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COMPLETELY SEPARATED TOPOLOGICAL GRAPH

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Abstract In this paper, we introduce a graph structure $G(\tau)$ on a topological space (X, τ), called completely separated topological graph. We show that (X, τ) is a discrete space if and only if $X \backslash \{p\}$ is a pendant vertex of $G(\tau)$ for all $p \in X$ and $G(\tau)$ is an edgeless graph if and only if every real-valued continuous function on X is constant. Also, if (X, τ_X) and (Y, τ_Y) are discrete topological spaces, then X and Y are homeomorphic if and only if $G(\tau_X)$ is isomorphic to $G(\tau_Y)$. Finally, for a discrete space (X, τ), $G(\tau)$ is planar if and only if $|X| \leq 3$.

1 Introduction

Graph theory apart from its combinatorial implication also helps to characterize various algebraic structures by studying graphs associated with them. Some researchers studied graphs associated with algebraic structures to obtain the interrelationships between graph properties and algebraic properties. Beck [1] initiated this idea. Even though his goal was to address the issues of coloring, his formal study eventually exposed the relationship between graph theory and algebra and advancing applications of one to the other.

Till now, a lot of research has been done in studying graphs defined on an algebraic structure such as groups, rings, vector space, modules, etc. In 2019, Muneshwar and Bondar [2] introduced the open subset inclusion graph $\gamma(\tau)$ on a finite topological space (X, τ), where the vertex set is the collection of proper open subsets of a topological space (X, τ) and two vertices A and B are adjacent if either $A \subset B$ or $B \subset A$. Again in the same year, Muneshwar and Bondar [3] studied another graph structure on a finite topological space (X, τ), called the intersection graph, where the vertex set is the collection of proper open subsets of X and two vertices A and B are adjacent if and only if $A \bigcap B \neq \phi$. Muneshwar and Bondar [2] [3] studied the graph properties of these graphs like connectedness, clique number, dominating number, etc.

In this paper, we introduce a graph defined on an arbitrary topological space (X, τ), called completely separated topological graph $G(\tau)$. The main objective of this paper is to not only study the graph properties of $G(\tau)$ but to investigate the relationships between topological properties of (X, τ) and graph-theoretic properties of $G(\tau)$. We show that for a T_1 topological space (X, τ), p is an isolated point of X if and only if $X \backslash \{p\}$ is a pendant vertex of $G(\tau)$ and (X, τ) is a discrete space if and only if $X \backslash \{p\}$ is a pendant vertex of $G(\tau)$ for all $p \in X$. We also show that if (X, τ_X) and (Y, τ_Y) are finite T_1 (or T_2) topological spaces, then X and Y are homeomorphic if and only if $G(\tau_X)$ is isomorphic to $G(\tau_Y)$. Also, for any topological space (X, τ) such that $|\tau|$ is finite, the dominating number of $G(\tau)$ is $|\tau| - 2$ if and only if every real-valued continuous function on X is constant.

For any two vertices A and B of a graph G, the distance $d(A, B)$ is the length of the shortest path between A and B. The diameter of a graph G is defined as $Diam(G) = sup\{d(A, B) : A,\ B \in V(G)\}$. For any two vertices A, B, $c(A, B)$ is the length of the shortest cycle containing both A and B. The girth of a graph is the length of its shortest cycle if it exists otherwise girth is ∞. A graph in which all vertices are pairwise adjacent is called a complete graph. The clique number of a graph G is defined as $\omega(G) = sup\{|V(H)| : H\ is\ a\ complete\ subgraph\ of\ G\}$, where $V(H)$ is the vertex set of H. A graph G is said to be triangulated (hypertriangulated) if each vertex (edge) is a vertex (an edge) of a triangle. The collection of all vertices in G adjacent to A is called an open neighbourhood of A. It is denoted by $N(A)$ and the cardinality of $N(A)$ is equal to the degree of the vertex A. The closed neighbourhood of A is the set $N(A) \bigcup \{A\}$ and is denoted by $N[A]$. We denote the set $\mathbb{R}$ equipped with usual topology by $\mathbb{R}_u$. Also, a topological space (X, τ) in which every member of τ is both open and closed, is called a clopen topological space [4].

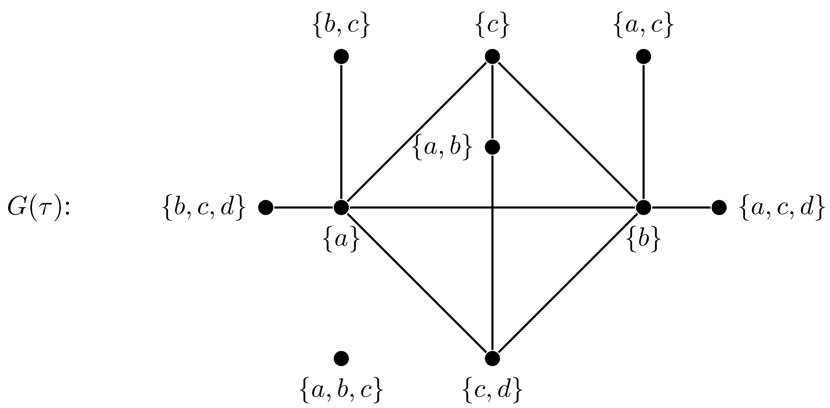
For undefined terms concerning topology and graph theory, we refer the reader to [5] and [6] respectively.

