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Gradient Remediability for Hyperbolic Systems Analysis, Approximations and Simulations

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Abstract In this paper, we introduce and characterize the notions of gradient remediability for a class of hyperbolic systems. We study with respect to the gradient observation and by an appropriate choice of efficient gradient actuators, the possibility of gradient compensation of known or unknown disturbances. We also examine the relation between these notions and the gradient controllability. Main properties concerning these concepts are considered. We applied it in a one-dimensional domain and showed how to find the optimal control ensuring the gradient remediability. Numerical examples and simulations confirm the results obtained.

1 Introduction

The problem described by the hyperbolic equations intervenes in different areas and occupies a great importance in the practice. This work is an extension of previous works that concern the analysis of this class of systems within certain concepts consisting of a set of notions, such as controllability [12, 17, 18, 11, 14], detectability, observability [21, 22, 13] and remediability [1, 3], which enable a better knowledge and understanding of the system. In addition to these concepts, others are very important in practical applications like gradient controllability [20, 9, 15], gradient detectability [6] and gradient observability [7, 8].

We recall that the notion of gradient remediability depends on the possible existence of appropriate gradient efficient actuators (input operator) guaranteeing the remediability of any disturbance present in the considered systems with respect to the gradient observation (output operator). This problem is particularly motivated by pollution problems and so-called space compensation or remediability problems. The notions of remediability and efficient actuators have been introduced and studied first for discrete systems and then for continuous systems of a finite time horizon and other situations (regional and asymptotic cases, internal or boundary actions of disturbances) [2, 4, 5]. In this work, we focused on the concept and properties of gradient remediability for hyperbolic systems. For some related studies see [10, 16, 19].

This paper is organized as follows: In the second section, we presented the considered systems. In the third section, we defined and characterized the concepts of exact and weak gradient remediability for hyperbolic systems. Next, and in the fourth section, we introduced the concepts of exact and weak gradient controllability and studied their relation with exact and weak gradient remediability. In the fifth section, we presented the characterization of the gradient remediability and showed the effect of the structure and the location of the actuators and sensors on it. In the sixth section, we applied in a one-dimensional domain. The seventh section is consecrated to the problem of gradient remediability with minimum energy, we demonstrated the existence of the optimal control ensuring the gradient compensation of any disturbance (known or unknown) by an appropriate choice of efficient gradient actuators. In the last section, we presented an illustrative example as well as approximations and numerical results.

2 Considered System

We consider the system described by the hyperbolic equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = Ay(x,t) + Bu(t) + f(x,t) & Q \\ y(x,0) = y^0(x), \ \dfrac{\partial y}{\partial t}(x,0) = y^1(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{2.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with regular boundary $\Gamma = \partial\Omega$, $Q = \Omega \times]0,T[$ and $\Sigma = \partial\Omega \times]0,T[$ for $T > 0$, $A = \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}\dfrac{\partial}{\partial x_j}\right)$ is a second-order elliptic linear operator with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and verified the elliptic conditions

$$\begin{cases} a_{ij} \in L^\infty(\Omega), \ 1 \le i,j \le n, \text{ with } a_{ij} = a_{ji}, \ 1 \le i,j \le n, \\ \\ \exists \alpha > 0, \ \forall \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, \ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha \sum_{i=1}^{n} |\xi_i|^2, \text{ pp. in } \Omega. \end{cases}$$

$B \in \mathscr{L}(\mathcal{U}, X)$, $u \in L^2(0,T;\mathcal{U})$, $\mathcal{U}$ the control space (Hilbert space), $X = H_0^1(\Omega)$ the state space and $f \in L^2(0,T;X)$ the disturbance term (generally unknown).
The system (2.1) is augmented by the output equation

$$z(t) = C\nabla y(t), \tag{2.2}$$

where $C \in \mathscr{L}\left(\left(L^2(\Omega)\right)^n, \mathcal{O}\right)$, $\mathcal{O}$ is the observation space (Hilbert space) and ∇ the gradient operator given by the formula

$$\begin{aligned} \nabla \ : \ & H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n \\ & y \mapsto \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \cdots, \frac{\partial y}{\partial x_n}\right). \end{aligned}$$

The operator $\overline{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined by

$$\overline{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ Ay_1 \end{bmatrix}, \quad \forall (y_1, y_2) \in D\left(\overline{A}\right) = D(A) \times H_0^1(\Omega), \tag{2.3}$$

is closed linear operator with domain $D\left(\overline{A}\right)$ dense in the Hilbert space $\overline{X} = H_0^1(\Omega) \times H_0^1(\Omega)$ with the inner product

$$\langle (y_1, y_2), (z_1, z_2) \rangle_{\overline{X}} = \langle \sqrt{-A}y_1, \sqrt{-A}z_1 \rangle_\Omega + \langle y_2, z_2 \rangle_\Omega.$$

In addition, if the operator A admits an orthonormal basis of eigenfunctions $\left(w_{m_j}\right)_{\substack{m \ge 1 \\ 1 \le j \le r_m}}$ associated to eigenvalues $\lambda_m < 0$ with multiplicity r_m, we have

$$Ay = \sum_{m \ge 1} \lambda_m \sum_{j=1}^{r_m} \langle y, w_{m_j} \rangle_\Omega w_{m_j}, \quad \forall y \in D(A),$$

and

$$\sqrt{-A}y = \sum_{m \ge 1} \sqrt{-\lambda_m} \sum_{j=1}^{r_m} \langle y, w_{m_j} \rangle_\Omega w_{m_j}, \quad \forall y \in D(A).$$

The adjoint operator $\overline{A}^*$ of $\overline{A}$ is given by $\overline{A}^* = -\overline{A}$.
The operator $\overline{A}$ generates on $\overline{X}$ a C_0-semigroup $(S\,(t))_{t\geq 0}$ given by

$$S\,(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} W_1\,(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ W_2\,(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{pmatrix}, \tag{2.4}$$

with

$$W_1(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sum_{m\geq 1}\sum_{j=1}^{r_m}\left(\langle y_1, w_{m_j}\rangle_\Omega \cos(\sqrt{-\lambda_m}t) + \frac{1}{\sqrt{-\lambda_m}}\langle y_2, w_{m_j}\rangle_\Omega \sin(\sqrt{-\lambda_m}t)\right) w_{m_j},$$

and

$$W_2(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sum_{m\geq 1}\sum_{j=1}^{r_m}\left(-\sqrt{-\lambda_m}\langle y_1, w_{m_j}\rangle_\Omega \sin(\sqrt{-\lambda_m}t) + \langle y_2, w_{m_j}\rangle_\Omega \cos(\sqrt{-\lambda_m}t)\right) w_{m_j}.$$

The adjoint semigroup $(S^*(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ is given by $S^*\,(t) = S\,(-t)\,,\ \forall t \geq 0$.
Finally, we consider the operator

$$\begin{aligned} \overline{B} \ &: \ \mathcal{U} \to \overline{X} \\ & \quad u \mapsto \overline{B}u = \begin{pmatrix} 0 & Bu \end{pmatrix}^T, \end{aligned}$$

and the function $\bar{f} = \begin{pmatrix} 0 & f \end{pmatrix}^T \in L^2\left(0, T; \overline{X}\right)$.
If we put

$$\bar{y}\,(t) = \begin{pmatrix} y\,(t) & \dfrac{\partial y}{\partial t}\,(t) \end{pmatrix}^T,\ \bar{y}^0 = \begin{pmatrix} y^0 & y^1 \end{pmatrix}^T \text{ and } \frac{\partial \bar{y}}{\partial t}\,(t) = \begin{pmatrix} \dfrac{\partial y}{\partial t}\,(t) & \dfrac{\partial^2 y}{\partial t^2}\,(t) \end{pmatrix}^T,$$

the system (2.1) is equivalent to the following system

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t}(t) = \overline{A}\bar{y}(t) + \overline{B}u(t) + \bar{f}(t) & 0 < t < T \\ \bar{y}(0) = \bar{y}^0 \end{cases} \tag{2.5}$$

The unique solution of the system (2.5) is

$$\bar{y}\,(t) = S\,(t)\,\bar{y}^0 + \int_0^t S\,(t-s)\,\overline{B}u\,(s)\,\mathrm{d}s + \int_0^t S\,(t-s)\,\bar{f}\,(s)\,\mathrm{d}s.$$

The system (2.5) is augmented by the output equation

$$\bar{z}\,(t) = \overline{C}\overline{\nabla}\bar{y}\,(t)\,, \tag{2.6}$$

where $\overline{C} = \begin{pmatrix} C & 0 \end{pmatrix}$ and

$$\begin{aligned} \overline{\nabla} \ &: \ H_0^1\,(\Omega) \times H_0^1\,(\Omega) \to \left(L^2\,(\Omega)\right)^n \times \left(L^2\,(\Omega)\right)^n \\ & \quad (y_1, y_2) \mapsto \overline{\nabla}\,(y_1, y_2) = (\nabla y_1, \nabla y_2)\,. \end{aligned}$$

3 Gradient Remediability

We consider the following operators

$$\begin{aligned} H \ : \ L^2\,(0, T; \mathcal{U}) \ &\to \ \overline{X} \\ u \ &\mapsto \ Hu = \textstyle\int_0^T S\,(T-s)\,\overline{B}u\,(s)\,\mathrm{d}s, \end{aligned}$$

and

$$\begin{array}{rclcl}\widetilde{H} & : & L^2\left(0,T;\overline{X}\right) & \to & \overline{X} \\ & & \bar{f} & \mapsto & \widetilde{H}\bar{f} = \int_0^T S\left(T-s\right)\bar{f}\left(s\right)\mathrm{d}s.\end{array}$$

The system (2.1) is disturbed by the force f assumed unknown and excited by a control that will be chosen to compensate for the disturbance f.
In the autonomous case, this same system is written

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = Ay(x,t) & Q \\ y(x,0) = y^0(x),\ \dfrac{\partial y}{\partial t}(x,0) = y^1(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{3.1}$$

The system (3.1) is equivalent to the following system

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t}(t) = \bar{A}\bar{y}(t) & 0 < t < T \\ \bar{y}(0) = \bar{y}^0 \end{cases} \tag{3.2}$$

where the solution $\bar{y}$ is given by

$$\bar{y}\left(t\right) = S\left(t\right)\bar{y}^0,$$

and the output function is given by

$$\bar{z}\left(t\right) = \overline{C}\overline{\nabla}S\left(t\right)\bar{y}^0.$$

If the control u compensates the effect of disturbance f at the time T, we obtain

$$\overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} \ = \ 0,$$

Definition 3.1.

