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SOME NOVEL INEQUALITIES VIA TEMPERED FRACTIONAL INTEGRAL OPERATOR AND THEIR APPLICATIONS

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Abstract In this paper, we generate some innovative results on the integral inequalities of the Chebyshev type as well as the reverse Minkowski inequality using the tempered fractional integral operator. These inequalities extend some prior conclusions. As a direct result of our major findings, we demonstrate inequalities of Chebyshev type incorporating Riemann-Liouville fractional integral operators. These variations can lead to some intriguing outcomes in a few exceptional situations. We may also discover some applications of this inequality by using a particular function, which we then graphically display.

1 Introduction and Preliminaries

In order to deal with the differentiation and integrals of any fractional order, the fractional calculus is a valuable tool. The development of many fractional integral operators (FIO) and their applications across various scientific disciplines are the main topics of recent research. Ordinary differential equations with fractional derivatives can be used to simulate a wide range of mathematical, statistical, engineering, physical, chemical, and biological processes [1, 2, 3]. Numerous physicists and mathematicians contributed to the advancement of the theories of fractional calculus, including Riemann-Liouville (R-L), Caputo, Hilfer, Riesz, Hardamard, Erdêlyi-Kober, Saigo, and Marichev-Saigo operator, which has been the subject of in-depth research by numerous researchers (see [4, 5, 6, 7, 8]). The properties of the comformable fractional derivative operators (CFDO) introduced by Khalil et al. [9]. The properties of the fractional comformable investigated by Abdeljawad et al. [10]. In [11], Jarad created the fractional conformable integral derivative operator. Anderson and Ulness [12] doveloped the concept of conformable integral and derivative by employing local proportional derivatives. Abdeljawad and Baleanu et al. [13] have established fractional derivative operation with exponential Kernel and their discrete versions. Applications of integral inequalities have been developed for a huge variety of fractional integral operators in several scientific domains. Inequalities of the Riemann- Liouville fractional integral operator have been well studied by Belarbi et al. [4]. The authors established may inequalities such as Ostrowski type inequalities Hermite- Hadamard-Fejer inequalities [14], Chebyshev type inequalities, Grüss type [15, 16], Gronwall inequalities [17] via generalized fractional integral operators (see [18, 19, 20, 21, 22]).

The aim of this paper we established Chebyshev-type inequality and reverse Minkowski inequality via our newly defined tempered fractional integral operators (TFIO). In this section, we discuss basic preliminaries and notations; in the second section Chebyshev inequality on one sided tempered fractional integral operators obtained related theorems and results, third section we obtain some results using the reverse Minkowski inequality via tempered fractional integral operators and a related theorem referred to as the reverse Minkowski inequality; and in the last section, we are establishing a conclusion.
The inequality Chebyshev type [23] for the synchronous and integrable functions Ψ and Φ specified on $[a, b]$.

$$\frac{1}{(b-a)}\int_a^b \Phi(\varrho)\Psi(\varrho)d\varrho \geq \left(\frac{1}{(b-a)}\int_a^b \Phi(\varrho)d\varrho\right)\left(\frac{1}{(b-a)}\int_a^b \Psi(\varrho)d\varrho\right). \tag{1.1}$$

Two functions Φ and Ψ are considered as synchronous function on $[a, b]$, if

$$(\Phi(\varrho_1)-\Phi(\varrho_2))(\Psi(\varrho_1)-\Psi(\varrho_2)) \geq 0, \qquad (\varrho_1, \varrho_2 \in [a, b]). \tag{1.2}$$

In (see [24, 15]), researchers explored and doveloped different generalization of inequalities in (1.1) Minkowski inequality is the best- known inequalities (see [25, 18, 26, 19]), and it is a generalization of the recognized triangular inequality.

Minkowski inequality: If $p \geq 1 \in \mathbb{R}$, Φ, and Ψ are functions of class $\mathfrak{C}_p[a, b]$. Then the given inequality on $[a, b]$ is satisfied

$$\left(\int |\Phi+\Psi|^p d\varrho\right)^{\frac{1}{p}} \leq \left(\int |\Phi|^p d\varrho\right)^{\frac{1}{p}} + \left(\int |\Psi|^p d\varrho\right)^{\frac{1}{p}}. \tag{1.3}$$

The types of functions that we will take into consideration during this paper are described in the following definitions.

Definition 1.1. : The function $\Phi(\varrho)$ is considered in $\mathfrak{C}_p[a, b]$, if

$$\left(\int_a^b |\Phi(\varrho)|^p d\varrho\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty. \tag{1.4}$$

Definition 1.2. : The function $\Phi(\varrho)$ is considered in $\mathfrak{C}_{p,s}[a, b]$, $s \geq 0$, if

$$\left(\int_a^b |\Phi(\varrho)|^p \varrho^s d\varrho\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty. \tag{1.5}$$

Definition 1.3. : Let $[a,b] \in \mathbb{R}$ and $\zeta, \omega \in \mathbb{C}$ with $\Re(\zeta) > 0$ and $\Re(\omega) > 0$, then the left and right sided tempered fractional integral operators are respectively defined by

$$\left(\mathbb{I}_a^{\zeta,\omega}\mathfrak{f}\right)(\kappa) = e^{-\omega\kappa}\mathfrak{I}_{a,\kappa}^{\zeta}\left(e^{\omega\kappa}\mathfrak{f}(\kappa)\right) = \frac{1}{\Gamma(\zeta)}\int_a^{\kappa} e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}\mathfrak{f}(\varrho)d\varrho, \qquad a<\kappa, \tag{1.6}$$

and

$$\left(\mathbb{I}_b^{\zeta,\omega}\mathfrak{f}\right)(\kappa) = e^{-\omega\kappa}\mathfrak{I}_{b,\kappa}^{\zeta}\left(e^{\omega\kappa}\mathfrak{f}(\kappa)\right) = \frac{1}{\Gamma(\zeta)}\int_{\kappa}^{b} e^{-\omega(\varrho-\kappa)}(\varrho-\kappa)^{\zeta-1}\mathfrak{f}(\varrho)d\varrho, \qquad \kappa<b. \tag{1.7}$$

Now, Buschman at el. [27] was first studied on tempered fractional integral, but Meerschaert et al. [28] and Li et al. [29] have derived the associated tempered fractional more explicitly. Other than these many authors (see[30, 31, 16]) were also discussed using many inequalities via Tempered fractional integral operator.

Special case: when $\mathfrak{f}(\kappa) = (\kappa-a)^n$, then

$$\begin{aligned}\left(\mathbb{I}_a^{\zeta,\omega}(\kappa-a)^n\right) &= \frac{1}{\Gamma(\zeta)}\int_a^{\kappa} e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}(\varrho-a)^n d\varrho \\ &= \frac{1}{\Gamma(\zeta)}\int_a^{\kappa} e^{-\omega((\kappa-a)-(\varrho-a))}((\kappa-a)-(\varrho-a))^{\zeta-1}(\varrho-a)^n d\varrho \\ &= e^{-\omega(\kappa-a)}\frac{1}{\Gamma(\zeta)}\int_a^{\kappa}\sum_{r=0}^{\infty}\frac{(\omega(\varrho-a))^r}{r!}((\kappa-a)-(\varrho-a))^{\zeta-1}(\varrho-a)^n d\varrho.\end{aligned}$$