2 The completely separated topological graph

Definition 2.1. Let (X, τ) be any topological space. The completely separated topological graph denoted by $G(\tau)$ is a graph whose vertex set is the collection of all proper open subsets of X with two vertices $A,\ B$ adjacent if and only if there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, $r,\ s \in \mathbb{R}$. Also, $f(A) \leq r$ means $f(a) \leq r$ for every $a \in A$ and $f(B) \geq s$ means $f(b) \geq s$ for every $b \in B$.

Remark 2.2. If (X, τ) is an indiscrete space, then it has no proper open subsets. Hence, we will consider (X, τ) as a non-indiscrete topological space and $|X| > 1$. It is easy to see that a topological space (X, τ) is discrete if and only if $\{p\}$ is a vertex of $G(\tau)$, for all $p \in X$. Also, (X, τ) is T_1 if and only if $X \backslash \{p\}$ is a vertex in $G(\tau)$, for all $p \in X$.

Example 2.3. Consider a topological space (X, τ), where $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then, it is easy to see that the characteristic functions of $\{a\},\ \{b\},\ \{c, d\}$ are continuous. The graph $G(\tau)$ is shown in $Fig(i)$. From $Fig(i)$, it is clear that $G(\tau)$ is not a connected graph. Hence, $G(\tau)$ may not be a connected graph for arbitrary space (X, τ). However, in section 3, we will show that for a discrete space (X, τ), $G(\tau)$ is a connected graph.



$Fig(i)$

Theorem 2.4. *Let (X, τ) be any topological space and $A,\ B$ be any two vertices in $G(\tau)$. If $A,\ B$ are adjacent, then $A \bigcap B = \phi$.*

Proof. Suppose there exists $x \in A \bigcap B$. As $A,\ B$ are adjacent there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, $r,\ s \in \mathbb{R}$. This implies that $f(x) \leq r$ and $f(x) \geq s$ which is a contradiction as $r < s$. Hence, $A \bigcap B = \phi$. □

Remark 2.5. The converse of Theorem 2.4 is not generally true. For if $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, then, it is easy to see that every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant. Here, $A = \{a\}$ and $B = \{b\}$ are disjoint but they are not adjacent.

Theorem 2.6. *Let (X, τ) be any topological space. If $G(\tau)$ is a complete graph, then $G(\tau)$ is either K_1 or K_2, where K_n is a complete graph with n vertices.*

Proof. If (X, τ) has only one proper open subset of X, then $G(\tau)$ has only one vertex. So, $G(\tau) = K_1$. Suppose (X, τ) has more than one proper open subset. Let A and B be two distinct proper open subsets of X. As $G(\tau)$ is complete, by Theorem 2.4, $A \bigcap B = \phi$. Suppose there exists another proper open subset C of X. Then, by Theorem 2.4, $C \bigcap A = \phi$ and $C \bigcap B = \phi$. This shows that $C \bigcap (A \bigcup B) = (C \bigcap A) \bigcup (C \bigcap B) = \phi$. If $A \bigcup B = X$, then this will contradict that C is a proper subsets of X as $C \bigcap (A \bigcup B) = \phi$. If $A \bigcup B \neq X$, then $A \bigcup B$ is also a proper open subset of X. But this contradicts that $G(\tau)$ is complete as $A \bigcup B$ is not adjacent to A and B. This shows that $G(\tau)$ has only two vertices A and B. Hence, $G(\tau) = K_2$. □

Corollary 2.7. *Let (X, τ) be a discrete topological space. Then, $G(\tau)$ is a complete graph if and only if $|X| = 2$.*

Theorem 2.8. *Let (X, τ) be a T_1 topological space and $p \in X$. Then, p is an isolated point of X if and only if $X \backslash \{p\}$ is a pendant vertex of $G(\tau)$.*

Proof. Suppose $p \in X$ is an isolated point. As X is also T_1 there exists a continuous function that separates $\{p\}$ and $X \backslash \{p\}$. Suppose there exists another vertex $A \neq \{p\}$ which is adjacent to $X \backslash \{p\}$. This implies that $A \bigcap (X \backslash \{p\}) = \phi$, which is not possible as $A \neq \{p\}$. Hence, $X \backslash \{p\}$ is a pendant vertex. Conversely, suppose $X \backslash \{p\}$ is a pendant vertex. Then, there exists only one vertex A which is adjacent to $X \backslash \{p\}$. This implies that $A \bigcap (X \backslash \{p\}) = \phi$. This shows that $A = \{p\}$. So, p is an isolated point of X as A is open. □