(i) A disturbance $\bar{f}$ is exactly G-remediable on $[0,T]$, if

$$\exists\, u \in\ L^2\left(0,T;\mathcal{U}\right) \text{ such that } \overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = 0.$$

(ii) A disturbance $\bar{f}$ is weakly G-remediable on $[0,T]$, if

$$\forall\, \varepsilon > 0,\ \exists\, u \in L^2\left(0,T;\mathcal{U}\right) \text{ such that } \|\overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f}\|_{\mathcal{O}} < \varepsilon.$$

(iii) The system (2.5)-(2.6) is exactly (resp. weakly) G-remediable on $[0,T]$, if for every $\bar{f}$ in $L^2\left(0,T;\overline{X}\right)$, $\bar{f}$ is a disturbance exactly (resp. weakly) G-remediable on $[0,T]$.

Proposition 3.2.

(i) A disturbance $\bar{f}$ is exactly G-remediable on $[0,T]$ if and only if

$$\overline{C}\overline{\nabla}\widetilde{H}\bar{f} \in \mathrm{Im}(\overline{C}\overline{\nabla}H).$$

(ii) A disturbance $\bar{f}$ is weakly G-remediable on $[0,T]$ if and only if

$$\overline{C}\overline{\nabla}\widetilde{H}\bar{f} \in \overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)}.$$

Proof.

(i) A disturbance $\bar{f}$ is exactly G-remediable on $[0,T]$ if and only if

$$\exists u \in L^2(0,T;\mathcal{U}) \text{ such that } \overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = 0,$$

i.e. if and only if

$$\exists u_1 \in L^2(0,T;\mathcal{U}) \text{ such that } \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = \overline{C}\overline{\nabla}Hu_1,$$

where $u_1 = -u \in L^2(0,T;\mathcal{U})$, this is equivalent to saying that $\overline{C}\overline{\nabla}\widetilde{H}\bar{f}\ \in\ \mathrm{Im}(\overline{C}\overline{\nabla}H)$.

(ii) A disturbance $\bar{f}$ is weakly G-remediable on $[0,T]$ if and only if

$$\forall\varepsilon>0,\ \exists u\in L^2(0,T;\mathcal{U})\ \text{ such that }\ \|\overline{C}\overline{\nabla}\widetilde{H}\bar{f}+\overline{C}\overline{\nabla}Hu\|_{\mathcal{O}}<\varepsilon,$$

i.e. if and only if

$$\forall\varepsilon>0,\ \exists u_1\in L^2(0,T;\mathcal{U})\ \text{ such that }\ \|\overline{C}\overline{\nabla}\widetilde{H}\bar{f}-\overline{C}\overline{\nabla}Hu_1\|_{\mathcal{O}}<\varepsilon,$$

where $u_1=-u\in L^2(0,T;\mathcal{U})$, this is equivalent to saying that $\overline{C}\overline{\nabla}\widetilde{H}\bar{f}\in\overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)}$.

□

Lemma 3.3.
Let V, W and Z be reflexive Banach spaces, $P\in\mathscr{L}(V,Z)$ and $Q\in\mathscr{L}(W,Z)$. Then the following properties are equivalent

i. $\mathrm{Im}\,P\subset\mathrm{Im}\,Q$.

ii. $\exists\gamma>0$ *such that* $\|P^*z^*\|_{V^*}\le\gamma\|Q^*z^*\|_{W^*},\quad\forall z^*\in Z^*$.

Proposition 3.4.
The following properties are equivalent

(i) The system (2.5)-(2.6) *is exactly G-remediable on* $[0,T]$.

(ii) $\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H})\subset\mathrm{Im}(\overline{C}\overline{\nabla}H)$.

(iii) There exists $\gamma>0$ such that for every $\sigma\in\mathcal{O}^*$, we have

$$\|S^*(T-.)\,\overline{\nabla}^*\overline{C}^*\sigma\|_{L^2(0,T;\overline{X}^*)}\le\gamma\|\overline{B}^*S^*(T-.)\,\overline{\nabla}^*\overline{C}^*\sigma\|_{L^2(0,T;\mathcal{U}^*)}.$$

Proof.

- $1\Leftrightarrow 2$: By using the Proposition 3.2, the definition of the exactly G-remediability on $[0,T]$ of the system (2.5)-(2.6) is equivalent to

$$\overline{C}\overline{\nabla}\widetilde{H}\bar{f}\in\mathrm{Im}(\overline{C}\overline{\nabla}H),\forall\bar{f}\in L^2(0,T;\overline{X}),$$

  i.e.

$$\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H})\subset\mathrm{Im}(\overline{C}\overline{\nabla}H).$$

- $2\Leftrightarrow 3$: In Lemma 3.3, we put

$$P=\overline{C}\overline{\nabla}\widetilde{H}\quad\text{ and }\quad Q=\overline{C}\overline{\nabla}H,$$

  where

$$\widetilde{H}^*=S^*(T-.)\quad\text{ and }\quad H^*=\overline{B}^*S^*(T-.)\,.$$

□

Proposition 3.5.
The following properties are equivalent

(i) The system (2.5)-(2.6) *is weakly G-remediable on* $[0,T]$.

(ii) $\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H})\subset\overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)}$.

(iii) $\ker(\overline{B}^*\widetilde{H}^*\overline{\nabla}^*\overline{C}^*)=\ker(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*)$.

Proof.

- $1\Leftrightarrow 2$: By using the Proposition 3.2, the definition of the weakly G-remediability on $[0,T]$ of the system (2.5)-(2.6) is equivalent to

$$\overline{C}\overline{\nabla}\widetilde{H}\bar{f}\in\overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)},\quad\forall\bar{f}\in L^2(0,T;\overline{X}),$$

  i.e.

$$\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H})\subset\overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)}.$$

- $2 \Rightarrow 3$: Let $\sigma \in \ker(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*)$, then $\sigma \in \ker(H^* \overline{\nabla}^* \overline{C}^*)$.
  We have
  $$\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H}) \subset \overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)},$$
  then
  $$\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H}) \subset [\ker(H^* \overline{\nabla}^* \overline{C}^*)]^{\perp},$$
  hence
  $$\langle \overline{C}\overline{\nabla}\widetilde{H}\bar{f}, \sigma\rangle_{\mathcal{O}\times\mathcal{O}^*} = 0, \quad \forall \bar{f} \in L^2(0,T;\overline{X})$$
  then $\sigma \in [\mathrm{Im}(\overline{C}\overline{\nabla}\widetilde{H})]^{\perp}$, this gives $\sigma \in \ker(\widetilde{H}^* \overline{\nabla}^* \overline{C}^*)$.
- $3 \Rightarrow 2$: Let $\sigma \in \mathcal{O}^*$ such that $H^* \overline{\nabla}^* \overline{C}^* \sigma = 0$, we have
  $$\begin{aligned} H^* \overline{\nabla}^* \overline{C}^* \sigma = 0 &\Rightarrow \overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = 0 \\ &\Rightarrow \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = 0 \\ &\Rightarrow \langle \overline{C}\overline{\nabla}\widetilde{H}\bar{f}, \sigma\rangle_{\mathcal{O}\times\mathcal{O}^*} = 0, \quad \forall \bar{f} \in L^2(0,T;\overline{X}) \end{aligned}$$
  then
  $$\overline{C}\overline{\nabla}\widetilde{H}\bar{f} \in [\ker(H^* \overline{\nabla}^* \overline{C}^*)]^{\perp} = \overline{\mathrm{Im}(\overline{C}\overline{\nabla}H)}, \quad \forall \bar{f} \in L^2(0,T;\overline{X}).$$

$\square$

4 Gradient Remediability and Gradient Controllability

In this paragraph, we mentioned the concepts of exact and weak G-controllability and studied the relationship between it and the exact and weak G-remediability.

Definition 4.1.

(i) The system (2.5) is exactly G-controllable on $[0,T]$ if
$$\forall y^d = (y_1^d, y_2^d) \in \left((L^2(\Omega))^n\right)^2, \exists u \in L^2(0,T;\mathcal{U}) \text{ such that } \overline{\nabla}\bar{y}\,(T) = y^d.$$

(ii) The system (2.5) is weakly G-controllable on $[0,T]$ if
$$\forall \varepsilon > 0, \forall y^d = (y_1^d, y_2^d) \in \left((L^2(\Omega))^n\right)^2, \exists u \in L^2(0,T;\mathcal{U}) \text{ such that}$$
$$\|\overline{\nabla}\bar{y}(T) - y^d\|_{((L^2(\Omega))^n)^2} < \varepsilon.$$

Proposition 4.2.
If the system (2.5) is exactly (resp. weakly) G-controllable on $[0,T]$, then system (2.5)-(2.6) is exactly (resp. weakly) G-remediable on $[0,T]$.