On setting $u = \frac{\varrho-a}{\kappa-a}$, we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}(\kappa-a)^n\right) = e^{-\omega(\kappa-a)}\frac{1}{\Gamma(\zeta)}\sum_{r=0}^{\infty}\left(\frac{\omega^r}{r!}\right)\int_0^1((\kappa-a)u)^{r+n}(1-u)^{\zeta-1}(\kappa-a)^{\zeta-1}(\kappa-a)du,$$

by solving the above equation and using definition of beta function, we get

$$\begin{aligned}\left(\mathbb{I}_a^{\zeta,\omega}(\kappa-a)^n\right) &= e^{-\omega(\kappa-a)}\frac{1}{\Gamma(\zeta)}\sum_{r=0}^{\infty}\left(\frac{\omega^r(\kappa-a)^{r+\zeta+n}}{r!}\right)\int_0^1 u^{r+n}(1-u)^{\zeta-1}du \\ &= e^{-\omega(\kappa-a)}\frac{(\kappa-a)^{\zeta+n}}{\Gamma(\zeta)}\sum_{r=0}^{\infty}\frac{\omega^r(\kappa-a)^r}{r!}\frac{\Gamma(r+n+1)\Gamma(\zeta)}{\Gamma(r+n+\zeta+1)}.\end{aligned}$$

$$\left(\mathbb{I}_a^{\zeta,\omega}(\kappa-a)^n\right) = e^{-\omega(\kappa-a)}(\kappa-a)^{\zeta+n}\sum_{r=0}^{\infty}\frac{(\omega(\kappa-a))^r \quad \Gamma(r+n+1)}{r!\Gamma(r+n+\zeta+1)}, \tag{1.8}$$

and when $a=0$ then

$$\left(\mathbb{I}_0^{\zeta,\omega}(\kappa^n)\right) = e^{-\omega\kappa}(\kappa)^{\zeta+n}\sum_{r=0}^{\infty}\frac{(\omega\kappa)^r \quad \Gamma(r+n+1)}{r!\Gamma(r+n+\zeta+1)}. \tag{1.9}$$

Remark 1.4. : Setting $\omega = 0$ in the equation (1.6) and (1.7) reduced to left and right R-L fractional integral operators respectively, for $\zeta \in \mathfrak{C}$ and $\Re(\zeta) > 0$.

$$\left(\mathbb{I}_a^{\zeta}\mathfrak{f}\right)(\kappa) = \mathfrak{I}_{a,\kappa}^{\zeta}\left(\mathfrak{f}(\kappa)\right) = \frac{1}{\Gamma(\zeta)}\int_a^{\kappa}(\kappa-\varrho)^{\zeta-1}\mathfrak{f}(\varrho)d\varrho, \qquad a<\kappa, \tag{1.10}$$

and

$$\left(\mathbb{I}_b^{\zeta}\mathfrak{f}\right)(\kappa) = \mathfrak{I}_{b,\kappa}^{\zeta}\left(\mathfrak{f}(\kappa)\right) = \frac{1}{\Gamma(\zeta)}\int_{\kappa}^{b}(\varrho-\kappa)^{\zeta-1}\mathfrak{f}(\varrho)d\varrho, \qquad \kappa<b. \tag{1.11}$$

The tempered fractional integral (1.6), satisfies the following semigroup property for $\Re(\gamma) > 0, \Re(\zeta) > 0$:

$$\mathbb{I}_a^{\gamma,\omega}\left(\mathbb{I}_a^{\zeta,\omega}\mathfrak{f}(\kappa)\right) = \mathbb{I}_a^{\gamma+\zeta,\omega}\mathfrak{f}(\kappa). \tag{1.12}$$

The aim of this paper is divided into two section. The first section Chebyshev and second section Minkowski inequalities via tempered fractional integral are discussed.

2 Main result I

In this part, we established Chebyshev inequalities via tempered fractional operator.

Theorem 2.1. *Suppose Φ and Ψ are two integrable functions which are synchronous on $[a,\infty)$. Then the subsequent inequality holds for all $\kappa \in [a,b]$ and $\zeta,\omega \in \mathbb{C}$ with $\Re(\zeta) > 0$ and $\Re(\omega) > 0$:*

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) \geq \left[\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)\right]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa). \tag{2.1}$$

Proof. $[a, \infty)$ is synchronous for Φ and Ψ, thus

$$(\Phi(\varrho_1) - \Phi(\varrho_2))(\Psi(\varrho_1) - \Psi(\varrho_2)) \geq 0, \tag{2.2}$$

or equivalent

$$\Phi(\varrho_1)\Psi(\varrho_1) + \Phi(\varrho_2)\Psi(\varrho_2) \geq \Phi(\varrho_1)\Psi(\varrho_2) + \Phi(\varrho_2)\Psi(\varrho_1), \tag{2.3}$$

multiplying both sides of (2.3) by $\frac{1}{\Gamma(\zeta)} e^{-\omega(\kappa-\varrho_1)}(\kappa - \varrho_1)^{\zeta-1}$ and integrating the resulting inequality with respect to ϱ_1 over (a, κ), we attain

$$\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_1)}(\kappa-\varrho_1)^{\zeta-1}\Phi(\varrho_1)\Psi(\varrho_1)d\varrho_1 + \frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_1)}(\kappa-\varrho_1)^{\zeta-1}\Phi(\varrho_2)\Psi(\varrho_2)d\varrho_1$$
$$\geq \frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_1)}(\kappa-\varrho_1)^{\zeta-1}\Phi(\varrho_1)\Psi(\varrho_2)d\varrho_1 + \frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_1)}(\kappa-\varrho_1)^{\zeta-1}\Phi(\varrho_2)\Psi(\varrho_1)d\varrho_1,$$

it follows that

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) + \Phi(\varrho_2)\Psi(\varrho_2)\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_1)}(\kappa-\varrho_1)^{\zeta-1}d\varrho_1 \geq \Psi(\varrho_2)\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa) + \Phi(\varrho_2)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa),$$

thus we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) + \Phi(\varrho_2)\Psi(\varrho_2)\left(\mathbb{I}_a^{\zeta,\omega}1\right)(\kappa) \geq \Psi(\varrho_2)\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa) + \Phi(\varrho_2)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa), \tag{2.4}$$

again multiplied both sides of (2.4) by $\frac{1}{\Gamma(\zeta)} e^{-\omega(\kappa-\varrho_2)}(\kappa - \varrho_2)^{\zeta-1}$ and integrating the resulting inequality w.r.t ϱ_2 over (a, κ), we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa)\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\zeta-1}d\varrho_2 + \left(\mathbb{I}_a^{\zeta,\omega}1\right)(\kappa)\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\zeta-1}\Phi(\varrho_2)\Psi(\varrho_2)d\varrho_2$$
$$\geq \left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\zeta-1}\Psi(\varrho_2)d\varrho_2 + \left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa)\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\zeta-1})\Phi(\varrho_2)d\varrho_2,$$

or equivalent to

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa), \tag{2.5}$$

or

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) \geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa), \tag{2.6}$$

setting $n = 0$, then equation (1.8) reduced into

$$\left(\mathbb{I}_a^{\zeta,\omega}(1)\right) = e^{-\omega(\kappa-a)}(\kappa-a)^\zeta \sum_{r=0}^\infty \frac{(\omega(\kappa-a))^r\Gamma(r+1)}{r!\Gamma(r+\zeta+1)}, \tag{2.7}$$

equation (2.6) becomes

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa) \geq \left[e^{-\omega(\kappa-a)}(\kappa-a)^\zeta \sum_{r=0}^\infty \frac{(\omega(\kappa-a))^r\Gamma(r+1)}{r!\Gamma(r+\zeta+1)}\right]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa). \tag{2.8}$$

