ is a semiring. $\square$

2.1 Distributive lattice of group-semirings

Let $(D, +, \cdot)$ be a distributive lattice and $\{G_\alpha, \alpha \in D\}$ be a family of group-semirings. Then

$$S = \bigcup \{G_\alpha, \alpha \in D\}$$

is called a distributive lattice of group-semirings G_α if, for $x \in G_\alpha$ and $y \in G_\beta$,

$$x + y \in G_{\alpha+\beta} \text{ and } x \cdot y \in G_{\alpha\beta}.$$

Also, $S = \bigcup\{G_\alpha, \alpha \in D\}$ is said to be a strong distributive lattice of group-semirings G_α if S is an SDL of semirings $\{G_\alpha\}$ such that each G_α is a group-semiring. Since each group-semiring is a left zero semigroup under addition, the addition in S is given by

$$x + y = x\psi_{\alpha,\alpha+\beta} + y\psi_{\beta,\alpha+\beta} = x\psi_{\alpha,\alpha+\beta}.$$

Now we look into some properties of the homomorphisms $\phi_{\alpha,\beta}$ and $\psi_{\beta,\alpha}$ whenever S is an SDL of group-semirings.

Theorem 2.2. Let $(D, +, \cdot)$ be a distributive lattice and $S = \bigcup\{G_\alpha : \alpha \in D\}$ be an SDL of group-semirings G_α with connecting homomorphisms $\{\phi_{\alpha,\beta} : G_\alpha \to G_\beta$, for $\beta \leq \alpha\}$ and $\{\psi_{\beta,\alpha} : G_\beta \to G_\alpha$, for $\beta \leq \alpha\}$. For each $\alpha \in D$, let e_α denote the identity in the group G_α. Then,

(i) If $\beta \leq \alpha$ in D, then $e_\alpha e_\beta = e_\beta$ and $e_\alpha + e_\beta = e_\alpha$.

(ii) If $x \in G_\alpha$ and $y, y' \in G_\beta$ for some $\alpha, \beta \in D$ then, $x + y = x + y'$.

(iii) For $x \in G_\alpha$ and $y \in G_\beta$ where $\beta \leq \alpha$,

$$x\phi_{\alpha,\beta} = xe_\beta \text{ and } y\psi_{\beta,\alpha} = y + e_\alpha.$$

(iv) $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ is an isomorphism for any $\beta \leq \alpha$ in D and $\psi_{\beta,\alpha} = (\phi_{\alpha,\beta})^{-1}$.

(v) If $\alpha + \beta \leq \delta$ and $\gamma \leq \alpha\beta$, then $\phi_{\alpha,\gamma}\psi_{\gamma,\beta} = \psi_{\alpha,\delta}\phi_{\delta,\beta}$.

Proof. From the description of sum and product in SDL of group-semirings, we have, for $\beta \leq \alpha$,

$$\begin{aligned} e_\alpha e_\beta &= e_\alpha \phi_{\alpha,\alpha\beta} e_\beta \phi_{\beta,\alpha\beta} \\ &= e_\alpha \phi_{\alpha,\beta} e_\beta \phi_{\beta,\beta} && \text{since } \alpha\beta = \beta \\ &= e_\beta e_\beta && \text{since } \phi_{\alpha,\beta} \text{ is a homomorphism} \\ &= e_\beta. \end{aligned}$$

Similarly, we can show that $e_\alpha + e_\beta = e_\alpha$. This proves (i).

Let $x \in G_\alpha$ and $y, y' \in G_\beta$ for some $\alpha, \beta \in D$. Since S is an SDL of group semirings, we have
$x + y = x\psi_{\alpha,\alpha+\beta} = x + y'$. Hence (ii).