Proof.
It suffices to apply the fact that $\overline{C}$ is a continuous operator. $\square$

5 Gradient Efficient Actuators and Sensors

In the case where $\mathcal{U} = \mathbb{R}^p$ i.e. the system (2.1) is excited by p zone actuators $(\Omega_i, a_i)_{1\le i\le p}$, where $a_i \in L^2(\Omega_i)$, $\Omega_i = \mathrm{supp}\,(a_i) \subset \Omega$, for $i = 1, 2, \ldots, p$, the operator $\overline{B}$ is given by
$$\begin{aligned} \overline{B} \;&:\; \mathbb{R}^p \to \overline{X} \\ u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) &\mapsto \overline{B}u(t) = \begin{pmatrix} 0 & \sum_{i=0}^{p} \chi_{\Omega_i}(x) a_i(x) u_i(t) \end{pmatrix}^T, \end{aligned}$$
and its adjoint is
$$\overline{B}^*(z_1, z_2) = \begin{pmatrix} \langle a_1, z_2\rangle_{\Omega_1} & \langle a_2, z_2\rangle_{\Omega_2} & \ldots & \langle a_p, z_2\rangle_{\Omega_p} \end{pmatrix}^T \in \mathbb{R}^p. \tag{5.1}$$

Corollary 5.1.
The system (2.5)-(2.6) *is exactly G-remediable on* $[0,T]$ *if and only if*

$$\exists\gamma>0:\int_0^T\left[\|\sqrt{-A}W_1\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}+\|W_2\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}\right]\mathrm{d}s$$
$$\leq\gamma\sum_{i=1}^{p}\int_0^T\langle g_i,W_2\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\rangle^2_{\Omega}\mathrm{d}s,\ \forall\sigma\in\mathbb{R}^q.$$

Proof.
Since the Proposition 3.5, the system (2.5)-(2.6) is exactly G-remediable on $[0,T]$ if and only if

$$\exists\gamma>0:\|S^*\left(T-.\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2\left(0,T;\overline{X}\right)}\leq\gamma\|\overline{B}^*S^*\left(T-.\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(0,T;\mathbb{R}^p)},\ \forall\sigma\in\mathbb{R}^q.$$

Firstly, we have

$$\|S^*\left(T-.\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2\left(0,T;\overline{X}\right)}=\int_0^T\|S\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{\overline{X}}\mathrm{d}s$$
$$=\int_0^T\left[\|\sqrt{-A}W_1\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}+\|W_2\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}\right]\mathrm{d}s,$$

and we have

$$\overline{B}^*S^*\left(T-.\right)\overline{\nabla}^*\overline{C}^*\sigma$$
$$=\Big(\langle a_1,W_2\left(.-T\right)\overline{\nabla}^*\overline{C}^*\sigma\rangle\quad\langle a_2,W_2\left(.-T\right)\overline{\nabla}^*\overline{C}^*\sigma\rangle\quad\cdots\quad\langle a_p,W_2\left(.-T\right)\overline{\nabla}^*\overline{C}^*\sigma\rangle\Big)^T,$$

then

$$\|\overline{B}^*S^*\left(T-.\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(0,T;\mathbb{R}^p)}=\sum_{i=1}^{p}\int_0^T\langle a_i,W_2\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\rangle^2_{\Omega}\mathrm{d}s.$$

Hence the result. □

Corollary 5.2.
The system (2.5)-(2.6) *is exactly G-remediable on* $[0,T]$ *if and only if*

$$\exists\gamma>0:T\sum_{m\geq1}(-\lambda_m)\sum_{j=1}^{r_m}\langle C^*\sigma,\nabla w_{m_j}\rangle^2_{(L^2(\Omega))^n}\leq\gamma\sum_{i=1}^{p}\int_0^T\Big[\sum_{m\geq1}\sqrt{-\lambda_m}\times$$
$$\sum_{j=1}^{r_m}\sin(\sqrt{-\lambda_m}(T-s))\sum_{j=1}^{r_m}\langle C^*\sigma,\nabla w_{m_j}\rangle_{(L^2(\Omega))^n}\langle a_i,w_{m_j}\rangle_{\Omega_i}\Big]^2\mathrm{d}s,\ \forall\sigma\in\mathbb{R}^q.$$

Proof.
We have

$$\|\sqrt{-A}W_1\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}$$
$$=\|\sum_{m\geq1}\sqrt{-\lambda_m}\sum_{j=1}^{r_m}\langle W_1(s-T)\overline{\nabla}^*\overline{C}^*\sigma,w_{m_j}\rangle w_{m_j}\|^2_{L^2(\Omega)}$$
$$=\|\sum_{m\geq1}\sqrt{-\lambda_m}\sum_{j=1}^{r_m}\Big(\langle\nabla^*C^*\sigma,w_{m_j}\rangle_{\Omega}\cos(\sqrt{-\lambda_m}(s-T))\Big)w_{m_j}\|^2_{L^2(\Omega)}$$
$$=\sum_{m\geq1}(-\lambda_m)\sum_{j=1}^{r_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle^2_{\Omega}\cos^2(\sqrt{-\lambda_m}(s-T)),$$

and

$$
\begin{aligned}
&\|W_2\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}\\
=&\|\sum_{m\geq 1}\sum_{j=1}^{r_m}-\sqrt{-\lambda_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle_{\Omega}\sin\left(\sqrt{-\lambda_m}\left(s-T\right)\right)w_{m_j}\|^2_{L^2(\Omega)}\\
=&\sum_{m\geq 1}\left(-\lambda_m\right)\sum_{j=1}^{r_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle^2_{\Omega}\sin^2\left(\sqrt{-\lambda_m}\left(s-T\right)\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_0^T\left[\|\sqrt{-A}W_1\left(s-T\right)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}+\|W_2(s-T)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(\Omega)}\right]\mathrm{d}s\\
=&\int_0^T\left[\sum_{m\geq 1}(-\lambda_m)\sum_{j=1}^{r_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle^2_{\Omega}\right]\mathrm{d}s\\
=&T\sum_{m\geq 1}(-\lambda_m)\sum_{j=1}^{r_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle^2_{\Omega}=T\sum_{m\geq 1}(-\lambda_m)\sum_{j=1}^{r_m}\langle C^*\sigma,\nabla w_{m_j}\rangle^2_{\Omega}.
\end{aligned}
$$

And we have

$$
\begin{aligned}
&\sum_{i=1}^{p}\int_0^T\langle a_i,W_2(s-T)\overline{\nabla}^*\overline{C}^*\sigma\rangle^2_{\Omega}\mathrm{d}s\\
=&\sum_{i=1}^{p}\int_0^T\langle\chi_{\Omega_i}a_i,\sum_{m\geq 1}\sum_{j=1}^{r_m}-\sqrt{-\lambda_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle_{\Omega}\sin(\sqrt{-\lambda_m}(s-T))w_{m_j}\rangle^2_{\Omega}\mathrm{d}s\\
=&\sum_{i=1}^{p}\int_0^T\left[\sum_{m\geq 1}\sum_{j=1}^{r_m}-\sqrt{-\lambda_m}\langle\nabla^*C^*\sigma,w_{m_j}\rangle_{\Omega}\langle\chi_{\Omega_i}a_i,w_{m_j}\rangle_{\Omega}\sin(\sqrt{-\lambda_m}(s-T))\right]^2\mathrm{d}s\\
=&\sum_{i=1}^{p}\int_0^T\left[\sum_{m\geq 1}\sum_{j=1}^{r_m}-\sqrt{-\lambda_m}\langle C^*\sigma,\nabla w_{m_j}\rangle_{\Omega}\langle\chi_{\Omega_i}a_i,w_{m_j}\rangle_{\Omega}\sin(\sqrt{-\lambda_m}(s-T))\right]^2\mathrm{d}s.
\end{aligned}
$$

Hence, the result follows immediately from Corollary 5.1. □

Now and in the case where $\mathcal{O}=\mathbb{R}^q$ i.e. the output of the system (2.6) is given by q sensors $(D_i,\mathfrak{s}_i)_{1\leq i\leq q}$, where $\mathfrak{s}_i\in L^2\left(D_i\right)$, $D_i=\operatorname{supp}\left(\mathfrak{s}_i\right)\subset\Omega$, for $i=1,2,\ldots,q$ and $D_i\cap D_j=\emptyset$ for $i\neq j$, the operator $\overline{C}=\begin{pmatrix}C & 0\end{pmatrix}$ is given by

$$
\begin{aligned}
C\quad:\quad&\left(L^2\left(\Omega\right)\right)^n\to\mathbb{R}^q\\
&y\mapsto Cy=\left(\sum_{i=1}^{n}\langle\mathfrak{s}_1,y_i\rangle_{D_1}\quad\sum_{i=1}^{n}\langle\mathfrak{s}_2,y_i\rangle_{D_2}\quad\ldots\quad\sum_{i=1}^{n}\langle\mathfrak{s}_q,y_i\rangle_{D_q}\right)^T,
\end{aligned}
$$

and its adjoint is $\overline{C}^*=\begin{pmatrix}C^* & 0\end{pmatrix}^T$, with for $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_q)\in\mathbb{R}^q$

$$
C^*\sigma=\left(\sum_{i=1}^{q}\chi_{D_i}(x)\sigma_i\mathfrak{s}_i(x)\quad\sum_{i=1}^{q}\chi_{D_i}(x)\sigma_i\mathfrak{s}_i(x)\quad\ldots\quad\sum_{i=1}^{q}\chi_{D_i}(x)\sigma_i\mathfrak{s}_i(x)\right)^T. \tag{5.2}
$$