For $a = 0$, the equation (2.8) reduce to

$$\left(\mathbb{I}_0^{\zeta,\omega}\Phi\Psi\right)(\kappa) \geq \left[e^{-\omega\kappa}(\kappa)^\zeta \sum_{r=0}^\infty \frac{(\omega\kappa)^r\Gamma(r+1)}{r!\Gamma(r+\zeta+1)}\right]^{-1}\left(\mathbb{I}_0^{\zeta,\omega}\Phi\right)(\kappa)\left(\mathbb{I}_0^{\zeta,\omega}\Psi\right)(\kappa). \tag{2.9}$$

□

Theorem 2.2. *Suppose Φ and Ψ be two integrable function that are synchronous on $[a, \infty)$. Then the subsequent inequality holds for all $\kappa \in [a, b]$ and $\zeta, \eta, \omega \in \mathbb{C}$ with $\Re(\zeta) > 0, \Re(\eta) > 0$ and $\Re(\omega) > 0$:*

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}(1))(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Phi\Psi)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Psi)(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Phi)(\kappa). \tag{2.10}$$

Proof. Multiplying both sides of equation (2.4) by $\frac{1}{\Gamma(\eta)} e^{-\omega(\kappa-\varrho_2)}(\kappa - \varrho_2)^{\eta-1}$ and integrating the resulting inequality with respect to ϱ_2 over (a, κ), we get

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa)\frac{1}{\Gamma(\eta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\eta-1}d\varrho_2 + \left(\mathbb{I}_a^{\zeta,\omega}1\right)(\kappa)\frac{1}{\Gamma(\eta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\eta-1})\Phi(\varrho_2)\Psi(\varrho_2)d\varrho_2$$
$$\geq \left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)\frac{1}{\Gamma(\eta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\eta-1})\Psi(\varrho_2)d\varrho_2 + \left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa)\frac{1}{\Gamma(\eta)}\int_a^\kappa e^{-\omega(\kappa-\varrho_2)}(\kappa-\varrho_2)^{\eta-1})\Phi(\varrho_2)d\varrho_2,$$

or equivalent

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}(1))(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Phi\Psi)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Psi)(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa)(\mathbb{I}_a^{\eta,\omega}\Phi)(\kappa). \tag{2.11}$$

□

Theorem 2.3. *Let $(\varphi_j)_{j=1,2,3...n}$ be n positive increasing functions on $[a,\infty)$. Then for $\kappa \in [a,b], \zeta \in (0,1], \omega, \zeta \in \mathbb{C}$, with $\Re(\omega) > 0$ and $\Re(\zeta) > 0$, we have*

$$\mathbb{I}_a^{\zeta,\omega}\left(\prod_{j=1}^{n}\varphi_j\right)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)^{1-n}\prod_{j=1}^{n}\left(\mathbb{I}_a^{\zeta,\omega}\varphi_j\right)(\kappa). \tag{2.12}$$

Proof. In order to demonstrate that this theorem is correct, we will use induction on $n \in N$. i.e. for $n = 1$, we have

$$\mathbb{I}_a^{\zeta,\omega}(\varphi_1)(\kappa) \geq \mathbb{I}_a^{\zeta,\omega}\varphi_1(\kappa), \quad \kappa > 0$$

i.e. holds. Now for $n = 2$, since φ_1 and φ_2 are positive and increasing functions, therefore we have

$$(\varphi_1(\kappa) - \varphi_1(\theta))(\varphi_2(\kappa) - \varphi_2(\theta)) \geq 0.$$

Consequently, by using theorem 2.1, we have

$$\mathbb{I}_a^{\zeta,\omega}(\varphi_1\varphi_2)(\kappa) \geq [\left(\mathbb{I}_a^{\zeta,\omega}1\right)]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}\varphi_1\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\varphi_2\right)(\kappa).$$

Now consider the induction hypothesis

$$\mathbb{I}_a^{\zeta,\omega}\left(\prod_{j=1}^{n-1}\varphi_j\right)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}1\right)^{2-n}\prod_{j=1}^{n-1}\left(\mathbb{I}_a^{\zeta,\omega}\varphi_j\right)(\kappa), \tag{2.13}$$

since $\varphi_j : j = 1, 2, ...n$ are positive increasing functions on $\mathbb{R}^+$, therefore $\Phi : \prod_{j=1}^{n-1}\varphi_j$ is increasing on $\mathbb{R}^+$. Let $\Psi = \varphi_n$ and applying theorem 2.1 for the functions Φ and Ψ, we have

$$\mathbb{I}_a^{\zeta,\omega}\left(\prod_{j=1}^{n}\varphi_j\right)(\kappa) = \mathbb{I}_a^{\zeta,\omega}\left(\prod_{j=1}^{n-1}\varphi_j\right)(\varphi_n)(\kappa),$$

or equivalent

$$\mathbb{I}_a^{\zeta,\omega}(\Phi\Psi)(\kappa) \geq [\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}\prod_{j=1}^{n-1}(\varphi_j)(\kappa)\right)\left(\mathbb{I}_a^{\zeta,\omega}\varphi_n\right)(\kappa),$$

using equation (2.13), in the right hand side of the above equation, we yield

$$\geq [\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)]^{-1}\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)^{2-n}\prod_{j=1}^{n-1}\left(\mathbb{I}_a^{\zeta,\omega}\varphi_j\right)(\kappa)\left(\mathbb{I}_a^{\zeta,\omega}\varphi_n\right)(\kappa)$$

or

$$\mathbb{I}_a^{\zeta,\omega}\left(\prod_{j=1}^{n}\varphi_j\right)(\kappa) \geq \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)^{1-n}\prod_{j=1}^{n}\left(\mathbb{I}_a^{\zeta,\omega}\varphi_j\right)(\kappa). \tag{2.14}$$

□

Theorem 2.4. *Let Φ and Ψ be two functions defined on $[a,\infty)$ such that Φ is increasing and Ψ is differentiable with a real number with Ψ' bounded below and $m = \inf_{\kappa\geq a}\Psi'(\kappa)$, then*

$$\mathbb{I}_a^{\zeta,\omega}(\Phi\Psi)(\kappa) \geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) - m(\kappa-a)\sum_{r=0}^{\infty}\frac{(r+1)}{(r+\zeta+1)}\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) + m\mathbb{I}_a^{\zeta,\omega}((\kappa-a)\Psi(\kappa)) \tag{2.15}$$

Proof. Let $h = \Phi(\kappa) - m(\kappa - a)$. We discover that h is differentiable and increasing on $\mathbb{R}_0^+$ and also using theorem 2.1, we have

$$\mathbb{I}_a^{\zeta,\omega}(\Phi(\kappa) - m(\kappa-a))\Psi(\kappa)] \geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}(\Phi(\kappa) - m(\kappa-a))\mathbb{I}_a^{\zeta,\omega}(\Psi(\kappa)),$$

$$\geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) - m[\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}(\kappa-a)\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa), \tag{2.16}$$

from equation (1.8), setting $n = 1$

$$\left(\mathbb{I}_a^{\zeta,\omega}(\kappa-a)\right) = e^{-\omega(\kappa-a)}(\kappa-a)^{\zeta+1}\sum_{r=0}^{\infty}\frac{(\omega(\kappa-a))^r\Gamma(r+2)}{r!\Gamma(r+\zeta+2)}, \tag{2.17}$$

also, setting $n = 0$, in equation (1.8) then reduced into

$$\left(\mathbb{I}_a^{\zeta,\omega}(1)\right) = e^{-\omega(\kappa-a)}(\kappa-a)^{\zeta}\sum_{r=0}^{\infty}\frac{(\omega(\kappa-a))^r\Gamma(r+1)}{r!\Gamma(r+\zeta+1)}. \tag{2.18}$$

Thus form equation (2.16), we have

$$\mathbb{I}_a^{\zeta,\omega}(\Phi(\kappa) - g(\kappa-a))\Psi(\kappa)] \geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) - m(\kappa-a)\times\sum_{r=0}^{\infty}\frac{(r+1)}{(r+\zeta+1)}\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa),$$

or equivalent to

$$\mathbb{I}_a^{\zeta,\omega}(\Phi\Psi)(\kappa) \geq [\mathbb{I}_a^{\zeta,\omega}(1)]^{-1}\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) - m(\kappa-a)\times\sum_{r=0}^{\infty}\frac{(r+1)}{(r+\zeta+1)}\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa) + m\mathbb{I}_a^{\zeta,\omega}((\kappa-a)\Psi(\kappa)).$$

□

3 Main result II

In this section we discuss about Minkowski inequality and it's reverse via tempered fractional operator.