Corollary 2.9. *(X, τ) is a discrete space if and only if $X \backslash \{p\}$ is a pendant vertex of $G(\tau)$ for all $p \in X$.*

Theorem 2.10. *Let (X, τ) be any topological space. Then, $G(\tau)$ is an edgeless graph if and only if every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Proof. First, we assume that $G(\tau)$ is an edgeless graph. Suppose there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ which is not constant. Then, there exists $x,\ y \in X$ such that $f(x) = r$ and $f(y) = s$ for some distinct $r,\ s \in \mathbb{R}$. Without loss of generality, we assume $r < s$. Fix a point $q \in (r, s)$. As f is continuous, $f^{-1}((-\infty, q))$ and $f^{-1}((q, \infty))$ are non-trivial open subsets of X and are adjacent in $G(\tau)$. This contradicts the fact that $G(\tau)$ is an edgeless graph. The converse part is trivial. □

Example 2.11. Consider $X = \mathbb{R}$ with the co-finite topology τ. It is known that every continuous function from X to $\mathbb{R}_u$ is constant. So, by Theorem 2.10, $G(\tau)$ is an edgeless graph.

Remark 2.12. If (X, τ) is a hyperconnected space [7], i.e., a space in which no two non-empty open sets are disjoint, then $G(\tau)$ is an edgeless graph. The proof follows from the fact that every continuous function from a hyperconnected space to a T_2 space is constant.

Theorem 2.13. *Let (X, τ) be any topological space such that there exists a proper clopen subset of X. Then, $G(\tau)$ is not an edgeless graph.*

Proof. Suppose A is a proper clopen subset of X, then the characteristic function of A is continuous. This implies that A and $X \backslash A$ are adjacent in $G(\tau)$. This shows that $G(\tau)$ is not an edgeless graph. □

Corollary 2.14. *If (X, τ) is a disconnected topological space, then $G(\tau)$ is not an edgeless graph.*

Remark 2.15. The converse of the claim in Corollary 2.14 is not true. Consider $X = \mathbb{R}_u$. It requires a little effort to see that the completely separated topological graph is not edgeless but X is a connected space.

Theorem 2.16. *Let (X, τ_X) and (Y, τ_Y) be any two topological spaces such that X and Y are homeomorphic. Then, $G(\tau_X)$ is isomorphic to $G(\tau_Y)$.*

Proof. Suppose $h : X \longrightarrow Y$ is a homeomorphic map. Consider the map $\overline{h} : V(G(\tau_X)) \longrightarrow V(G(\tau_Y))$ define by $\overline{h}(A) = h(A)$, $\forall\ A \in V(G(\tau_X))$. Clearly, $\overline{h}$ is bijective as h is bijective. Next, we show that $\overline{h}$ preserves adjacency. Suppose A and B are adjacent in $G(\tau_X)$. Then, there exists a continuous map $f : X \longrightarrow \mathbb{R}_u$ such that $f(A) \leq r < s \leq f(B)$, for some $r,\ s \in \mathbb{R}$. Since h is a homeomorphism, $\overline{h}(A)$ and $\overline{h}(B)$ are open in Y. Clearly, $f \circ h^{-1} : Y \longrightarrow \mathbb{R}_u$ is continuous and $f \circ h^{-1}(\overline{h}(A)) = f(A) \leq r < s \leq f(B) = f \circ h^{-1}(\overline{h}(B))$. This show that $\overline{h}(A)$ and $\overline{h}(B)$ are adjacent in $G(\tau_Y)$. Conversely, suppose $\overline{A}$ and $\overline{B}$ are adjacent in $G(\tau_Y)$. As h is a homeomorphic map, we have $\overline{A} = h(A)$ and $\overline{B} = h(B)$ for some $A,\ B \in V(G(\tau_X))$. We need to show that A and B are adjacent in $G(\tau_X)$. As $\overline{A} = h(A)$ and $\overline{B} = h(B)$ are adjacent, there exists a continuous map $g : Y \longrightarrow \mathbb{R}_u$ such that $g(\overline{A} = h(A)) \leq r < s \leq g(\overline{B} = h(B))$, $r,\ s \in \mathbb{R}$. Clearly, $g \circ h : X \longrightarrow \mathbb{R}_u$ is continuous and $g \circ h(A) = g(h(A)) = g(\overline{A}) \leq r < s \leq g(\overline{B}) = g(h(B)) = g \circ h(B)$. This shows that A and B are adjacent in $G(\tau_X)$. □

Remark 2.17. The converse of Theorem 2.16 is not generally true. Consider $X = \{a, b\}$ with $\tau_X = \{\phi, \{a\}, \{b\}, X\}$ and $Y = \{a, b, c, d\}$ with $\tau_Y = \{\phi, \{a, b\}, \{c, d\}, X\}$. Then, it is apparent that both $G(\tau_X)$ and $G(\tau_Y)$ are isomorphic to the complete graph K_2. This shows that $G(\tau_X)$ and $G(\tau_Y)$ are isomorphic to each other. But (X, τ_X) and (Y, τ_Y) are not homeomorphic.