Corollary 5.3.
The system (2.5)-(2.6) is exactly G-remediable on $[0,T]$ if and only if

$$
\begin{aligned}
\exists\gamma>0:T\sum_{m\geq 1}(-\lambda_m)\sum_{j=1}^{r_m}\left[\sum_{l=1}^{q}\sigma_l\sum_{k=1}^{n}\langle\mathfrak{s}_l,\frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_l}\right]\leq\gamma\sum_{i=1}^{p}\int_0^T\Bigg[\sum_{m\geq 1}\sqrt{-\lambda_m}\\
\times\sin(\sqrt{-\lambda_m}(T-s))\sum_{j=1}^{r_m}\sum_{l=1}^{q}\sigma_l\sum_{k=1}^{n}\langle\mathfrak{s}_l,\frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_l}\langle a_i,w_{m_j}\rangle_{\Omega_i}\Bigg]^2\mathrm{d}s,\ \forall\sigma\in\mathbb{R}^q.
\end{aligned}
$$

Proof.
It suffices to use the relation (5.2). □

▷ Now, we introduce the notion of the *gradient efficient actuators*

Definition 5.4. The actuators $(\Omega_i, a_i)_{1\leq i\leq p}$, $a_i \in L^2(\Omega_i)$ are said to be *gradient efficient* if the system (2.5)-(2.6) so excited is weakly G-remediable on $[0, T]$.

In order to give it a characterization, we appreciate the following two definitions, for $m \geq 1$

- The matrix A_m of order $(p \times r_m)$ defined by

$$A_m = \left(\langle a_i, w_{m_j}\rangle_{\Omega_i}\right)_{ij}, \quad 1 \leq i \leq p \quad \text{and} \quad 1 \leq j \leq r_m.$$

- The matrix S_m of order $(q \times r_m)$ defined by

$$S_m = \left(\sum_{k=1}^{n} \langle \mathcal{S}_i, \frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_i}\right)_{ij}, \quad 1 \leq i \leq p \quad \text{and} \quad 1 \leq j \leq r_m.$$

Proposition 5.5.
The actuators $(\Omega_i, a_i)_{1\leq i\leq p}$, $a_i \in L^2(\Omega_i)$ are gradient efficient if and only if

$$\ker(\nabla^* C^*) = \bigcap_{m\geq 1} \ker(A_m g_m).$$

Where, for $\sigma \in \mathbb{R}^q$ and $m \geq 1$,

$$g_m(\sigma) = \left(\langle \nabla^* C^* \sigma, w_{m_1}\rangle_\Omega \quad \langle \nabla^* C^* \sigma, w_{m_2}\rangle_\Omega \quad \dots \quad \langle \nabla^* C^* \sigma, w_{m_{r_m}}\rangle_\Omega\right)^T \in \mathbb{R}^{r_m}.$$

Proof.
Since the Proposition 5.3, the actuators $(\Omega_i, a_i)_{1\leq i\leq p}$, $a_i \in L^2(\Omega_i)$ are gradient efficient if and only if

$$\ker\left(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right) = \ker\left(\widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right).$$

Let $\sigma \in \mathbb{R}^q$, we have

$$\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = \overline{B}^* S(. - T)\, \overline{\nabla}^* \overline{C}^* \sigma$$

$$= \left(\langle a_1, W_2(. - T)\, \overline{\nabla}^* \overline{C}^* \sigma\rangle \quad \langle a_2, W_2(. - T)\, \overline{\nabla}^* \overline{C}^* \sigma\rangle \quad \cdots \quad \langle a_p, W_2(. - T)\, \overline{\nabla}^* \overline{C}^* \sigma\rangle\right)^T,$$

then

$$\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = \begin{pmatrix} \sum\limits_{m\geq 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}(T - .)\right) \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_1, w_{m_j}\rangle_{\Omega_1} \\ \sum\limits_{m\geq 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}(T - .)\right) \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_2, w_{m_j}\rangle_{\Omega_2} \\ \vdots \\ \sum\limits_{m\geq 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}(T - .)\right) \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_p, w_{m_j}\rangle_{\Omega_p} \end{pmatrix},$$

and we have $\forall m \geq 1$,

$$A_m g_m(\sigma) = \begin{pmatrix} \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_1, w_{m_j}\rangle_{\Omega_1} \\ \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_2, w_{m_j}\rangle_{\Omega_2} \\ \vdots \\ \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_p, w_{m_j}\rangle_{\Omega_p}. \end{pmatrix}$$

If we assume that $\sigma \in \bigcap\limits_{m\geq 1} \ker\left(A_m g_m\right)$, then $\sigma \in \ker\left(A_m g_m\right)$, $\forall m \geq 1$ and

$$\begin{aligned}
&\sigma \in \ker\left(A_m g_m\right),\ \forall m \geq 1\\
&\Rightarrow \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \sigma, w_{m_j}\right\rangle_\Omega \left\langle a_i, w_{m_j}\right\rangle_{\Omega_i} = 0,\ \forall i \in \{1,2,\ldots,p\},\ \forall m \geq 1\\
&\Rightarrow \sum_{m\geq 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}\,(T-.)\right) \sum_{j=1}^{r_m} \left\langle \nabla^* C^* \sigma, w_{m_j}\right\rangle_\Omega \left\langle a_i, w_{m_j}\right\rangle_{\Omega_i} = 0,\\
&\quad \forall i \in \{1,2,\ldots,p\}\\
&\Rightarrow \overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = 0 \Rightarrow\ \sigma \in \ker\left(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right),
\end{aligned}$$

then $\bigcap\limits_{m\geq 1} \ker\left(A_m g_m\right) \subset \ker\left(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right)$.

And if we assume that $\sigma \in \ker\left(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right)$, then $\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = 0$ and

$$\sum_{m\geq 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}\,(T-.)\right) \sum_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1,2,\ldots,p\}$$

since for T large enough, the sets $\left(\sin\left(\sqrt{-\lambda_n}\,(T-.)\right)\right)_{n\geq 1}$ form a complete orthogonal set of $L^2(0,T)$, then we have

$$\begin{aligned}
&\sum_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \langle a_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1,2,\ldots,p\},\ \forall m \geq 1\\
&\Rightarrow A_m g_m(\sigma) = 0,\ \forall m \geq 1\\
&\Rightarrow \sigma \in \ker\left(A_m g_m\right) = 0,\ \forall m \geq 1\\
&\Rightarrow \sigma \in \bigcap_{m\geq 1} \ker\left(A_m g_m\right),
\end{aligned}$$

then $\ker\left(\overline{B}^* \widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right) \subset \bigcap\limits_{m\geq 1} \ker\left(A_m g_m\right)$.

On the other hand, we have for every $\sigma \in \mathbb{R}^q$,

$$\begin{aligned}
\widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma &= S^*(T-.)\, \overline{\nabla}^* \overline{C}^* \sigma\\
&= S(.-T)\, \overline{\nabla}^* \overline{C}^* \sigma\\
&= \begin{pmatrix} W_1(.-T)\, \overline{\nabla}^* \overline{C}^* \sigma \\ W_2(.-T)\, \overline{\nabla}^* \overline{C}^* \sigma \end{pmatrix}\\
&= \begin{pmatrix} \sum\limits_{m\geq 1} \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \cos\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \\ \sum\limits_{m\geq 1} \sum\limits_{j=1}^{r_m} \sqrt{-\lambda_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \end{pmatrix},
\end{aligned}$$

We have

$$\begin{aligned}
&\sigma \in \ker\left(\widetilde{H}^* \overline{\nabla}^* \overline{C}^*\right) \Rightarrow \widetilde{H}^* \overline{\nabla}^* \overline{C}^* \sigma = 0\\
&\Rightarrow \begin{pmatrix} \sum\limits_{m\geq 1} \sum\limits_{j=1}^{r_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \cos\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \\ \sum\limits_{m\geq 1} \sum\limits_{j=1}^{r_m} \sqrt{-\lambda_m} \langle \nabla^* C^* \sigma, w_{m_j}\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

Since for T large enough, the sets

$$\left(\sin\left(\sqrt{-\lambda_n}\,(T-\cdot)\right)\right)_{n\geq 1} \text{ and } \left(\cos\left(\sqrt{-\lambda_n}\,(T-\cdot)\right)\right)_{n\geq 1}$$

forms a complete orthogonal set of $L^2(0,T)$, then we have

$$\sigma \in \ker\left(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*\right) \Rightarrow \nabla^* C^*\sigma = 0 \Leftrightarrow \sigma \in \ker(\nabla^* C^*),$$

then

$$\ker\left(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*\right) \subset \ker(\nabla^* C^*).$$

And we have

$$\begin{aligned}
&\sigma \in \ker(\nabla^* C^*) \Rightarrow \nabla^* C^*\sigma = 0\\
&\Rightarrow \begin{pmatrix} \sum_{m\geq 1}\sum_{j=1}^{r_m} \langle \nabla^* C^*\sigma, w_{m_j}\rangle_\Omega \cos\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \\ \sum_{m\geq 1}\sum_{j=1}^{r_m} \sqrt{-\lambda_m}\langle \nabla^* C^*\sigma, w_{m_j}\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\,(T-.)\right) w_{m_j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

then

$$\sigma \in \ker(\nabla^* C^*) \Rightarrow \sigma \in \ker\left(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*\right),$$

so

$$\ker(\nabla^* C^*) \subset \ker\left(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*\right),$$

and therefore

$$\ker(\nabla^* C^*) = \ker\left(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*\right).$$