Theorem 3.1. *Let* $\Phi, \Psi \in \mathfrak{C}_{1,s}[a,\kappa], s \in \mathbb{R}/\{0\}$ *and* $p \geq 1$, *be two positive functions in* $[0,\infty)$ *such that, for all* $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ *and* $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. *If* $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ *for* $m, M \in \mathbb{R}^+$ *and for all* $\varrho \in [a,\kappa]$, *then*

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{1}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{1}{p}} \leq c_1\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right)^{\frac{1}{p}}, \tag{3.1}$$

where $c_1 = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. using the given condition $\frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \quad a \leq \varrho \leq \kappa$, we get

$$\Phi(\varrho) \leq M\Psi(\varrho) + M\Phi(\varrho) - M\Phi(\varrho),$$

or

$$(M+1)\Phi(\varrho) \leq M\left(\Psi(\varrho)+\Phi(\varrho)\right),$$

which is equivalent to

$$(M+1)^p\Phi^p(\varrho) \leq M^p\left(\Psi(\varrho)+\Phi(\varrho)\right)^p, \tag{3.2}$$

by multiplying both sides of (3.2) with $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ)

$$\begin{aligned}&(M+1)^p\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}\Phi^p(\varrho)d\varrho\\ &\leq M^p\frac{1}{\Gamma(\zeta)}\int_a^\kappa e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}\left(\Psi(\varrho)+\Phi(\varrho)\right)^p d\varrho,\end{aligned}$$

or

$$(M+1)^p\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right) \leq M^p\left(\mathbb{I}_a^{\zeta,\omega}\left(\Psi+\Phi\right)^p(\kappa)\right),$$

thus

$$\left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}} \leq \left(\frac{M}{M+1}\right)\left[\mathbb{I}_a^{\zeta,\omega}\left(\Psi+\Phi\right)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.3}$$

On the other hand, as $m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)}$, it follows that $m\Psi(\varrho) \leq \Phi(\varrho)$,

$$\Psi(\varrho) \leq \frac{1}{m}\Phi(\varrho) + \frac{1}{m}\Psi(\varrho) - \frac{1}{m}\Psi(\varrho),$$

or equivalent to

$$\left(1+\frac{1}{m}\right)^p\Psi^P(\varrho) \leq \left(\frac{1}{m}\right)^p\left(\Phi(\varrho)+\Psi(\varrho)\right)^p, \tag{3.4}$$

now, multiplying both sides of (3.4) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality w.r.t ϱ over (a,κ),we obtain

$$\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}} \leq \left(\frac{1}{m+1}\right)\left[\mathbb{I}_a^{\zeta,\omega}\left(\Psi+\Phi\right)^p(\kappa)\right]^{\frac{1}{p}}, \tag{3.5}$$

by adding equation (3.3) and equation (3.5), we obtain the result given in (3.1)

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{1}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{1}{p}} \leq \frac{M(m+1)+(M+1)}{(m+1)(M+1)}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right)^{\frac{1}{p}}. \tag{3.6}$$

Thus equation (3.6) is knows as the reverse Minkowski inequality involving tempered fractional operator.

□

Theorem 3.2. *Let* $\Phi, \Psi \in \mathbb{C}_{1,s}[a,\kappa], s \in \mathbb{R}/\{0\}$ *and* $p \geq 1$, *are two positive functions in* $[0,\infty)$ *s.t., for all* $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ *and* $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. *If* $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ *for* $m, M \in \mathbb{R}^+$ *and for all* $\varrho \in [a,\kappa]$, *then*

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{2}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{2}{p}} \geq c_2\left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}}\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}}, \tag{3.7}$$

where $c_2 = \frac{(m+1)(M+1)}{M} - 2$.

Proof. Taking the product between equation (3.3) and equation (3.5), we have

$$\left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}}\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}} \leq \frac{M}{(M+1)(m+1)}\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi+\Psi\right)^p(\kappa)\right)^{\frac{2}{p}}, \tag{3.8}$$

involving the Minkowski inequality, on the right side of equation (3.8),we obtain

$$\frac{(M+1)(m+1)}{M}\left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}}\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}} \leq \left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{1}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{1}{p}}\right]^2, \tag{3.9}$$

from equation (3.9), we get

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{2}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{2}{p}} \geq \left(\frac{(M+1)(m+1)}{M} - 2\right)\left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}}\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.10}$$

□

Now, other inequalities of the Minkowski reverse type are given in the result that we present below

Theorem 3.3. *Let* $\Phi, \Psi \in \mathbb{C}_{1,s}[a,\kappa]$, $s \in \mathbb{R}/[0]$ *and* $p, q \geq 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *be two positive functions in* $[0,\infty)$ *s.t., for all* $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ *and* $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. *If* $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ *for* $m, M \in \mathbb{R}^+$ *and for all* $\varrho \in [a,\kappa]$, *then*

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{1}{p}} \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\mathbb{I}_a^{\zeta,\omega}\Phi^{\frac{1}{p}}(\kappa)\Psi^{\frac{1}{q}}(\kappa)\right). \tag{3.11}$$

Proof. : Since $\frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \quad \varrho \in [a,\kappa]$, we have $\Phi(\varrho) \leq M\Psi(\varrho)$,

or equivalent

$$\Psi^{\frac{1}{q}}(\varrho) \geq M^{\frac{-1}{q}}\Phi^{\frac{1}{q}}(\varrho), \tag{3.12}$$

multiplying both sides of equation (3.12) by $\Phi^{\frac{1}{p}}(\varrho)$, we can rewrite it as follows

$$\Phi^{\frac{1}{p}}(\varrho)\Psi^{\frac{1}{q}}(\varrho) \geq M^{\frac{-1}{q}}\Phi^{\frac{1}{q}}(\varrho)\Phi^{\frac{1}{p}}(\varrho), \tag{3.13}$$

now, multiplying both sides of (3.13) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ) and using the relation $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$M^{\frac{-1}{q}}\left(\mathbb{I}_a^{\zeta,\omega}\Phi\right)(\kappa) \leq \left(\mathbb{I}_a^{\zeta,\omega}\right)\left(\Phi^{\frac{1}{p}}(\kappa)\Psi^{\frac{1}{q}}(\kappa)\right),$$

or equivalently

$$M^{\frac{-1}{pq}}\left[\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\right]^{\frac{1}{p}} \leq \left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi^{\frac{1}{p}}(\kappa)\Psi^{\frac{1}{q}}(\kappa)\right)\right]^{\frac{1}{p}}. \tag{3.14}$$