Corollary 2.18. *Let (X, τ_X) and (Y, τ_Y) be two $T_1(or\ T_2)$ finite topological spaces. Then, X and Y are homeomorphic if and only if $G(\tau_X)$ is isomorphic to $G(\tau_Y)$.*

Proof. Suppose $G(\tau_X)$ is isomorphic to $G(\tau_Y)$. It is sufficient to prove $|X| = |Y|$. As X and Y are finite, X and Y are discrete spaces. If $|X| \neq |Y|$, then the number of vertices of $G(\tau_X)$ is not equal to the number of vertices of $G(\tau_Y)$, which contradict the fact that $G(\tau_X)$ is isomorphic to $G(\tau_Y)$. This shows that $|X| = |Y|$. As X and Y are discrete spaces, this shows that X and Y are homeomorphic. □

Remark 2.19. Corollary 2.18 is still true if X and Y are infinite discrete topological spaces. However, for an infinite $T_1(or\ T_2)$ non-discrete topological space, it is not known if the converse part of the Corollary 2.18 holds.

Theorem 2.20. *For any topological space (X, τ), $G(\tau)$ is never a cycle C_n, $\forall\ n \geq 3$.*

Proof. **Case I:** We will first show that $G(\tau)$ is not C_3. Suppose $G(\tau)$ is C_3, then $G(\tau)$ is a complete graph. By Theorem 2.6, $G(\tau)$ is either K_1 or K_2, which is a contradiction. Hence, $G(\tau)$ is not C_3.
Case II: Suppose $G(\tau)$ is a cycle C_4 say $A - B - C - D - A$. By Theorem 2.4, $A \bigcap B = \phi$ and $A \bigcap D = \phi$. This implies that $A \bigcap (B \bigcup D) = (A \bigcap B) \bigcup (A \bigcap D) = \phi$. This shows that $B \bigcup D$ is a proper open subset of X. If $B \bigcup D$ is different from B, D, and C, then this will contradict the fact that $G(\tau)$ has only four vertices. If $B \bigcup D = C$, then this will contradict the fact that C is adjacent to both B and D. So, either $B \bigcup D = B$ or $B \bigcup D = D$ which gives D is a subset of B or B is a subset of D. Similarly, we have either $A \bigcup C = A$ or $A \bigcup C = C$, then C is a subset of A or A is a subset of C. We have the following two subcases:
subcase i: If $B \bigcup D = B$, then either $A \bigcup C = A$ or $A \bigcup C = C$. Consider the case $B \bigcup D = B$ and $A \bigcup C = A$, i.e., D and C are proper subset of B and A respectively. This implies that $D \bigcup C$ is a proper open subset of X as $B \bigcap C = \phi$. It is easy to see that $D \bigcup C$ is an open set different from A, B, C, and D. This contradicts that $G(\tau)$ has only four vertices. Similarly if $B \bigcup D = B$ and $A \bigcup C = C$, then we get $D \bigcup A$ is a proper open subset of X, which is a contradiction.
subcase ii: : If $B \bigcup D = D$, then either $A \bigcup C = A$ or $A \bigcup C = C$. In the same ways as in **subcase i**, we will get a contradiction. Hence, $G(\tau)$ is not C_4 for any topological space (X, τ).
Case III: Suppose $G(\tau)$ is a cycle C_n for $n \geq 5$. Then, there exist four vertices A, B, C, D (say) such that $-A - B - C - D-$ is part of the cycle C_n. By Theorem 2.4, $B \bigcap A = \phi$ and $B \bigcap C = \phi$. This implies that $B \bigcap (A \bigcup C) = (B \bigcap A) \bigcup (B \bigcap C) = \phi$. This shows that $A \bigcup C$ is a proper open subset of X. Clearly, $A \bigcup C$ is different from B and D as C is adjacent to both B and D. So, we have either $A \bigcup C = A$ or $A \bigcup C = C$ or $A \bigcup C$ is a new vertex different from A, B, C, D. If $A \bigcup C$ is a new vertex different from A, B, C, D, then as it is not adjacent to A there exists another vertex N which is adjacent to $A \bigcup C$. So, there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(N) \leq r < s \leq f(A \bigcup C)$. This shows that $f(N) \leq r < s \leq f(A)$