□ □

Corollary 5.6.
If there exists $m_0 \geq 1$ such that rank $\left(S_{m_0}^T\right) = q$*, then the actuators $(\Omega_i, a_i)_{1\leq i\leq p}$, $a_i \in L^2(\Omega_i)$ are gradient efficient if and only if*

$$\bigcap_{m\geq 1} \ker\left(A_m S_m^T\right) = \{0\}.$$

Proof.
Let $\sigma \in \mathbb{R}^q$, then

$$\begin{aligned}
&\sigma \in \bigcap_{m\geq 1} \ker\left(A_m S_m^T\right)\\
&\Leftrightarrow \sigma \in \ker\left(A_m S_m^T\right),\ \forall m \geq 1\\
&\Leftrightarrow A_m S_m^T \sigma = 0,\ \forall m \geq 1\\
&\Leftrightarrow \sum_{l=1}^{q}\sum_{j=1}^{r_m} \sigma_l \langle \mathfrak{s}_l, \sum_{k=1}^{n} \frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_l} \langle a_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1,2,\ldots,p\},\ \forall m \geq 1\\
&\Leftrightarrow \sum_{j=1}^{r_m} \langle \nabla^* C^*\sigma, w_{m_j}\rangle_\Omega \langle a_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1,2,\ldots,p\},\ \forall m \geq 1\\
&\Leftrightarrow (A_m g_m)(\sigma) = 0, \forall m \geq 1\\
&\Leftrightarrow \sigma \in \ker(A_m g_m), \forall m \geq 1\\
&\Leftrightarrow \sigma \in \bigcap_{m\geq 1} \ker(A_m g_m),
\end{aligned}$$

this gives

$$\bigcap_{m\geq 1} \ker\left(A_m S_m^T\right) = \bigcap_{m\geq 1} \ker(A_m g_m).$$

On the other hand, we have $\sigma \in \ker(\nabla^* C^*) \Leftrightarrow \nabla^* C^* \sigma = 0$ and by using (5.2), we obtain

$$\langle \nabla^* C^* \sigma, w_{m_j} \rangle_\Omega = \sum_{l=1}^{q} \sigma_l \langle \mathcal{S}_l, \sum_{k=1}^{n} \frac{\partial w_{m_j}}{\partial x_k} \rangle_{D_l}, \ \forall j \in \{1, 2, \ldots, r_m\}, \ \forall m \geq 1$$

then

$$\left(\langle \nabla^* C^* \sigma, w_{m_1} \rangle_\Omega \quad \langle \nabla^* C^* \sigma, w_{m_2} \rangle_\Omega \quad \ldots \quad \langle \nabla^* C^* \sigma, w_{m_{r_m}} \rangle_\Omega \right)^T = S_m^T \sigma, \ \forall m \geq 1$$

and

$$\nabla^* C^* \sigma = 0 \Rightarrow S_m^T \sigma = 0, \forall m \geq \ 1,$$

then

$$\sigma \in \ker(\nabla^* C^*) \Rightarrow \sigma \in \ker(S_m^T), \forall m \geq \ 1.$$

since for $m_0 \geq 1$, we have $\operatorname{rank}\left(S_{m_0}^T\right) = q$, then

$$\ker\left(S_{m_0}^T\right) = \{0\},$$

this gives $\sigma = 0$, i.e. $\ker(\nabla^* C^*) = \{0\}$ and from the proposition 5, we have the result. □

Corollary 5.7.
If there exists $m_0 \geq 1$ *such that* $\operatorname{rank}\left(S_{m_0}^T\right) = q$ *and if*

$$\operatorname{rank}\left(A_{m_0} S_{m_0}^T\right) = q \tag{5.3}$$

or

$$\operatorname{rank}\left(A_{m_0}\right) = r_{m_0} \tag{5.4}$$

then the actuators $(\Omega_i, a_i)_{1 \leq i \leq p}$, $a_i \in L^2(\Omega_i)$ *are gradient efficient.*

Proof.

- If there exists $m_0 \geq 1$ such that

$$\operatorname{rank}\left(S_{m_0}^T\right) = q \quad \text{and} \quad \operatorname{rank}\left(A_{m_0} S_{m_0}^T\right) = q.$$

  The matrix $\left(A_{m_0} S_{m_0}^T\right)$ is of order $(p \times q)$ then from the rank-nullity theorem, we have

$$\operatorname{rank}\left(A_{m_0} S_{m_0}^T\right) + \dim\left(\ker\left(A_{m_0} S_{m_0}^T\right)\right) = q,$$

  then

$$\dim\left(\ker\left(A_{m_0} S_{m_0}^T\right)\right) = 0,$$

  which is equivalent to

$$\ker\left(A_{m_0} S_{m_0}^T\right) = \{0\},$$

  then

$$\bigcap_{m \geq 1} \ker\left(A_m S_m^T\right) = \{0\},$$

  that is equivalent, from the corollary 5.6, to the gradient efficient of actuators $(\Omega_i, a_i)_{1 \leq i \leq p}$.
- Now, we suppose that

$$\operatorname{rank}\left(S_{m_0}^T\right) = q \text{ and } \operatorname{rank}\left(A_{m_0}\right) = r_{m_0}.$$

  The matrix $\left(S_{m_0}^T\right)$ is order $(r_{m_0} \times q)$ then from the rank-nullity theorem, we have

$$\operatorname{rank}\left(S_{m_0}^T\right) + \dim\left(\ker\left(S_{m_0}^T\right)\right) = q,$$

  then

$$\dim\left(\ker\left(S_{m_0}^T\right)\right) = 0,$$

that is equivalent to

$$\ker\left(S_{m_0}^T\right) = \{0\}, \tag{5.5}$$

the same, the matrix (A_{m_0}) is order $(p \times r_{m_0})$, then from rank-nullity theorem, we have

$$\operatorname{rank}(A_{m_0}) + \dim(\ker(A_{m_0})) = r_{m_0},$$

then

$$\dim(\ker(A_{m_0})) = 0,$$

that is equivalent to

$$\ker(A_{m_0}) = \{0\}, \tag{5.6}$$

On the other hand, let $\sigma \in \ker\left(A_{m_0}S_{m_0}^T\right)$ then $\left(A_{m_0}S_{m_0}^T\right)\sigma = 0$ which gives

$$A_{m_0}\left(S_{m_0}^T\sigma\right) = 0.$$

from (5.5), we obtain $S_{m_0}^T\sigma = 0$ and from (5.6), we obtain $\sigma = 0$. then

$$\ker\left(A_{m_0}S_{m_0}^T\right) = \{0\},$$

hence therefore $\bigcap\limits_{m\geq 1}\ker\left(A_mS_m^T\right) = \{0\}$, that is equivalent, from the corollary 5.6, to the gradient efficient of actuators $(\Omega_i, a_i)_{1\leq i\leq p}$.

□

6 Application

We consider the system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = \Delta y(x,t) + \sum\limits_{i=1}^{p}\chi_{\Omega_i}a_i(x)\,u_i(t) + f(x,t) & Q \\ y(x,0) = y^0(x), \ \dfrac{\partial y}{\partial t}(x,0) = y^1(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with a sufficiently regular boundary, and we consider the system (6.1) is augmented by the output equation

$$z = \left(\sum_{i=1}^{n}\langle \mathcal{S}_1, \frac{\partial y}{\partial x_i}\rangle_{D_1} \quad \sum_{i=1}^{n}\langle \mathcal{S}_2, \frac{\partial y}{\partial x_i}\rangle_{D_2} \quad \cdots \quad \sum_{i=1}^{n}\langle \mathcal{S}_q, \frac{\partial y}{\partial x_i}\rangle_{D_q}\right)^T \tag{6.2}$$

There exists an orthonormal basis of eigenfunctions $\left(w_{m_j}\right)_{\substack{m\geq 1\\ 1\leq j\leq r_m}}$ of Δ associated to eigenvalues $(\lambda_m)_{m\geq 1}$ with multiplicity r_m and given by $\Delta w_{m_j} = \lambda_m w_{m_j}$, $\forall m \geq 1$ and $j = 1, 2, \ldots, r_m$. For $\Omega =]0,1[$, the eigenfunctions of Δ are

$$w_m(x) = \sqrt{2}\sin(m\pi x),\ \forall m \geq 1,$$

and the simple associated eigenvalues are

$$\lambda_m = -m^2\pi^2,\ \forall m \geq 1,$$

The semigroup generated by Δ is

$$S(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sum\limits_{m\geq 1}(\langle y_1, w_m\rangle_\Omega \cos(m\pi t) + \dfrac{1}{m\pi}\langle y_2, w_m\rangle_\Omega \sin(m\pi t))w_m \\ \sum\limits_{m\geq 1}(-m\pi\langle y_1, w_m\rangle_\Omega \sin(m\pi t) + \langle y_2, w_m\rangle_\Omega \cos(m\pi t))w_m \end{pmatrix}$$

If $D = \text{supp}\,(\mathcal{S}) \subset]0,1[$, $(q=1)$ and if there exists m_0 such that $\left\langle \mathcal{S}, \dfrac{\partial w_{m_0}}{\partial x} \right\rangle_D \neq 0$, then, from the Corollary 5.6, an actuator (Ω_1, a_1) is gradient efficient if and only if

$$\langle a_1, w_{m_0} \rangle_{\Omega_1} = \int_{\Omega_1} a_1\,(x) \sin\,(m_0 \pi x)\,\mathrm{d}x \neq 0.$$

Thus for example, if $a_1 = w_{m_0}$ then the actuator (Ω_1, a_1) is gradient efficient.