On the other hand $m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)}$, it follows that

$$m^{\frac{1}{p}}\Psi^{\frac{1}{p}}(\varrho) \leq \Phi^{\frac{1}{p}}(\varrho), \tag{3.15}$$

further, by multiplying both sides of equation (3.15) by $\Psi^{\frac{1}{q}}(\varrho)$ and involving the relation $\frac{1}{p}+\frac{1}{q}=1$, it yield

$$m^{\frac{1}{p}}\Psi(\varrho) \leq \Phi^{\frac{1}{p}}(\varrho)\Psi^{\frac{1}{q}}(\varrho), \tag{3.16}$$

now, multiplying both sides of (3.16) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we obtain

$$m^{\frac{1}{p}}\left(\mathbb{I}_a^{\zeta,\omega}\Psi\right)(\kappa) \leq \mathbb{I}_a^{\zeta,\omega}\left(\Phi^{\frac{1}{p}}(\kappa)\Psi^{\frac{1}{q}}(\kappa)\right),$$

or equivalently

$$m^{\frac{1}{pq}}\left[\mathbb{I}_a^{\zeta,\omega}\Psi(\kappa)\right]^{\frac{1}{q}} \leq \left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi^{\frac{1}{p}}(\kappa)\Psi^{\frac{1}{q}}(\kappa)\right)\right]^{\frac{1}{q}}. \tag{3.17}$$

Finally, we multiplied equation (3.14) by equation (3.17) and using the relation $\frac{1}{p}+\frac{1}{q}=1$, we obtain required inequality (3.11). □

Theorem 3.4. *Let* $\Phi, \Psi \in \mathbb{C}_{1,s}[a,\kappa]$, $s \in \mathbb{R}/[0]$ *and* $p, q \geq 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *are two positive functions in* $[0,\infty)$ *s.t., for all* $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ *and* $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. *If* $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ *for* $m, M \in \mathbb{R}^+$ *and for all* $\varrho \in [a,\kappa]$, *then*

$$\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\Psi(\kappa) \leq c_3\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi^p+\Psi^p\right)(\kappa)\right) + c_4\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi^q+\Psi^q\right)(\kappa)\right), \tag{3.18}$$

where $c_3 = \frac{2^{p-1}M^p}{p(M+1)^p}$ *and* $c_4 = \frac{2^{q-1}}{q(m+1)^q}$.

Proof. Using $\Phi(\varrho) \leq M\Psi(\varrho)$ for $\varrho \in (a,\kappa)$, we discover the subsequent inequality:

$$\Phi(\varrho)(M+1) \leq M\left(\Phi(\varrho)+\Psi(\varrho)\right),$$

or equivalent to

$$(M+1)^p\Phi^p(\varrho) \leq M^p(\Phi+\Psi)^p(\varrho), \tag{3.19}$$

now, multiplying both sides of (3.19) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ),we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa) \leq \frac{M^p}{(M+1)^p}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p\right)(\kappa). \tag{3.20}$$

Also, we have $0 < m \leqslant \frac{\Phi(\varrho)}{\Psi(\varrho)}, \quad \varrho \in [a,\kappa]$, it follows

$$(m+1)^q\Psi^q(\varrho) \leq (\Phi+\Psi)^q(\varrho), \tag{3.21}$$

further, we multiplying both sides of (3.21) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Psi^q\right)(\kappa) \leq \frac{1}{(m+1)^q}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^q\right)(\kappa), \tag{3.22}$$

Now, applying young's inequality

$$\Phi(\varrho)\Psi(\varrho) \leq \frac{\Phi^p(\varrho)}{p} + \frac{\Psi^q(\varrho)}{q}, \tag{3.23}$$

multiplying both sides of (3.23) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi(\kappa)\right) \leq \frac{1}{p}\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa) + \frac{1}{q}\left(\mathbb{I}_a^{\zeta,\omega}\Psi^q\right)(\kappa), \tag{3.24}$$

using equation (3.20) and equation (3.22) in equation(3.24), we obtain

$$\left(\mathbb{I}_a^{\zeta,\omega}\Phi\Psi(\kappa)\right) \leq \frac{M^p}{p(M+1)^p}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p\right)(\kappa) + \frac{1}{q(m+1)^q}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^q\right)(\kappa), \tag{3.25}$$

now, using inequality $(x+y)^s \leq 2^{s-1}(x^s+y^s), \quad s>1, \quad x,y>0$, we have

$$\mathbb{I}_a^{\zeta,\omega}(\Phi\Psi)(\kappa) \leq \frac{2^{p-1}M^p}{p(M+1)^p}\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi^p+\Psi^p\right)(\kappa)\right) + \frac{2^{q-1}}{q(m+1)^q}\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi^q+\Psi^q\right)(\kappa)\right). \tag{3.26}$$

□

Theorem 3.5. *Let $\Phi,\Psi \in \mathbb{C}_{1,s}[a,\kappa]$, $s \in \mathbb{R}/\{0\}$ and $p,q \geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$, be two positive functions in $[0,\infty)$ s.t., for all $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ and $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. If $0 < c < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ for $c,m,M \in \mathbb{R}^+$ and for all $\varrho \in [a,\kappa]$, then*

$$\frac{M+1}{M-c}\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right)^{\frac{1}{p}} \leq \left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)^{\frac{1}{p}} + \left(\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right)^{\frac{1}{p}} < \frac{m+1}{m-c}\left(\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right)^{\frac{1}{p}}. \tag{3.27}$$

Proof. By applying the hypothesis, $0 < c < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$, we obtain the subsequent inequalities

$$(M+1)(m-c) \leq (m+1)(M-c), \tag{3.28}$$

or equivalent to

$$\frac{(M+1)}{(M-c)} \leq \frac{(m+1)}{(m-c)}, \tag{3.29}$$

further, we have

$$m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \quad (m-c) \leq \frac{\Phi(\varrho)-c\Psi(\varrho)}{\Psi(\varrho)} \leq (M-c),$$

which implies that

$$\frac{\Phi(\varrho)-c\Psi(\varrho)}{(m-c)} \geq \Psi(\varrho) \geq \frac{\Phi(\varrho)-c\Psi(\varrho)}{(M-c)},$$

or equivalent to

$$\frac{(\Phi(\varrho)-c\Psi(\varrho))^p}{(M-c)^p} \leq \Psi^p(\varrho) \leq \frac{(\Phi(\varrho)-c\Psi(\varrho))^p}{(m-c)^p}. \tag{3.30}$$

Again, we have $\frac{1}{M} \leq \frac{\Psi(\varrho)}{\Phi(\varrho)} \leq \frac{1}{m}$,

$$\frac{(m-c)}{cm} \leq \frac{\Phi(\varrho)-c\Psi(\varrho)}{c\Phi(\varrho)} \leq \frac{(M-c)}{cM},$$

or

$$\left(\frac{M}{M-c}\right)^p(\Phi(\varrho)-c\Psi(\varrho))^p \leq \Phi^p(\varrho) \leq \left(\frac{m}{m-c}\right)^p(\Phi(\varrho)-c\Psi(\varrho))^p. \tag{3.31}$$

Now, multiplying both sides of (3.30) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we obtain

$$\frac{1}{(M-c)^p}\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p \leq \mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) \leq \frac{1}{(m-c)^p}\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p,$$

or

$$\frac{1}{(M-c)}\left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right]^{\frac{1}{p}} \leq \left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}} \leq \frac{1}{(m-c)}\left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right]^{\frac{1}{p}}. \tag{3.32}$$