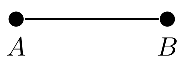
and $f(N) \leq r < s \leq f(C)$. This implies that N is adjacent to A, C, and $A \bigcup C$, which is not possible as $G(\tau)$ is a cycle. So, we have either $A \bigcup C = A$ or $A \bigcup C = C$. Similarly, we have $B \bigcup D = B$ or $B \bigcup D = D$. If $A \bigcup C = A$ and $B \bigcup D = B$, then in the same way as in case II, we have $C \bigcup D$ is a proper open set and is different from A, B, C, and D. So, $C \bigcup D$ is also a vertex of $G(\tau)$. As $G(\tau)$ is a cycle there exists vertex K which is adjacent to $C \bigcup D$. So, there exists a continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(K) \leq r < s \leq f(C \bigcup D)$. This shows that $f(K) \leq r < s \leq f(C)$ and $f(K) \leq r < s \leq f(D)$. This implies that K is adjacent to C, D, and $C \bigcup D$, which is not possible as K, C, D, and $C \bigcup D$ are vertices on the cycle C_n. Similarly, if $A \bigcup C = A$ and $B \bigcup D = D$, we get a contradiction. Hence, $G(\tau)$ is never a cycle C_n. □

3 Properties of $G(\tau)$ for discrete space

In this section, we discuss properties of $G(\tau)$ when τ is a discrete topology. In a discrete space (X, τ), every function from X to $\mathbb{R}_u$ is continuous. So, to prove two subsets are adjacent, it is enough to show that they are disjoint.

When $X = \{a, b\}$, then vertices of $G(\tau)$ are $A = \{a\}$ and $B = \{b\}$ and the graph $G(\tau)$ is as shown in $Fig(ii)$.

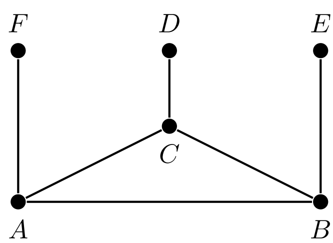
$G(\tau)$:



$Fig(ii)$

When $X = \{a, b, c\}$, then vertices are $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a, b\}$, $E = \{a, c\}$, $F = \{b, c\}$ and the graph $G(\tau)$ is as shown in $Fig(iii)$.

$G(\tau)$:



$Fig(iii)$

Theorem 3.1. *Let (X, τ) be a discrete topological space and A, B be any two proper open subsets of X. Then,*
(i) $d(A, B) = 1$ if and only if $A \bigcap B = \phi$.
(ii) $d(A, B) = 2$ if and only if $A \bigcap B \neq \phi$ and $A \bigcup B \neq X$.
(iii) $d(A, B) = 3$ if and only if $A \bigcap B \neq \phi$ and $A \bigcup B = X$.

Proof. (i) Trivial.
(ii) Suppose $d(A, B) = 2$, then by (i) $A \bigcap B \neq \phi$ and there exists an open subset C of X which is adjacent to both A and B. This implies that $C \bigcap A = \phi$ and $C \bigcap B = \phi$. This implies that $(C \bigcap A) \bigcup (C \bigcap B) = \phi$. So, $C \bigcap (A \bigcup B) = \phi$. This shows that $A \bigcup B \neq X$. Conversely, suppose $A \bigcap B \neq \phi$ and $A \bigcup B \neq X$, then $A - (X \backslash (A \bigcup B)) - B$ is a path of length 2. This shows that $d(A, B) = 2$.
(iii) If $d(A, B) = 3$, then by (i), $A \bigcap B \neq \phi$. Further, from (ii), it follows that $A \bigcup B = X$. Conversely, by (i) $d(A, B) \neq 1$ and by (ii) $d(A, B) \neq 2$. Clearly, $A - (X \backslash A) - (X \backslash B) - B$ is a path of length 3 as $(X \backslash A) \bigcap (X \backslash B) = \phi$. Hence, $d(A, B) = 3$. □

Corollary 3.2. *Let (X, τ) be a discrete topological space. Then, $G(\tau)$ is a connected graph with*

$$Diam(G(\tau)) = \begin{cases} 1, & \textit{if} \quad |X| = 2 \\ 3, & \textit{if} \quad |X| \geq 3 \end{cases} \tag{3.1}$$

Theorem 3.3. *Let (X, τ) be a discrete topological space. Then, $girth(G(\tau)) = 3$ if and only if $|X| \geq 3$.*

Proof. The proof is trivial. □

Theorem 3.4. *Let (X, τ) be a discrete topological space and A, B be any two proper open subsets of X such that $A \bigcup B \neq X$. Then,*
(i) For $|X| \geq 3$, $c(A, B) = 3$ if and only if $A \bigcap B = \phi$.
(ii) For $|X| \geq 4$, $c(A, B) = 4$ if and only if $A \bigcap B \neq \phi$ and $|X \backslash (A \bigcup B)| \geq 2$.