7 Exact Gradient Remediability With Minimal Energy

In this section we consider the following exact gradient remediability problem:
For $f \in L^2\left(0,T; H_0^1\,(\Omega)\right)$, does there exist an optimal control $u^* \in L^2\,(0,T;\mathcal{U})$ such that $\bar{z}_{u^*,f}\,(T) = \overline{C}\overline{\nabla} S\,(T)\,\bar{y}^0$, i.e. minimizing the function $J\,(u) = \|u\|^2_{L^2(0,T;\mathcal{U})}$ on the set

$$\mathcal{U}_{ad} = \left\{ u \in L^2\,(0,T;\mathcal{U}) : \overline{C}\overline{\nabla} H u + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = 0 \right\}.$$

For $\sigma \in \mathcal{O}^* \simeq \mathcal{O}$, let

$$N(\sigma) = \left[\int_0^T \|\overline{B}^* S^*(T-s)\overline{\nabla}^*\overline{C}^*\sigma\|^2_{\mathcal{U}^*}\,\mathrm{d}s \right]^{\frac{1}{2}},$$

Not that N is a semi-norme, but not necessarity a norm.

Lemma 7.1.
If $\ker(\nabla^* C^*) = \{0\}$*, then the following conditions are equivalent*

(i) The system (2.5)-(2.6) *is weakly G-remediable on* $[0,T]$.

(ii) $\ker(H^*\overline{\nabla}^*\overline{C}^*) = \{0\}$.

(iii) N *is a norme on* $\mathcal{O}$.

Proof.
• (1) $\Leftrightarrow$ (2): From the proposition 3.5, the system (2.5)-(2.6) is weakly G-remediable on $[0,T]$ if and only if $\ker(H^*\overline{\nabla}^*\overline{C}^*) = \ker(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*)$, since the $\ker(\widetilde{H}^*\overline{\nabla}^*\overline{C}^*) = \ker(\nabla^* C^*)$ and $\ker(\nabla^* C^*) = \{0\}$ then $\ker(H^*\overline{\nabla}^*\overline{C}^*) = \{0\}$.
• (2) $\Rightarrow$ (3): We have $N\,(\sigma) = \|H^*\overline{\nabla}^*\overline{C}^*\sigma\|_{L^2(0,T;\mathcal{U}^*)}$, then

$$N\,(\sigma) = \{0\} \Rightarrow H^*\overline{\nabla}^*\overline{C}^*\sigma = 0 \Rightarrow \sigma \in \ker(H^*\overline{\nabla}^*\overline{C}^*) \Rightarrow \sigma = 0.$$

• (3) $\Rightarrow$ (2): We have $\sigma \in \ker(H^*\overline{\nabla}^*\overline{C}^*) \Rightarrow N(\sigma) = 0 \Rightarrow \sigma = 0$. □

Let us consider the operateur $\Lambda = \overline{C}\;\overline{\nabla} H H^*\overline{\nabla}^*\overline{C}^*$ for $\sigma \in \mathcal{O}^* \simeq \mathcal{O}$, we have $\Lambda\sigma = \overline{C}\;\overline{\nabla} H H^*\overline{\nabla}^*\overline{C}^*\sigma \in \mathcal{O}$.

Lemma 7.2.
If $\ker(\nabla^* C^*) = \{0\}$, *then the following conditions are equivalent*

(i) the system (2.5)-(2.6) *is weakly G-remediable on* $[0,T]$.

(ii) the operateur Λ *is positive definite.*

Proof.
For $\sigma \in \mathcal{O}^* \simeq \mathcal{O}$, we have

$$\begin{aligned}\langle \Lambda\sigma, \sigma\rangle_{\mathcal{O}} &= \langle \overline{C}\overline{\nabla} H H^*\overline{\nabla}^*\overline{C}^*\sigma, \sigma\rangle_{\mathcal{O}} = \langle H^*\overline{\nabla}^*\overline{C}^*\sigma, H^*\overline{\nabla}^*\overline{C}^*\sigma\rangle_{L^2(0,T;\mathcal{U}^*)} \\ &= \|H^*\overline{\nabla}^*\overline{C}^*\sigma\|^2_{L^2(0,T;\mathcal{U}^*)} = [N(\sigma)]^2,\end{aligned}$$

then (1) $\Leftrightarrow$ (2). □

Suppose that N is a norme on $\mathcal{O}$ and let $\mathcal{F}$ be the completion of the space $\mathcal{O}$ with the norm N, then $\mathcal{F}$ is a Hilbert space with the inner product defined by

$$\langle \sigma, \theta \rangle_{\mathcal{F}} = \langle H^* \overline{\nabla}^* \overline{C}^* \sigma, H^* \overline{\nabla}^* \overline{C}^* \theta \rangle_{L^2(0,T;\mathcal{U}^*)}, \forall \sigma, \theta \in \mathcal{F}.$$

Proposition 7.3.

(i) $\mathcal{O} \subset \mathcal{F}$ *with continuos injection.*

(ii) $\langle \Lambda\sigma, \theta \rangle_{\mathcal{O}} = \langle \sigma, \theta \rangle_{\mathcal{F}}$, $\forall \sigma, \theta \in \mathcal{O}$.

(iii) There exists an isomorphism extention unique of Λ *from* $\mathcal{F}$ *to* $\mathcal{F}^*$ *such that*

$$\|\Lambda\sigma\|_{\mathcal{F}^*} = \|\sigma\|_{\mathcal{F}}, \ \forall \sigma \in \mathcal{F}.$$

Proof.

(i) For $\sigma \in \mathcal{O}$, we have

$$\begin{aligned}
&\|\sigma\|_{\mathcal{F}} = \|H^* \overline{\nabla}^* \overline{C}^* \sigma\|_{L^2(0,T;\mathcal{U}^*)} \\
&\Rightarrow \|\sigma\|^2_{\mathcal{F}} = \int_0^T \|\overline{B}^* S^*(T-s) \overline{\nabla}^* \overline{C}^* \sigma\|^2_{\mathcal{U}^*} \, \mathrm{d}s \\
&\Rightarrow \|\sigma\|^2_{\mathcal{F}} \leq \Big(\int_0^T \|\overline{B}^* S^*(T-s) \overline{\nabla}^* \overline{C}^* \|^2_{\mathcal{U}^*} \mathrm{d}s \Big) . \|\sigma\|^2_{\mathcal{O}} \leq \gamma \|\sigma\|^2_{\mathcal{O}}.
\end{aligned}$$

(ii) Let $\sigma, \theta \in \mathcal{O}$, we have

$$\langle \Lambda\sigma, \theta \rangle_{\mathcal{O}} = \langle \overline{C} \overline{\nabla} H H^* \overline{\nabla}^* \overline{C}^* \sigma, \theta \rangle_{\mathcal{O}} = \langle \sigma, \theta \rangle_{\mathcal{F}}.$$

(iii) Let $\sigma \in \mathcal{F}$, the linear operator

$$\begin{array}{rcccl}
\Lambda\sigma & : & \mathcal{O} & \longrightarrow & \mathbb{R} \\
 & & \theta & \longmapsto & (\Lambda\sigma)(\theta) = \langle \Lambda\sigma, \theta \rangle_{\mathcal{O}}
\end{array}$$

is continuous on $\mathcal{O}$ for the topology of $\mathcal{F}$, because

$$|(\sigma\theta)(\theta)| = |\langle \Lambda\sigma, \theta \rangle_{\mathcal{O}}| = |\langle \sigma, \theta \rangle_{\mathcal{F}}| \leq \|\sigma\|_{\mathcal{F}} . \|\theta\|_{\mathcal{F}}.$$

then it can be continuously extended in a unique way to $\mathcal{F}$, hence $\Lambda\sigma \in \mathcal{F}^*$ and $\|\Lambda\sigma\|_{\mathcal{F}^*} = \|\sigma\|_{\mathcal{F}}$, and so the operator Λ is an isomorphism from $\mathcal{F}$ to $\mathcal{F}^*$.

□

The following result, for the exact gradient remediability with minimal energy.

Proposition 7.4.
For $f \in L^2(0,T; H^1_0(\Omega))$, *the equation*

$$\Lambda\sigma_f = -\overline{C} \overline{\nabla} \widetilde{H} \bar{f},$$

has a unique solution $\sigma_f \in \mathcal{F}$, *and the control*

$$u_{\sigma_f}(t) = \overline{B}^* S^*(T-t) \overline{\nabla}^* \overline{C}^* (\sigma_f), \quad \forall t \in [0,T]$$

satisfies $\overline{C} \overline{\nabla} H u_{\sigma_f} + \overline{C} \overline{\nabla} \widetilde{H} \bar{f} = 0$. *Moreover,* u_{σ_f} *is optimal, with* $\|u_{\sigma_f}\|_{L^2(0,T;\mathcal{U})} = \|\sigma_f\|_{\mathcal{F}}$.