Now let $\beta \leq \alpha$ and $x \in G_\alpha$, then

$$\begin{aligned} xe_\beta &= x\phi_{\alpha,\beta}e_\beta\phi_{\beta,\beta} && \text{since } \alpha\beta = \beta \\ &= x\phi_{\alpha,\beta}e_\beta && \text{since } \phi_{\beta,\beta} \text{ is identity map} \\ &= x\phi_{\alpha,\beta}. \end{aligned}$$

Likewise, we can observe that, for $y \in G_\beta$,

$$y + e_\alpha = y\psi_{\beta,\alpha}.$$

This proves (iii). Now consider $\beta \leq \alpha$ and $x \in G_\alpha$, then

$$x\phi_{\alpha,\beta}\psi_{\beta,\alpha} = xe_\beta + e_\alpha.$$

Now by (ii), we have

$$\begin{aligned} xe_\beta + e_\alpha &= xe_\beta + x && \text{since } e_\alpha, x \in G_\alpha \\ &= xe_\beta + xe_\alpha && \text{since } x = xe_\alpha \\ &= x(e_\beta + e_\alpha) \\ &= xe_\alpha && \text{since } e_\alpha + e_\beta = e_\alpha \\ &= x. \end{aligned}$$

By a similar manner we have, for $y \in G_\beta$,

$$y\psi_{\beta,\alpha}\phi_{\alpha,\beta} = (y + e_\alpha)e_\beta = y.$$

Thus $\phi_{\alpha,\beta}$ and $\psi_{\beta,\alpha}$ are mutually inverse isomorphisms.
Now let $\alpha + \beta \leq \delta$ and $\gamma \leq \alpha\beta$. Then $\gamma \leq \alpha \leq \delta$ and $\gamma \leq \beta \leq \delta$. So

$$\begin{aligned} \phi_{\delta,\gamma} &= \phi_{\delta,\alpha}\phi_{\alpha,\gamma} && \text{by (ii) of Definition 1.1} \\ &= \phi_{\delta,\beta}\phi_{\beta,\gamma} \end{aligned}$$

Hence we have

$$\begin{aligned} \phi_{\alpha,\gamma} &= (\phi_{\delta,\alpha})^{-1}\phi_{\delta,\gamma} \\ &= (\phi_{\delta,\alpha})^{-1}\phi_{\delta,\beta}\phi_{\beta,\gamma} \\ &= \psi_{\alpha,\delta}\phi_{\delta,\beta}(\psi_{\gamma.\beta})^{-1}. \end{aligned}$$

Therefore, $\phi_{\alpha,\gamma}\psi_{\gamma,\beta} = \psi_{\alpha,\delta}\phi_{\delta,\beta}$.

□

Corollary 2.3. Let $S = \bigcup\{G_\alpha, \alpha \in D\}$ be an SDL of group-semirings, then the homomorphisms $\phi_{\alpha,\beta}$ (similarly $\psi_{\beta,\alpha}$), for $\beta \leq \alpha \in D$, are uniquely determined.

We now provide a construction of SDL of group-semirings starting with a collection $\{G_\alpha : \alpha \in D\}$ of group-semirings. We can see that only one family of isomorphisms is sufficient to describe the structure. Further, the additional conditions on the homomorphisms used in [4], are not assumed here. Moreover, it turns out that all the groups are isomorphic to each other.

Theorem 2.4. Let $\{G_\alpha : \alpha \in D\}$ be a family of group-semirings and $\{\phi_{\alpha,\beta} : G_\alpha \to G_\beta \text{ for } \beta \leq \alpha\}$ be a family of isomorphisms satisfying the following.

(SDL1) $\phi_{\alpha,\alpha}$ is the identity map on G_α.

(SDL2) If $\gamma \leq \beta \leq \alpha$ in D then $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

Then $S = \bigcup\{G_\alpha, \alpha \in D\}$ is a semiring with sum defined by

$$x + y = x\left(\phi_{\alpha+\beta,\alpha}\right)^{-1}$$

and the product is given by

$$xy = x\phi_{\alpha,\alpha\beta}y\phi_{\beta,\alpha\beta}$$

for $x \in G_\alpha$ and $y \in G_\beta$.