Proof. (i) Suppose $c(A, B) = 3$, then it is obvious that $A \bigcap B = \phi$. Conversely, suppose $A \bigcap B = \phi$. Then, since $A \bigcup B \neq X$ we see that $(A, B, X \backslash (A \bigcup B))$ form a cycle of length 3. So, $c(A, B) = 3$.
(ii) Suppose $c(A, B) = 4$. By (i) $A \bigcap B \neq \phi$. Suppose $|X \backslash (A \bigcup B)| = 1$, then there exists $x \in X$ such that $X \backslash (A \bigcup B) = \{x\}$ (i.e., $A \bigcup B \bigcup \{x\} = X$). Since $c(A, B) = 4$, there exists two distinct vertices C_1 and C_2 which are adjacent to both A and B. This implies that $C_1 \bigcap A = \phi$ and $C_1 \bigcap B = \phi$. This implies that $C_1 \bigcap (A \bigcup B) = \phi$. This shows that $C_1 = \{x\}$. Similarly, we get $C_2 = \{x\}$ which contradicts the fact that C_1 and C_2 are distinct. Hence, $|X \backslash (A \bigcup B)| \geq 2$. Conversely, suppose $A \bigcap B \neq \phi$ and $|X \backslash (A \bigcup B)| \geq 2$. By (i), $c(A, B) > 3$. Let $p,\ q \in X \backslash (A \bigcup B)$ such that $p \neq q$, then $\{p\}, \{q\}$ are vertices. It is not difficult to see that $(A, \{p\}, B, \{q\})$ form a cycle of length 4. Hence, $c(A, B) = 4$. □

It is easy to see that for a discrete topological space (X, τ), $G(\tau)$ is not triangulated and not hypertriangulated. The next two theorems give some conditions that a vertex is a vertex of some triangle and an edge is an edge of some triangle.

Theorem 3.5. *Let (X, τ) be a discrete topological space such that $|X| \geq 3$ and A be any proper open subset of X. Then, A is a vertex of a triangle if and only if $|X \backslash A| \geq 2$.*

Proof. Suppose A is a vertex of a triangle with $|X \backslash A| = 1$, i.e., $X \backslash A = \{p\}$ for some $p \in X$. But p is an isolated point as X is discrete. This implies that A is a pendant vertex by Theorem 2.8. This contradicts the fact that A is a vertex of a triangle, that is, A is adjacent to two vertices. Hence, $|X \backslash A| \geq 2$. Conversely, let $p,\ q \in X \backslash A$ such that $p \neq q$. Clearly we see that $(\{p\}, \{q\}, A)$ form a triangle. □

Theorem 3.6. *Let (X, τ) be a discrete topological space such that $|X| \geq 3$ and A, B be any proper open subsets of X such that AB is an edge in $G(\tau)$. Then, the edge AB is an edge of a triangle if and only if $A \bigcup B \neq X$.*

Proof. Suppose AB is an edge of a triangle. Then, there exists a vertex C which is adjacent to both A and B and so, $C \bigcap A = \phi$ and $C \bigcap B = \phi$. This implies that $C \bigcap (A \bigcup B) = (C \bigcap A) \bigcup (C \bigcap B) = \phi$. This shows that $A \bigcup B \neq X$. Conversely, suppose $A \bigcup B \neq X$. Then, $X \backslash (A \bigcup B)$ is adjacent to both A and B. This shows that AB is an edge of a triangle. □

4 Open and closed neighbourhood of an open set in $G(\tau)$

In this section, we investigate the properties of an open neighbourhood $N(A)$ and closed neighbourhood $N[A]$ in the graph $G(\tau)$. From the definition of $G(\tau)$ it is clear that A is a proper open subset of X if and only if A is a vertex of $G(\tau)$. So, the set $N(A)$ can also be seen as the collection of proper open subsets of X that, when considered as vertices in the graph $G(\tau)$, are adjacent to A. Similarly, $N[A]$ can be seen as the collection of proper open subsets of X that, when considered as vertices in the graph $G(\tau)$, are adjacent to A including A. For a discrete space (X, τ), it is easy to see that $N(A) = \{B : \ B \in \tau \backslash \{\phi, X\} \ and \ B \bigcap A = \phi\}$.

Remark 4.1. Let (X, τ) be any topological space. Then, $N[A] = \{A\}$ for any vertex A if and only if every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant. This claim follows from Theorem 2.10.