Now, multiplying both sides of (3.31) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ),we obtain

$$\frac{M}{(M-c)}\left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right]^{\frac{1}{p}} \leq \left[\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right]^{\frac{1}{p}} \leq \frac{m}{(m-c)}\left[\mathbb{I}_a^{\zeta,\omega}\left(\Phi(\kappa)-c\Psi(\kappa)\right)^p\right]^{\frac{1}{p}}. \tag{3.33}$$

Then, adding equation (3.32) and equation (3.33), we obtain desire result (3.27). □

Theorem 3.6. *Let $\Phi,\Psi \in \mathbb{C}_{1,s}[a,\kappa]$, $s \in \mathbb{R}/\{0\}$ and $p \geq 1$ be two positive functions in $[0,\infty)$ s.t., for all $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ and $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. If $0 \leq a \leq \Phi(\varrho) \leq A$ and $0 \leq b \leq \Psi(\varrho) \leq B$ for $a,b,A,B \in \mathbb{R}^+$ and for all $\varrho \in [a,\kappa]$, then*

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa)\right]^{\frac{1}{p}} + \left[\left(\mathbb{I}_a^{\zeta,\omega}\Psi^p\right)(\kappa)\right]^{\frac{1}{p}} \leq c_5\left[\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right]^{\frac{1}{p}}, \tag{3.34}$$

with $c_5 = \frac{A(a+B)+B(A+b)}{(A+b)(a+B)}$.

Proof. Under the given condition, it follows that $0 \leq b \leq \Psi(\varrho) \leq B$, or

$$\frac{1}{b} \geq \frac{1}{\Psi(\varrho)} \geq \frac{1}{B}, \tag{3.35}$$

conducting the product between (3.35) and $0 \leq a \leq \Phi(\varrho) \leq A$, we have

$$\frac{A}{b} \geq \frac{\Phi(\varrho)}{\Psi(\varrho)} \geq \frac{a}{B}, \tag{3.36}$$

then

$$a\Psi(\varrho) \leq B\Phi(\varrho),$$

or equivalent to

$$(a+B)\Psi(\varrho) \leq B\left(\Phi(\varrho)+\Psi(\varrho)\right),$$

implies that,

$$\Psi^p(\varrho) \leq \left(\frac{B}{a+B}\right)^p \left(\Phi(\varrho)+\Psi(\varrho)\right)^p, \tag{3.37}$$

and

$$0 \leq a \leq \Phi(\varrho) \leq A,$$

or

$$\frac{1}{a} \geq \frac{1}{\Phi(\varrho)} \geq \frac{1}{A}, \tag{3.38}$$

conducting the product between (3.38) and $0 \leq b \leq \Psi(\varrho) \leq B$, we have

$$\frac{b}{A} \leq \frac{\Psi(\varrho)}{\Phi(\varrho)} \leq \frac{B}{a}, \tag{3.39}$$

then

$$b\Phi(\varrho) \leq A\Psi(\varrho),$$

or equivalent to

$$(b+A)\Phi(\varrho) \leq A\left(\Phi(\varrho)+\Psi(\varrho)\right),$$

implies that,

$$\Phi^p(\varrho) \leq \left(\frac{A}{b+A}\right)^p \left(\Phi(\varrho)+\Psi(\varrho)\right)^p, \tag{3.40}$$

now, multiplying both sides of (3.37) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ),we can written as

$$\left[\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa)\right]^{\frac{1}{p}} \leq \left(\frac{B}{a+B}\right)\left[\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.41}$$

Also, multiplying both sides of (3.40) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we can written as

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa)\right)\right]^{\frac{1}{p}} \leq \left(\frac{A}{b+A}\right)\left[\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.42}$$

Finally, adding equation (3.41) and equation (3.42), we obtain desire result.

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa)\right]^{\frac{1}{p}} + \left[\left(\mathbb{I}_a^{\zeta,\omega}\Psi^p\right)(\kappa)\right]^{\frac{1}{p}} \leq \frac{A(a+B)+B(A+b)}{(A+b)(a+B)}\left[\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.43}$$

□

Theorem 3.7. *Let $\Phi,\Psi \in \mathbb{C}_{1,s}[a,\kappa]$, $s \in \mathbb{R}/\{0\}$ and $p \geq 1$ be two positive functions in $[0,\infty)$ s.t., for all $\kappa > a$, $\mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ and $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. If $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $\varrho \in [a,\kappa]$, then*

$$\frac{1}{M}\left(\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\Psi(\kappa)\right) \leq \frac{1}{(m+1)(M+1)}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi+\Psi)^2(\kappa)\right) \leq \frac{1}{m}\left(\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\Psi(\kappa)\right). \tag{3.44}$$

Proof. Using the condition $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$, it follows that

$$m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M,$$

or

$$(m+1)\Psi(\varrho) \leq \Phi(\varrho)+\Psi(\varrho) \leq (M+1)\Psi(\varrho), \tag{3.45}$$

also, it follows that

$$\frac{1}{M} \leq \frac{\Psi(\varrho)}{\Phi(\varrho)} \leq \frac{1}{m},$$

or

$$\left(\frac{M+1}{M}\right)\Phi(\varrho) \leq \Phi(\varrho)+\Psi(\varrho) \leq \left(\frac{m+1}{m}\right)\Phi(\varrho), \tag{3.46}$$

conducting the product between equation (3.45) and equation (3.46), we have

$$\frac{\Phi(\varrho))\Psi(\varrho)}{M} \leq \frac{(\Phi(\varrho)+\Psi(\varrho))^2}{(M+1)(m+1)} \leq \frac{\Phi(\varrho)\Psi(\varrho)}{m}, \tag{3.47}$$

now, multiplying both sides of (3.47) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ),we can written as

$$\frac{1}{M}\left(\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\Psi(\kappa)\right) \leq \frac{1}{(m+1)(M+1)}\left(\mathbb{I}_a^{\zeta,\omega}(\Phi(\kappa)+\Psi(\kappa))^2\right) \leq \frac{1}{m}\left(\mathbb{I}_a^{\zeta,\omega}\Phi(\kappa)\Psi(\kappa)\right). \tag{3.48}$$

□

Theorem 3.8. *: Let $\Phi,\Psi \in \mathbb{C}_{1,s}[a,\kappa], s \in \mathbb{R}/\{0\}$ and $p \geq 1$ be two positive functions in $[0,\infty)$ s.t., for all $\kappa > a, \mathbb{I}_a^{\zeta,\omega}\Phi^p(\kappa) < \infty$ and $\mathbb{I}_a^{\zeta,\omega}\Psi^p(\kappa) < \infty$. If $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $\varrho \in [a,\kappa]$, then*

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa)\right]^{\frac{1}{p}} + \left[\left(\mathbb{I}_a^{\zeta,\omega}\Psi^p\right)(\kappa)\right]^{\frac{1}{p}} \leq 2\left[\mathbb{I}_a^{\zeta,\omega}h^p(\Phi\Psi)^p(\kappa)\right]^{\frac{1}{p}}, \tag{3.49}$$

where

$$h\left[\Phi(\kappa),\Psi(\kappa)\right] = max\left[\left(\frac{M}{m}+1\right)\Phi(\kappa) - M\Psi(\kappa), \frac{(m+M)\Psi(\kappa)-\Phi(\kappa)}{m}\right]. \tag{3.50}$$

Proof. Under the given condition $0 < m \leq \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \quad \varrho \in (a,\kappa)$

It can be written as

$$0 < m \leq M + m - \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \tag{3.51}$$

and

$$M + m - \frac{\Phi(\varrho)}{\Psi(\varrho)} \leq M, \tag{3.52}$$

from equation(3.51) and equation (3.52), we obtain

$$\Psi(\varrho) < \frac{(M+m)\Psi(\varrho)-\Phi(\varrho)}{m} \leq h\left(\Phi(\varrho),\Psi(\varrho)\right), \tag{3.53}$$

where $h\left[\Phi(\varrho),\Psi(\varrho)\right] = \max\left[\left(\frac{M}{m}+1\right)\Phi(\varrho) - M\Psi(\varrho), \frac{(m+M)\Psi(\varrho)-\Phi(\varrho)}{m}\right]$.