Let $\psi_{\beta,\alpha}$ denote the inverse of $\phi_{\alpha,\beta}$, for $\alpha \leq \beta$ in D. Towards proving the theorem we state the following lemma.

Lemma 2.5. If $\alpha + \beta \leq \delta$ and $\gamma \leq \alpha\beta$ then $\phi_{\alpha,\gamma}\psi_{\gamma,\beta} = \psi_{\alpha,\delta}\phi_{\delta,\beta}$.

Proof. Since $\gamma \leq \alpha, \beta \leq \delta$, we have

$$\phi_{\delta,\gamma} = \phi_{\delta,\alpha}\phi_{\alpha,\gamma} = \phi_{\delta,\beta}\phi_{\beta,\gamma}$$

Therefore $\phi_{\alpha,\gamma}\left(\phi_{\beta,\gamma}\right)^{-1} = \left(\phi_{\delta,\alpha}\right)^{-1}\phi_{\delta,\beta}$. □

Proof (of Theorem 2.4). From Definition 1.1, it follows that both $(S,\cdot)$ and $(S,+)$ are semigroups. To prove the distributive property, consider $x \in G_\alpha$, $y \in G_\beta$ and $z \in G_\sigma$. Then $x + y \in G_\delta$ and $(x+y)z \in G_\gamma$ where $\delta = \alpha + \beta$ and $\gamma = \delta\sigma$.

Now we have

$$\begin{aligned}
(x+y)z &= (x+y)\phi_{\delta,\gamma}z\phi_{\sigma,\gamma} \\
&= (x\psi_{\alpha,\delta})\phi_{\delta,\gamma}z\phi_{\sigma,\gamma} \\
&= x(\phi_{\alpha,\alpha\gamma}\psi_{\alpha\gamma,\gamma})z\phi_{\sigma,\gamma} && \text{by Lemma 2.5} \\
&= x\phi_{\alpha,\alpha\sigma}\psi_{\alpha\sigma,\gamma}z\phi_{\sigma,\gamma} && \text{since } \alpha\gamma = \alpha\sigma.
\end{aligned}$$

($\alpha\gamma = \alpha\delta\sigma = \alpha(\alpha+\beta)\sigma$ implies $\alpha\sigma \leq \alpha\gamma$ and $\gamma \leq \sigma$ implies $\alpha\gamma \leq \alpha\sigma$.)
Also $xz \in G_{\alpha\sigma},\ yz \in G_{\beta\sigma}$ and $xz + yz \in G_\gamma$, since $\alpha\sigma + \beta\sigma = (\alpha+\beta)\sigma = \delta\sigma = \gamma$. Then

$$\begin{aligned}
xz + yz &= x\phi_{\alpha,\alpha\sigma}z\phi_{\sigma,\alpha\sigma} + y\phi_{\beta,\beta\sigma}z\phi_{\sigma,\beta\sigma} \\
&= (x\phi_{\alpha,\alpha\sigma}z\phi_{\sigma,\alpha\sigma})\psi_{\alpha\sigma,\gamma} \\
&= x\phi_{\alpha,\alpha\sigma}\psi_{\alpha\sigma,\gamma}z\phi_{\sigma,\alpha\sigma}\psi_{\alpha\sigma,\gamma} && \text{since } \psi \text{ is a homomorphism} \\
&= x\phi_{\alpha,\alpha\sigma}\psi_{\alpha\sigma,\gamma}z\phi_{\sigma,\gamma} && \text{since } \alpha\sigma = \alpha\gamma \leq \gamma \leq \sigma.
\end{aligned}$$

It follows that $(x+y)z = xz + yz$. Similarly, we can prove that $z(x+y) = zx + zy$. □

The multiplicative reduct $(S,\cdot)$ of a distributive lattice of group-semirings $(S,+,\cdot)$ is a semilattice of groups and hence it is a strong semilattice of groups[2]. Thus the homomorphisms $\{\phi_{\alpha,\beta}\}$ is naturally determined. If all these homomorphisms are isomorphisms, then the semiring is an SDL of group-semirings. This leads to the following characterization theorem of SDL of group-semirings. Let e_α denote the multiplicative identity of the group-semiring G_α.