Theorem 4.2. *Let (X, τ) be a discrete topological space and A be any proper open subset of X. Then, $N[A]$ induced a complete subgraph if and only if $|X \backslash A| = 1$.*

Proof. Suppose $N[A]$ induces a complete subgraph. If $|X\backslash A| > 1$, then there exist $p,\ q \in X\backslash A$ such that $p \neq q$. As X is a discrete space, it is easy to see that $\{p\}$ and $\{p, q\}$ are adjacent to A, but not adjacent to each other. This contradicts the fact that $N[A]$ induces a complete subgraph. Hence, $|X\backslash A| = 1$. Conversely, suppose $|X\backslash A| = 1$. This implies that $A = X\backslash\{p\}$ for some $p \in X$. So, A is a pendant vertex, as p is an isolated point. This shows that $N[A] = \{\{p\}, A\}$ induces a complete subgraph with only two vertices and one edge. □

Theorem 4.3. *Let (X, τ) be a discrete topological space. Then, $N[A]$ forms a base for X if and only if A is a singleton subset of X, where $N[A] = \{A\} \bigcup \{B :\ B \in \tau\backslash\{\phi, X\}$ and $B \bigcap A = \phi\}$.*

Proof. Suppose $N[A]$ forms a base for X. Assuming A is not a singleton set, then there exists $p,\ q \in A$, such that $p \neq q$. Then, $\{p\} \notin N[A]$ and $\{q\} \notin N[A]$. But this contradicts that $N[A]$ is a base for discrete space X. Conversely, suppose A is singleton say $\{p\}$, then it is easy to see that $\{x\} \in N[A]$ for all $x \in X$. This shows that $\{\{x\} : x \in X\} \subseteq N[A]$ and hence $N[A]$ forms a base for X. □

Theorem 4.4. *Let (X, τ) be any topological space. If A is any proper open subset of X, then $deg(A) = |N(A)| \leq 2^{|X\backslash A|} - 1$, where $deg(A)$ denotes the degree of the vertex A.*

Proof. If B is any vertex adjacent to A, then $A \bigcap B = \phi$. This shows that $B \subseteq X\backslash A$. Thus, the only vertices adjacent to A are the non-empty subsets of $X\backslash A$. Hence, $|N(A)| \leq 2^{|X\backslash A|} - 1$. □

Corollary 4.5. *Let (X, τ) be a finite discrete topological space with $|X| = n$. If A is any proper open subset of X such that $|A| = m$, then $deg(A) = |N(A)| = 2^{n-m} - 1$.*

Remark 4.6. It is known that the number of subsets of cardinality m in the power set of a set of cardinality n is $\frac{n!}{m!(n-m)!}$. Hence, the number of vertices in $G(\tau)$ ((X, τ) is a finite discrete space) having degree $(2^{n-m} - 1)$ is $\frac{n!}{m!(n-m)!}$.

Corollary 4.7. *Let (X, τ) be a finite discrete topological space such that $|X| = n$. Then, $e = \frac{1}{2} \sum_{m=1}^{n-1} \frac{n!}{m!(n-m)!}(2^{n-m} - 1)$, where e denotes the number of edges of $G(\tau)$.*

Proof. The proof follows from Remark 4.6 and from the fact that the sum of the degree of all vertices is $2e$. □

Remark 4.8. Let (X, τ) be a discrete topological space. Then,
(i) Minimum degree of a vertex in $G(\tau)$ is 1.
(ii)Maximum degree of a vertex in $G(\tau)$ is $2^{|X|-1} - 1$.

Theorem 4.9. *Let (X, τ) be a discrete topological space and $A,\ B$ be any proper open subsets of X. Then, $N(A) \bigcap N(B) = N(A \bigcup B)$.*

Proof. Let $C \in N(A) \bigcap N(B)$. This implies that $C \bigcap A = \phi$ and $C \bigcap B = \phi$. This shows that $C \in N(A \bigcup B)$ since $C \bigcap (A \bigcup B) = (C \bigcap A) \bigcup (C \bigcap B) = \phi$. Conversely, suppose $C \in N(A \bigcup B)$. Then, $\phi = C \bigcap (A \bigcup B) = (C \bigcap A) \bigcup (C \bigcap B)$. This implies that $C \bigcap A = \phi$ and $C \bigcap B = \phi$. This shows that $C \in N(A) \bigcap N(B)$. □

Theorem 4.10. *Let (X, τ) be a discrete topological space and $A,\ B$ be any proper open subsets of X such that $A \bigcap B \neq \phi$. Then,*
(i) $N(A) \bigcup N(B) \subseteq N(A \bigcap B)$
(ii) $N(A) \bigcap N(B) \subseteq N(A \bigcap B)$

Proof. (i) Let $C \in N(A) \bigcup N(B)$. This implies that $C \bigcap A = \phi$ or $C \bigcap B = \phi$. But we have $C \bigcap (A \bigcap B) = (C \bigcap A) \bigcap (C \bigcap B) = \phi$. This shows that $N(A) \bigcup N(B) \subseteq N(A \bigcap B)$.
(ii) Let $C \in N(A) \bigcap N(B)$. Then, $C \bigcap A = \phi$ and $C \bigcap B = \phi$ and so $C \bigcap (A \bigcap B) = (C \bigcap A) \bigcap (C \bigcap B) = \phi$. This shows that $N(A) \bigcap N(B) \subseteq N(A \bigcap B)$. □

Remark 4.11. Equality of Theorem 4.10 does not hold in general. Let $a,\ b,\ c \in X$. Let $A = \{a, b\}$, $B = \{b, c\}$ and $C = \{a, c\}$. Clearly, $C \in N(A \bigcap B)$ but $C \notin N(A) \bigcup N(B)$ and $C \notin N(A) \bigcap N(B)$.