Proof.
The operator Λ is isomorphism from $\mathcal{F}$ to $\mathcal{F}^*$ then for $f \in L^2(0,T; H^1_0(\Omega))$, the equation

$$\Lambda\sigma_f = -\overline{C} \overline{\nabla} \widetilde{H} \bar{f}, \tag{7.1}$$

has a unique solution and for $u_{\sigma_f} = \overline{B}^* S^*(T-.)\overline{\nabla}^* \overline{C}^* \sigma_f$, we obtain

$$\Lambda\sigma_f = \overline{C}\overline{\nabla}\int_0^T S(T-s)\overline{B}\overline{B}^* S^*(T-s)\overline{\nabla}^*\overline{C}^*\sigma_f \mathrm{d}s = \overline{C}\overline{\nabla}Hu_{\sigma_f},$$

then

$$\overline{C}\overline{\nabla}Hu_{\sigma_f} + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = 0.$$

On the other hand, consider the set

$$\mathcal{U}_{ad} = \{u \in L^2(0,T;\mathcal{U}) \mid \overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f} = 0\},$$

$\mathcal{U}_{ad}$ is convex, closed and non-empty, because $u_{\sigma_f} \in \mathcal{U}_{ad}$.
Now we Consider the function

$$J(u) = \|\overline{C}\overline{\nabla}Hu + \overline{C}\overline{\nabla}\widetilde{H}\bar{f}\|_{\mathcal{O}}^2 + \|u\|^2_{L^2(0,T;\mathcal{U})}.$$

For $u \in \mathcal{U}_{ad}$, we have $J(u) = \|u\|^2_{L^2(0,T;\mathcal{U})}$. J is strictly convex on $\mathcal{U}_{ad}$, hence it admits a unique minimum in $u^* \in \mathcal{U}_{ad}$, i.e.

$$\exists! u^* \in \mathcal{U}_{ad} : J(u^*) = \min_{u \in \mathcal{U}_{ad}} J(u),$$

with u^* characterized by

$$\langle u^*, v - u^*\rangle \geq 0\,, \quad \forall v \in \mathcal{U}_{ad}.$$

For $v \in \mathcal{U}_{ad}$, we have

$$\begin{aligned}
\langle u_{\sigma_f}, v - u_{\sigma_f}\rangle_{L^2(0,T;\mathcal{U})} &= \int_0^T \langle u_{\sigma_f}(t), v(t) - u_{\sigma_f}(t)\rangle_{\mathcal{U}}\, \mathrm{d}t \\
&= \int_0^T \langle \overline{B}^* S^*(T-t)\,\overline{\nabla}^*\overline{C}^*\sigma_f, v(t) - u_{\sigma_f}(t)\rangle_{\mathcal{U}}\, \mathrm{d}t \\
&= \int_0^T \langle \overline{\nabla}^*\overline{C}^*\sigma_f, S(T-t)\overline{B}(v(t) - u_{\sigma_f}(t))\rangle_{\mathcal{U}}\, \mathrm{d}t \\
&= \langle \sigma_f, \overline{C}\overline{\nabla}\int_0^T S(T-t)\overline{B}(v(t) - u_{\sigma_f}(t))\, \mathrm{d}t\rangle_{\mathcal{U}} \\
&= \langle \sigma_f, \overline{C}\overline{\nabla}Hv - \overline{C}\overline{\nabla}Hu_{\sigma_f}\rangle_{\mathcal{U}} = 0.
\end{aligned}$$

Since u^* is unique, we have $u^* = u_{\sigma f}$ and $u_{\sigma f}$ is optimal with $\|u_{\sigma_f}\|^2_{L^2(0,T;\mathcal{U})} = \|\sigma_f\|^2_{\mathcal{F}}$. □

Remark 7.5. If $\mathcal{O} = \mathbb{R}^q$, we have $\mathcal{F} = \mathcal{F}^* = \mathcal{O} = \mathbb{R}^q$.

8 Approximations And Numerical Simulations

In this section we give an approximations of σ_f defined by (7.1) as a solutions of a finite dimension linear system $Ax = b$ and then the optimal control u_{σ_f}.

8.1 Approximation

Coefficients of the system

For $i, j \geq 1$, let

$$a_{ij} = \langle \Lambda e_i, e_j\rangle_{\mathbb{R}^q},$$

where $(e_i)_{1\leq i\leq q}$ is the canonical basis of $\mathbb{R}^q$, we have

$$\Lambda e_i = \overline{C}\overline{\nabla}\int_0^T S(T-s)\overline{B}\overline{B}^* S^*(T-s)\overline{\nabla}^*\overline{C}^* e_i\, \mathrm{d}s.$$

Then

$$\Lambda e_i = \begin{bmatrix} \sum\limits_{m'\geq 1}\sum\limits_{k'=1}^{r_{m'}}\sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m} W_1\left(m',k',m,k\right) \\ \sum\limits_{m'\geq 1}\sum\limits_{k'=1}^{r_{m'}}\sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m} W_2\left(m',k',m,k\right) \\ \vdots \\ \sum\limits_{m'\geq 1}\sum\limits_{k'=1}^{r_{m'}}\sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m} W_q\left(m',k',m,k\right) \end{bmatrix}$$

Where for $\alpha \in \{1, 2, ..., q\}$, we have

$$W_\alpha\left(m',k',m,k\right) = \sum_{\tau=1}^{p}\langle a_\tau, \omega_{m_k}\rangle_{\Omega_\tau}\langle a_\tau, \omega_{m'_{k'}}\rangle_{\Omega_\tau}\sum_{l=1}^{n}\langle \mathfrak{d}_i, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_i}\sum_{l'=1}^{n}\langle \mathfrak{d}_\alpha, \frac{\partial \omega_{m'_{k'}}}{\partial x_{l'}}\rangle_{D_\alpha}$$
$$\times\left(\frac{\sqrt{-\lambda_m}}{\sqrt{-\lambda_{m'}}}\int_0^T \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\sin\left(\sqrt{-\lambda_{m'}}\left(T-s\right)\right)\mathrm{d}s\right),$$

So

$$a_{ij} = \langle \Lambda e_i, e_j\rangle_{\mathbb{R}^q} = \sum_{m'\geq 1}\sum_{k'=1}^{r_{m'}}\sum_{m\geq 1}\sum_{k=1}^{r_m} W_j\left(m',k',m,k\right),$$

for M and N sufficiently large

$$a_{ij} \simeq \sum_{m'=1}^{M}\sum_{k'=1}^{r_{m'}}\sum_{m=1}^{N}\sum_{k=1}^{r_m}\sum_{\tau=1}^{p}\langle a_\tau, \omega_{m_k}\rangle_{\Omega_\tau}\left\langle a_\tau, \omega_{m'_{k'}}\right\rangle_{\Omega_\tau}\sum_{l=1}^{n}\langle \mathfrak{d}_i, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_i}\sum_{l'=1}^{n}\langle \mathfrak{d}_j, \frac{\partial \omega_{m'_{k'}}}{\partial x_{l'}}\rangle_{D_j}$$
$$\times\left(\frac{\sqrt{-\lambda_m}}{\sqrt{-\lambda_{m'}}}\int_0^T \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\sin\left(\sqrt{-\lambda_{m'}}\left(T-s\right)\right)\mathrm{d}s\right),$$

and for $j \geq 1$, let

$$b_j = -\left\langle \overline{C}\ \overline{\nabla}\widetilde{H}\bar{f}, e_j\right\rangle_{\mathbb{R}^q},$$

we have

$$\overline{C}\overline{\nabla}\widetilde{H}\bar{f} = \begin{bmatrix} \sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m}\sum\limits_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathfrak{d}_1, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_1}\int_0^T\left\langle f\left(s\right), \omega_{m_k}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\mathrm{d}s \\ \sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m}\sum\limits_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathfrak{d}_2, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_2}\int_0^T\left\langle f\left(s\right), \omega_{m_k}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\mathrm{d}s \\ \vdots \\ \sum\limits_{m\geq 1}\sum\limits_{k=1}^{r_m}\sum\limits_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathfrak{d}_q, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_q}\int_0^T\left\langle f\left(s\right), \omega_{m_k}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\mathrm{d}s \end{bmatrix}$$

then

$$b_j = -\sum_{m\geq 1}\sum_{k=1}^{r_m}\sum_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathfrak{d}_j, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_j}\int_0^T\left\langle f\left(s\right),\ \omega_{m_k}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\mathrm{d}s,$$

for N sufficiently large, we have

$$b_j \simeq -\sum_{m=1}^{N}\sum_{k=1}^{r_m}\sum_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathfrak{d}_j, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_j}\int_0^T\left\langle f\left(s\right), \omega_{m_k}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\left(T-s\right)\right)\mathrm{d}s.$$

Optimal control

For $j \geq 1$, the function coordinates u_{j,σ_f} of the optimal control

$$u_{\sigma_f} = \overline{B}^* S^* (T - .) \overline{\nabla}^* \overline{C}^* \sigma_f \in L^2 (0, T; \mathbb{R}^p),$$

are given by

$$\begin{aligned} u_{j,\sigma_f} &= \langle a_j, W_2 (. - T) \overline{\nabla}^* \overline{C}^* \sigma_f \rangle_{\Omega_j} \\ &= \sum_{m \geq 1} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \sum_{l'=1}^{q} -(\sigma_f)_{l'} \sqrt{-\lambda_m} \langle \delta_{l'}, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_{l'}} \langle a_j, \omega_{m_k} \rangle_{\Omega_j} \sin\left(\sqrt{-\lambda_m} (. - T) \right), \end{aligned}$$

for N sufficiently large, we have

$$u_{j,\sigma_f} \simeq \sum_{m=1}^{N} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \sum_{l'=1}^{q} -(\sigma_f)_{l'} \sqrt{-\lambda_m} \langle \delta_{l'}, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_{l'}} \langle a_j, \omega_{m_k} \rangle_{\Omega_j} \sin\left(\sqrt{-\lambda_m} (. - T) \right).$$