Theorem 2.6. A distributive lattice of group-semirings $S = \bigcup\{G_\alpha, \alpha \in D\}$ is an SDL of group-semirings if and only if the following holds:

- The map $x \mapsto xe_\beta : G_\alpha \to G_\beta$ is a group isomorphism for every $\beta \leq \alpha \in D$.

Proof. By (iii) and (iv) of Theorem 2.2 the necessary part follows.

Towards proving the sufficient part, assume that $S = \bigcup\{G_\alpha, \alpha \in D\}$ is a distributive lattice of group-semirings such that $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ defined by $(x)\phi_{\alpha,\beta} = xe_\beta$ is a group isomorphism, where $\alpha, \beta \in D$ and $\beta \leq \alpha$ in D. Now we have

$$(x)\phi_{\alpha,\alpha} = xe_\alpha = x$$

for all $x \in G_\alpha$. That is, $\phi_{\alpha,\alpha}$ is the identity homomorphism on G_α. Also for $\gamma \leq \beta \leq \alpha$,

$$\begin{aligned}(x)\phi_{\alpha,\beta}\phi_{\beta,\gamma} = xe_\beta\phi_{\beta,\gamma} &= xe_\beta e_\gamma \\ &= xe_\gamma \qquad\qquad \text{since } \gamma \leq \beta \\ &= (x)\phi_{\alpha,\gamma}\end{aligned}$$

Therefore $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for $\gamma \leq \beta \leq \alpha$. Thus, the multiplicative reduct of S is a strong semilattice of groups.

For $\beta \leq \alpha$ and $y \in G_\beta$, consider

$$\begin{aligned}(y + e_\alpha)e_\beta &= ye_\beta + e_\alpha e_\beta \\ &= ye_\beta \qquad\qquad \text{since } (S_\beta, +) \text{ is a left zero semigroup} \\ &= y\end{aligned}$$

It follows that the inverse of $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ is $\psi_{\beta,\alpha} : G_\beta \to G_\alpha$, defined by

$$y\psi_{\beta,\alpha} = y + e_\alpha, \text{ for } y \in G_\beta.$$

Now suppose $x \in G_\alpha$ and $y \in G_\beta$, for $\alpha, \beta \in D$. Then $x + y \in G_{\alpha+\beta}$. Consider

$$\begin{aligned}(x + y)\phi_{\alpha+\beta,\alpha\beta} &= (x + y)e_{\alpha\beta} \\ &= xe_{\alpha\beta} + ye_{\alpha\beta} \\ &= xe_{\alpha\beta}.\end{aligned}$$

Also, we have

$$\begin{aligned}(x + e_{\alpha+\beta})\phi_{\alpha+\beta,\alpha\beta} &= (x + e_{\alpha+\beta})e_{\alpha\beta} \\ &= xe_{\alpha\beta} + e_{\alpha+\beta}e_{\alpha\beta} \\ &= xe_{\alpha\beta}\end{aligned}$$

Since $\phi_{\alpha+\beta,\alpha\beta}$ is an isomrophism, we can say that

$$x + y = x + e_{\alpha+\beta} = x\psi_{\alpha,\alpha+\beta} = x\psi_{\alpha,\alpha+\beta} + y\psi_{\beta,\alpha+\beta}.$$

Hence we have two collections of homomorphisms $\{\phi_{\alpha,\beta}, \beta \leq \alpha \in D\}$ and $\{\psi_{\beta,\alpha}, \beta \leq \alpha \in D\}$ satisfying the conditions (i) to (iv) of Definition 1.1 such that the addition and multiplications in S are defined by (1.1b) and (1.1a) respectively. Thus S is an SDL of group-semirings. □