On the other hand, from the hypothesis, it also follows that

$$0 < \frac{1}{M} \leq \frac{\Psi(\varrho)}{\Phi(\varrho)} \leq \frac{1}{m}, \tag{3.54}$$

then

$$\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{\Psi(\varrho)}{\Phi(\varrho)}, \tag{3.55}$$

and

$$\frac{1}{M} + \frac{1}{m} - \frac{\Psi(\varrho)}{\Phi(\varrho)} \leq \frac{1}{m}, \tag{3.56}$$

from equation (3.55) and equation (3.56), which implies that

$$\frac{1}{M} \leq \frac{\left(\frac{1}{M}+\frac{1}{m}\right)\Phi(\varrho) - \Psi(\varrho)}{\Phi(\varrho)} \leq \frac{1}{m}, \tag{3.57}$$

or

$$\Phi(\varrho) \leq M\left(\frac{1}{M}+\frac{1}{m}\right)\Phi(\varrho) - M\Psi(\varrho),$$

$$= \frac{M(M+m)\Phi(\varrho) - \Psi(\varrho)M^2m}{mM},$$

or

$$\Phi(\varrho) \leq \left(\frac{M}{m}+1\right)\Phi(\varrho) - M\Psi(\varrho),$$

thus

$$\Phi(\varrho) \leq h\left[\Phi(\varrho)\Psi(\varrho)\right], \tag{3.58}$$

from equation (3.53) and equation (3.58) we can write

$$\Psi^p(\varrho) \leq h^p\left[\Phi(\varrho),\Psi(\varrho)\right], \tag{3.59}$$

$$\Phi^p(\varrho) \leq h^p\left[\Phi(\varrho),\Psi(\varrho)\right], \tag{3.60}$$

now, multiplying both sides of (3.59) and (3.60) by $\frac{1}{\Gamma(\zeta)}e^{-\omega(\kappa-\varrho)}(\kappa-\varrho)^{\zeta-1}$, then we integrate the resulting inequality with respect to ϱ over (a,κ), we can written as

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Psi^p\right)(\kappa)\right]^{\frac{1}{p}} \leq \left[\mathbb{I}_a^{\zeta,\omega}h^p(\Phi,\Psi)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.61}$$

$$\left[\left(\mathbb{I}_a^{\zeta,\omega}\Phi^p\right)(\kappa)\right]^{\frac{1}{p}} \leq \left[\mathbb{I}_a^{\zeta,\omega}h^p(\Phi,\Psi)^p(\kappa)\right]^{\frac{1}{p}}. \tag{3.62}$$

Now adding equation (3.61) and equation (3.62), we get desire result given in equation (3.49).

4 Applications

Let $u(x)$ and $v(x)$ be two integrable functions which are synchronous on $[a,b]$, then the theorem 2.1 and theorem 2.2 hold the following results such that

$$u(x) = e^{(x-a)} = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!},$$

and

$$v(x) = [1+(x-a)]^n = \sum_{r=0}^{\infty} \frac{(n)_r (x-a)^r}{r!},$$

therefore

$$\mathbb{I}_a^{\zeta,\omega} u(\kappa) = \mathbb{I}_a^{\zeta,\omega}\left(e^{(\kappa-a)}\right) = \frac{1}{\Gamma(\zeta)} \int_a^{\kappa} e^{-\omega(\kappa-\varrho)} (\kappa-\varrho)^{\zeta-1} \left\{ \sum_{r=0}^{\infty} \frac{(\varrho-a)^r}{r!} \right\} d\varrho,$$

now changing the summation and order of integration, we get

$$= \frac{1}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{1}{r!} \int_a^{\kappa} e^{-\omega((\kappa-a)-(\varrho-a))} ((\kappa-a)-(\varrho-a))^{\zeta-1} (\varrho-a)^r d\varrho,$$

$$= e^{-\omega(\kappa-a)} \frac{1}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{1}{r!} \int_a^{\kappa} \sum_{s=0}^{\infty} \frac{(\omega(\varrho-a))^s}{s!} ((\kappa-a)-(\varrho-a))^{\zeta-1} (\varrho-a)^r d\varrho,$$

$$= \frac{e^{-\omega(\kappa-a)}}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{s=0}^{\infty} \frac{\omega^s}{s!} \int_a^{\kappa} (\varrho-a)^{r+s} ((\kappa-a)-(\varrho-a))^{\zeta-1}) d\varrho,$$

setting $t = \frac{\varrho-a}{\kappa-a}$, we obtain

$$= \frac{e^{-\omega(\kappa-a)}}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{s=0}^{\infty} \frac{\omega^s}{s!} (\kappa-a)^{r+s+\zeta} \int_a^{\kappa} t^{r+s} (1-t)^{\zeta-1} dt,$$

$$\mathbb{I}_a^{\zeta,\omega}\left(e^{(\kappa-a)}\right) = e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\omega^s}{r!s!} (\kappa-a)^{r+s+\zeta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\zeta)}. \tag{4.1}$$

Also

$$\mathbb{I}_a^{\zeta,\omega} v(\kappa) = \mathbb{I}_a^{\zeta,\omega} [1+(\kappa-a)]^n = \frac{1}{\Gamma(\zeta)} \int_a^{\kappa} e^{-\omega(\kappa-\varrho)} (\kappa-\varrho)^{\zeta-1} \left\{ \sum_{r=0}^{\infty} \frac{(n)_r (\varrho-a)^r}{r!} \right\} d\varrho,$$

now changing the summation and order of integration, we get

$$= \frac{1}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \int_a^{\kappa} e^{-\omega((\kappa-a)-(\varrho-a))} ((\kappa-a)-(\varrho-a))^{\zeta-1} (\varrho-a)^r d\varrho,$$

$$= e^{-\omega(\kappa-a)} \frac{1}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \int_a^{\kappa} \sum_{s=0}^{\infty} \frac{(\omega(\varrho-a))^s}{s!} ((\kappa-a)-(\varrho-a))^{\zeta-1} (\varrho-a)^r d\varrho,$$

$$= \frac{e^{-\omega(\kappa-a)}}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \sum_{s=0}^{\infty} \frac{\omega^s}{s!} \int_a^{\kappa} (\varrho-a)^{r+s} ((\kappa-a)-(\varrho-a))^{\zeta-1}) d\varrho,$$

setting $z = \frac{\varrho-a}{\kappa-a}$, we obtain

$$= \frac{e^{-\omega(\kappa-a)}}{\Gamma(\zeta)} \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \sum_{s=0}^{\infty} \frac{\omega^s}{s!} (\kappa-a)^{r+s+\zeta} \int_a^{\kappa} z^{r+s} (1-z)^{\zeta-1} dz,$$

$$\mathbb{I}_a^{\zeta,\omega} [1+(x-a)]^n = e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(n)_r \omega^s}{r!s!} (\kappa-a)^{r+s+\zeta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\zeta)}. \tag{4.2}$$