Cost

The cost is given by

$$\begin{aligned} \|u_{\sigma_f}\|_{L^2(0,T;\mathbb{R}^p)} &= \left(\int_0^T \|\overline{B}^* S^* (T-s) \overline{\nabla}^* \overline{C}^* \sigma_f\|^2_{\mathbb{R}^p} \mathrm{d}s \right)^{\frac{1}{2}} \\ &= \left[\sum_{j=1}^{p} \int_0^T \left(\sum_{m \geq 1} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \sum_{l'=1}^{q} -(\sigma_f)_{l'} \sqrt{-\lambda_m} \langle \delta_{l'}, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_{l'}} \langle a_j, \omega_{m_k} \rangle_{\Omega_j} \sin(\sqrt{-\lambda_m}(s-T)) \right)^2 \mathrm{d}s \right]^{\frac{1}{2}}, \end{aligned}$$

for N sufficiently large

$$\|u_{\sigma_f}\|_{L^2(0,T;\mathbb{R}^p)} \simeq \left[\sum_{j=1}^{p} \int_0^T \left(\sum_{m=1}^{N} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \sum_{l'=1}^{q} -(\sigma_f)_{l'} \sqrt{-\lambda_m} \langle \delta_{l'}, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_{l'}} \langle a_j, \omega_{m_k} \rangle_{\Omega_j} \sin(\sqrt{-\lambda_m}(s-T)) \right)^2 \mathrm{d}s \right]^{\frac{1}{2}}.$$

8.2 The corresponding observation

The observation corresponding to the control is given by

$$z_{u_{\sigma_f},f} = \overline{C} \overline{\nabla} S(t) \bar{y}^0 + \overline{C} \overline{\nabla} \int_0^t S(t-s) \overline{B} u_{\sigma_f}(s) \mathrm{d}s + \overline{C} \overline{\nabla} \int_0^t S(t-s) \bar{f}(s) \mathrm{d}s,$$

and

$$\begin{aligned} z_{j,u_{\sigma_f},f} = &\sum_{m \geq 1} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \left[\langle y^0, \omega_{m_k} \rangle_\Omega \cos\left(\sqrt{-\lambda_m}\, t \right) + \frac{1}{\sqrt{-\lambda_m}} \langle y^1, \omega_{m_k} \rangle_\Omega \sin\left(\sqrt{-\lambda_m}\, t \right) \right] \\ &\times \langle \delta_j, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_j} \\ &+ \sum_{m \geq 1} \sum_{k=1}^{r_m} \sum_{i=1}^{p} \sum_{l=1}^{n} \frac{1}{\sqrt{-\lambda_m}} \langle \delta_j, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_j} \langle a_i, \omega_{m_k} \rangle_{\Omega_i} \int_0^t u_i(s) \sin\left(\sqrt{-\lambda_m} (t-s) \right) \mathrm{d}s \\ &+ \sum_{m \geq 1} \sum_{k=1}^{r_m} \sum_{l=1}^{n} \frac{1}{\sqrt{-\lambda_m}} \langle \delta_j, \frac{\partial \omega_{m_k}}{\partial x_l} \rangle_{D_j} \int_0^t \langle f(s), \omega_{m_k} \rangle_\Omega \sin\left(\sqrt{-\lambda_m} (t-s) \right) \mathrm{d}s, \end{aligned}$$

for N sufficiently large

$$
\begin{aligned}
z_{j,u_{\sigma_f},f} \simeq & \sum_{m=1}^{N}\sum_{k=1}^{r_m}\sum_{l=1}^{n}\left[\langle y^0,\omega_{m_k}\rangle_\Omega \cos\left(\sqrt{-\lambda_m}\,t\right)+\frac{1}{\sqrt{-\lambda_m}}\langle y^1,\omega_{m_k}\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\,t\right)\right] \\
& \times \langle \mathcal{S}_j, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_j} \\
& + \sum_{m=1}^{M}\sum_{k=1}^{r_m}\sum_{i=1}^{p}\sum_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathcal{S}_j, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_j}\langle a_i,\omega_{m_k}\rangle_{\Omega_i}\int_0^t u_i(s)\sin\left(\sqrt{-\lambda_m}\,(t-s)\right)\mathrm{d}s \\
& + \sum_{m=1}^{M}\sum_{k=1}^{r_m}\sum_{l=1}^{n}\frac{1}{\sqrt{-\lambda_m}}\langle \mathcal{S}_j, \frac{\partial \omega_{m_k}}{\partial x_l}\rangle_{D_j}\int_0^t \langle f(s),\omega_{m_k}\rangle_\Omega \sin\left(\sqrt{-\lambda_m}\,(t-s)\right)\mathrm{d}s.
\end{aligned}
$$

8.3 Numerical simulations

For $\Omega =]0,1[$, we consider the system

$$
\begin{cases}
\dfrac{\partial^2 y}{\partial t^2}(x,t) = \Delta y(x,t) + \sum\limits_{i=1}^{p}\chi_{\Omega_i}a_i(x)u_i(t) + f(x,t) & Q \\
y(x,0) = \dfrac{\partial y}{\partial t}(x,0) = 0 & \Omega \\
y(\xi,t) = 0 & \Sigma
\end{cases}
$$

This system is equivalent to

$$
\begin{cases}
\dfrac{\partial \bar{y}}{\partial t}(x,t) = \bar{\Delta}\bar{y}(x,t) + \begin{pmatrix} 0 & \sum\limits_{i=1}^{p}\chi_{\Omega_i}a_i(x)u_i(t)\end{pmatrix}^T + \begin{pmatrix} 0 & f(x,t)\end{pmatrix}^T & Q \\
\bar{y}(x,0) = \dfrac{\partial \bar{y}}{\partial t}(x,0) = 0 & \Omega \\
\bar{y}(\xi,t) = 0 & \Sigma
\end{cases}
$$

is augmented by the output equation

$$z(t) = C\nabla y(t) = \overline{C}\overline{\nabla}\bar{y}(t).$$

The eigenfunctions are defined by

$$\omega_m(x) = \sqrt{2}\sin(m\pi x),\ m \geq 1,$$

the eigenvalues are simple and given by

$$\lambda_m = -m^2\pi^2,\ m \geq 1.$$

By choosing

- A sesor $(D,\mathcal{S})$: With $D =]0,1[$ and $\mathcal{S}(x) = \sqrt{2}x^2$
- An efficient actuator (Ω, a) : With $\Omega =]0,1[$ and $a(x) = \cos\left(\frac{\pi}{3}x\right)$,
- A disturbed function : $f(x,t) = 240\exp\left(-\left(\frac{t}{10}+x\right)\right),\ t > 0.$

And for $T = 50$, numerical results are obtained which show the theoretical results previously obtained.
Hence, in Figure 1, the representations of the discrete observation $z_{u,f}$ corresponding to the control optimal $u = u_{\sigma_f}$ and the disturbance f, the observation corresponding to the disturbance f and without control $u = 0$ and the $z_{0,0}$ which represent the normal observation, that is $u = 0$ and $f = 0$.

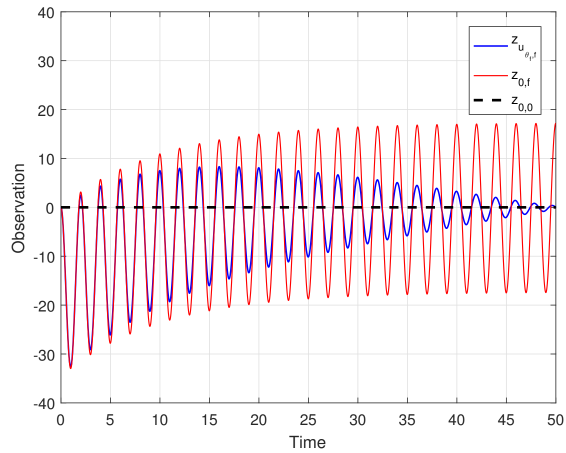

Figure 1. Representation of $z_{u_{\sigma_f},f}$(blue),$z_{0,f}$ (red) and $z_{0,0}$(black).

The Figure 1 show that the disturbance f is compensate by the control optimal u_{σ_f} at the time T $(T = 50)$ that is, we have $z_{u_{\sigma_f}} \equiv z_{0,0}\,(t)$, and the optimal control u_{σ_f} ensuring the gradient remediability of the disturbance f, is represented in Figure 2.

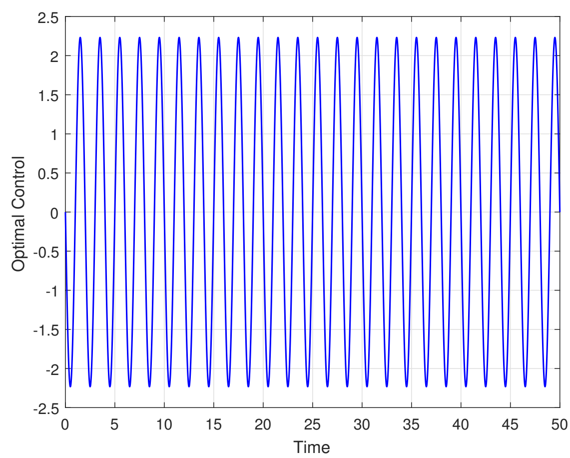

Figure 2. Representation of the optimal control u_{σ_f}.

9 Conclusion

Gradient remediability of hyperbolic systems is considered. The relationship with the notions of gradient controllability is studied. We showed that if any hyperbolic system is gradient controllable, it is gradient remediable. The role of actuators and sensors in the gradient compensation of any disturbance is examined. An algorithm has been successfully implemented and we illustrated this with an example and numerical simulations. The obtained results are related to the choice of convenient gradient efficient actuators. Many questions remain open, such as the case of the regional gradient remediability of hyperbolic systems. These questions are still under consideration and the results will appear in a separate paper.

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