Because of Theorem 2.2, in an SDL of group-semirings, all the groups are isomorphic. Moreover the collection $\{\phi_{\alpha,\beta}, \beta \leq \alpha \in D\}$ induce an isomorphism between any two groups; in particular $\phi_{\alpha,\alpha\beta}\phi^{-1}_{\beta,\alpha\beta}$ is an isomorphism from G_α to G_β for any $\alpha, \beta \in D$. We can use these isomorphisms to show that an SDL of group semirings $S = \bigcup\{G_\alpha, \alpha \in D\}$ is isomorphic to the direct product of the distributive lattice D and a group semiring G_α. Thus a structure theorem for SDL of group-semirings is obtained, which can be phrased as follows.

Theorem 2.7. A semiring S is an SDL of group-semirings if and only if S is the direct product of a distributive lattice and a group-semiring.

Proof. Let $S = \bigcup\{G_\alpha, \alpha \in D\}$ be an SDL of group-semirings with connecting homomorphisms $\{\phi_{\alpha,\beta}; \beta \leq \alpha\}$. Define $\lambda : S \to D \times G_\alpha$ as follows. For $x \in S$,

$$\lambda(x) = (\beta, x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta})$$

where x belongs to G_β for some $\beta \in D$. We can show that λ is an isomorphism from S to $D \times G_\alpha$. Note that the following diagram commutes for any $\alpha, \beta \in D$.

$$\begin{tikzcd}
\boxed{G_\alpha} \arrow[dr, "\phi_{\alpha,\alpha\beta}"] \arrow[ddr, bend right, "\phi_{\alpha,\alpha\beta\gamma}"'] & & \boxed{G_\beta} \arrow[dl, "\phi_{\beta,\alpha\beta}"'] \arrow[ddl, bend left, "\phi_{\beta,\alpha\beta\gamma}"] \\
& \boxed{G_{\alpha\beta}} \arrow[d, "\phi_{\alpha\beta,\alpha\beta\gamma}"] & \\
& \boxed{G_{\alpha\beta\gamma}} &
\end{tikzcd}$$

Consider $x, y \in S$, where $x \in G_\beta$ and $y \in G_\gamma$. Then $xy = (x\phi_{\beta,\beta\gamma})(y\phi_{\gamma,\beta\gamma}) \in G_{\beta\gamma}$. We have

$$\begin{aligned}
\lambda(xy) &= (\beta\gamma, (xy)\phi_{\beta\gamma,\alpha\beta\gamma}\phi^{-1}_{\alpha,\alpha\beta\gamma}) \\
\text{and } \lambda(x)\lambda(y) &= (\beta, x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta})(\gamma, y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}) \\
&= (\beta\gamma, x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta}y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}).
\end{aligned}$$

Now we show that

$$(xy)\phi_{\beta\gamma,\alpha\beta\gamma}\phi^{-1}_{\alpha,\alpha\beta\gamma} = (x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta})(y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma})$$

For that, we consider

$$\begin{aligned}
\left(x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta}y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}\right)\phi_{\alpha,\alpha\beta\gamma} &= \left(x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta}\right)\phi_{\alpha,\alpha\beta\gamma} \\
&\qquad \left(y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}\right)\phi_{\alpha,\alpha\beta\gamma} \\
&= x\phi_{\beta,\alpha\beta}\phi_{\alpha\beta,\alpha\beta\gamma}y\phi_{\gamma,\alpha\gamma}\phi_{\alpha\gamma,\alpha\beta\gamma} \\
&= x\phi_{\beta,\alpha\beta\gamma}y\phi_{\gamma,\alpha\beta\gamma}
\end{aligned}$$

and

$$\begin{aligned}
xy\phi_{\beta\gamma,\alpha\beta\gamma} &= (x\phi_{\beta,\beta\gamma}y\phi_{\gamma,\beta\gamma})\phi_{\beta\gamma,\alpha\beta\gamma} \\
&= x\phi_{\beta,\alpha\beta\gamma}y\phi_{\gamma,\alpha\beta\gamma}
\end{aligned}$$