□

Proposition 4.1. *Let $u(\kappa) = e^{(\kappa-a)}$and $v(\kappa) = [1+(\kappa-a)]^n$ two integrable functions which are synchronous on $[a,\infty)$. Then the theorem 2.1 holds inequality for all $\kappa \in [a,b]$ and $\zeta, \omega \in \mathbb{C}$ with $\Re(\zeta) > 0$ and $\Re(\omega) > 0$:*

$$\left(\mathbb{I}_a^{\zeta,\omega} uv\right)(\kappa) \geq [\left(\mathbb{I}_a^{\zeta,\omega}(1)\right)]^{-1} \left\{ e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\omega^s}{r!s!} (\kappa-a)^{r+s+\zeta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\zeta)} \right\}^2 (n)_r. \tag{4.3}$$

Proposition 4.2. *Let $u(\kappa) = e^{(\kappa-a)}$and $v(\kappa) = [1+(\kappa-a)]^n$ be two integrable functions which are synchronous on $[a,\infty)$. Then the theorem 2.2 holds inequality for all $\kappa \in [a,b]$ and $\zeta, \eta, \omega \in \mathbb{C}$ with $\Re(\zeta) > 0, \Re(\eta) > 0$ and $\Re(\omega) > 0$:*

$$\left(\mathbb{I}_a^{\zeta,\omega} uv\right)(\kappa) \left(\mathbb{I}_a^{\eta,\omega}(1)\right)(\kappa) + \left(\mathbb{I}_a^{\zeta,\omega}(1)\right)(\kappa) \left(\mathbb{I}_a^{\eta,\omega} uv\right)(\kappa) \geq e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\omega^s}{r!s!} (\kappa-a)^{r+s+\zeta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\zeta)}$$
$$\times e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(n)_r \omega^s}{r!s!} (\kappa-a)^{r+s+\eta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\eta)} + e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(n)_r \omega^s}{r!s!} (\kappa-a)^{r+s+\zeta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\zeta)}$$
$$\times e^{-\omega(\kappa-a)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\omega^s}{r!s!} (\kappa-a)^{r+s+\eta} \frac{\Gamma(r+s+1)}{\Gamma(r+s+\eta)}. \tag{4.4}$$

4.1 Graphical representations

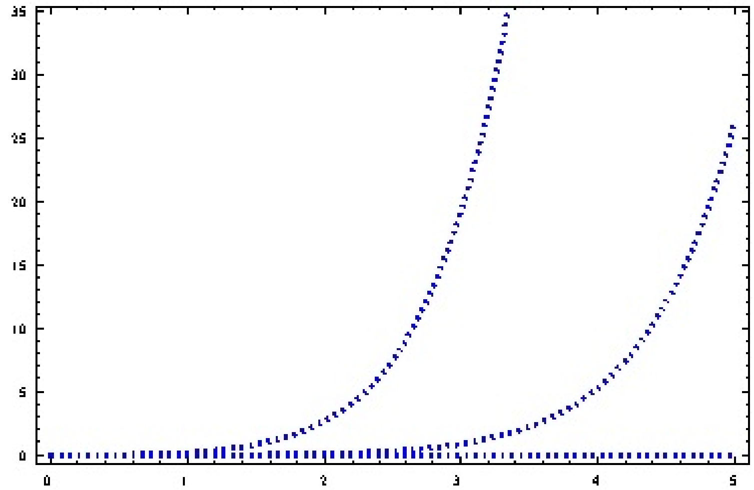

Figure 1. Curve between κ and $\mathbb{I}_a^{w,p}(e^{(\kappa-a)})\mathbb{I}_a^{w,p}[1+(\kappa-a)]^n(\mathbb{I}_a^{w,p}[1])^{-1}$ with different value of p with $a=1, n=2$.

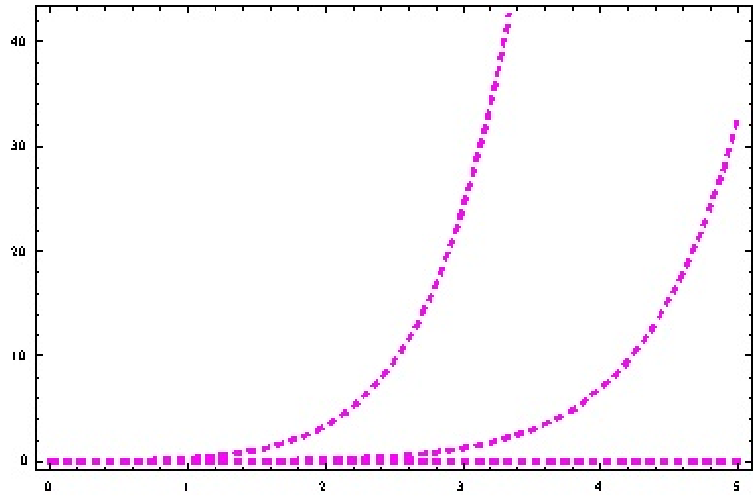

Figure 2. Curve between κ and $\mathbb{I}_a^{w,p}(e^{(\kappa-a)}[1+(\kappa-a)]^n)$ with different value of p with $a=1, n=2$.

4.2 Result and discussion

In this paper,we present result of theorem 2.1 graphically in which figure 1 shows right hand side of equation (2.1), whereas figure 2 shows left hand side of equation (2.1). In figure 3 we show theorem 2.1 in combined form of figure 1 and 2 with fixed value of $a = 1$ and $n = 2$. The figure 3 verifies the inequality since for a given value of κ, the left hand side of inequality always greater than equal to right hand side whereas for small value of κ equality holds.

5 Conclusion remarks

In this current paper, we introduced Chebshev inequality by using the tempered fractional integral operator as a particular case, $\omega = 0$ and $a = 0$ then the inequality (2.1), (2.10), (2.12), and (2.15) involving fractional integral will leads to R-L fractional integral operator defined as Belarbi and Dahamani et al. [4]. Some applications will be discussed in the propositions 4.1 and 4.2. Also important inequalities Minkowski inequality involving tempered fractional integral operator generalized the reverse Minkowaski inequality and some important relation.

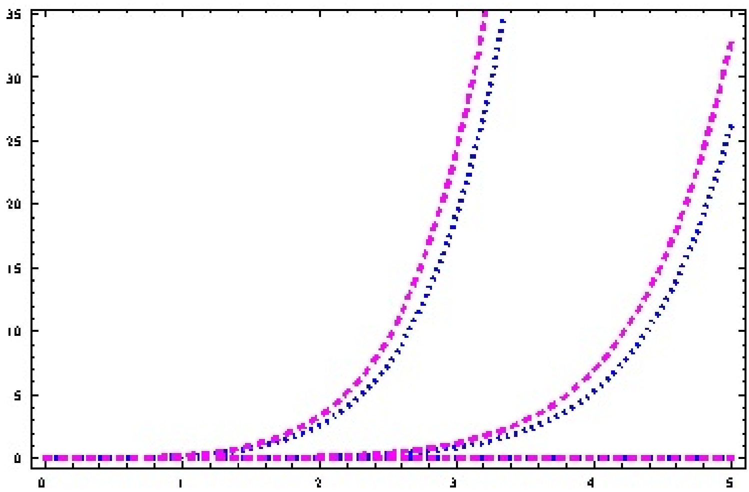

Figure 3. Comparison between $\mathbb{I}_a^{w,p}(e^{(\kappa-a)})\mathbb{I}_a^{w,p}[1+(\kappa-a)]^n(\mathbb{I}_a^{w,p}[1])^{-1}$ and $\mathbb{I}_a^{w,p}(e^{(\kappa-a)}[1+(\kappa-a)]^n)$.

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