This proves

$$(\beta, x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta})(\gamma, y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}) = (\beta\gamma, xy\phi_{\beta\gamma,\alpha\beta\gamma}\phi^{-1}_{\alpha,\alpha\beta\gamma})$$

and hence $\lambda(xy) = \lambda(x)\lambda(y)$. Similarly, we get

$$(\beta, x\phi_{\beta,\alpha\beta}\phi^{-1}_{\alpha,\alpha\beta}) + (\gamma, y\phi_{\gamma,\alpha\gamma}\phi^{-1}_{\alpha,\alpha\gamma}) = \left(\beta+\gamma, (x+y)\phi_{\beta+\gamma,\alpha(\beta+\gamma)}\phi^{-1}_{\alpha,\alpha(\beta+\gamma)}\right)$$

Thus $\lambda : S \to D \times G_\alpha$ is a semiring isomorphism. □

The following observations of semirings which are SDLs of group-semirings are easy to verify.

Theorem 2.8. Let $S = \bigcup\{G_\alpha, \alpha \in D\}$ be an SDL of group-semirings and $E(S)$ denotes the set of all multiplicative idempotents of S. Then we have,

(i) $E(S) = \{e_\alpha : \alpha \in D\}$ where e_α is the identity of the group G_α,

(ii) $E(S)$ is a sub-semiring of S and $(E(S), +, \cdot)$ is a distributive lattice isomorphic to D and

(iii) $(S, \cdot)$ is an inverse semigroup.

(iv) $(S, \cdot, +)$ is also a semiring. □

Now we show that, if S is an SDL of group-semirings, we can recover the distributive lattice D and the group-semiring G_α from the structure of S. The following theorem provides the details.

Theorem 2.9. Let S be an SDL of group-semirings. Then we have the following:

(a) The map $\chi : x \mapsto xx^{-1}$ is a semiring homomorphism from S to $E(S)$.

(b) $\ker \chi = \{(x, y) : \chi(x) = \chi(y)\}$ is a congruence on S such that;

  (i) each congruence class is a group-semiring, denoted as G_α and

  (ii) $D = S/\ker(\chi)$ is a distributive lattice.

(c) $S = \bigcup\{G_\alpha, \alpha \in D\}$

Proof. Let S be an SDL of group-semirings. $(S, \cdot)$ is an inverse semigroup, and hence χ is well defined. Since all the idempotents are central in $(S, \cdot)$, we have $(xy)(y^{-1}x^{-1})xy = xy$. Thus $y^{-1}x^{-1}$ is the inverse of xy for all $x, y \in S$. Therefore we get

$$\chi(xy) = xy\,(xy)^{-1} = xyy^{-1}x^{-1} = yy^{-1}xx^{-1} = xx^{-1}yy^{-1} = \chi(x)\chi(y).$$

Also, $(x+y)(x+y)^{-1}$ and $xx^{-1}+yy^{-1}$ are idempotents and belong to the same group-semiring, they are equal. Thus we have

$$\chi(x + y) = \chi(x) + \chi(y).$$

Therefore χ is a semiring homomorphism and hence $\ker \chi$ is a congruence on S. Since the χ image of each element $x \in S$ is the identity of the group-semiring containing x, the congruence classes are the group-semirings. Thus $S/\ker \chi = E(S)$ and the theorem holds. □

Conclusion remarks

This paper comprehensively examines the structure of Strong Distributive Lattices (SDLs) of group-semirings, building upon the foundational works of Bendelt-Petrich and S. Ghosh. This research provides a thorough understanding of the fundamental properties, construction methodologies, and characterization of SDLs within the context of group-semirings.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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