5 Dominating number and clique number

In a graph G, a dominating set is a set D of vertices such that any vertex in G either belongs to D or is adjacent to at least one member of D. The dominating number of a graph G is defined as $dt(G) = inf\{|D| : D\ is\ a\ dominating\ set\ of\ G\}$.

Theorem 5.1. *Let (X, τ) be a clopen topological space. Then, $dt(G(\tau)) = 1$ if and only if $\tau = \{\phi, A, B, X\}$, where $A \bigcap B = \phi$ and $A \bigcup B = X$.*

Proof. Suppose $dt(G(\tau)) = 1$ and $D = \{A\}$ is a dominating set for some clopen set A. Let B be any vertex such that $B \neq A$. As D is a dominating set, B is adjacent to A. By Theorem 2.4, $B \bigcap A = \phi$. As (X, τ) is a clopen topological space, $X \backslash B$ is also a vertex. This shows that $X \backslash B$ is not adjacent to A as $(X \backslash B) \bigcap A \neq \phi$. By definition of dominating set, we have $A = X \backslash B$. This shows that $B = X \backslash A$ and $A \bigcup B = X$. Suppose there exists another dominating set $D' = \{A'\}$ where $A \neq A'$. As A is a vertex and D' is a dominating set, then in the same way we get $A' = X \backslash A$ and $A = X \backslash A'$. This shows that $A' = B$. Hence, A and B are the only two proper open subsets of X and $\tau = \{\phi, A, B, X\}$. The converse part follows from the fact that the characteristic function of A is continuous and hence $G(\tau)$ is a complete graph with two vertices A and B. □

Remark 5.2. Theorem 5.1 is not generally true if (X, τ) is not clopen topological space. For example, consider $X = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$. Then, $dt(G(\tau)) = 1$ with dominating set $D = \{\{a\}\}$, but τ contain only one proper open subset.

Theorem 5.3. *Let (X, τ) be any topological space such that $|\tau|$ is finite. Then, $dt(G(\tau)) = |\tau| - 2$ if and only if every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant.*

Proof. Suppose $dt(G(\tau)) = |\tau| - 2$. If possible there exists a non-constant continuous function $f : X \longrightarrow \mathbb{R}_u$ such that $f(p) = r$ and $f(q) = s$ for some $p,\ q \in X$ and $r,\ s \in \mathbb{R}$, $r \neq s$. Without loss of generality, we assume $r < s$ and fix $t \in (r, s)$, then $A = f^{-1}((-\infty, t))$ and $B = f^{-1}((t, \infty))$ are proper open subsets which are adjacent to each other in $G(\tau)$. It is easy to see that $D = \tau \backslash \{\phi, X, A\}$ is a dominating set with $|D| = |\tau| - 3$ which is less than $dt(G(\tau))$, which is not possible. So, every continuous function $f : X \longrightarrow \mathbb{R}_u$ is constant. Conversely, by Theorem 2.10, $G(\tau)$ is an edgeless graph. Hence, $dt(G(\tau)) = |\tau| - 2$. □

Theorem 5.4. *Let (X, τ) be a discrete topological space. Then,*

$$dt(G(\tau)) = \begin{cases} 1, & \text{if} \quad |X| = 2 \\ |X|, & \text{if} \quad |X| \geq 3 \end{cases} \tag{5.1}$$

Proof. If $|X| = 2$, then by Theorem 5.1 $dt(G(\tau)) = 1$. If $|X| > 2$, then for each dominating set D and for each $p \in X$, we have either $\{p\}$ or $X \backslash \{p\}$ lies in D. This shows that cardinality of D is at least $|X|$. So, by definition of dominating number, we get $dt(G(\tau)) = |X|$. □

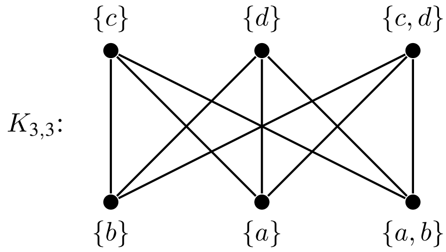
Theorem 5.5. *Let (X, τ) be a discrete topological space. Then, the clique number of $G(\tau)$ is $\omega(G(\tau)) = |X|$.*

Proof. Let $H(\tau)$ be a subgraph induced by vertex set $V(H(\tau)) = \{\{x\} : x \in X\}$. Then, as X is a discrete space, every vertex $\{x\}$ is adjacent to $\{y\}$ for all $x \neq y,\ x,\ y \in X$. This shows that $H(\tau)$ is a complete graph with $|X|$ number of vertices. It is also easy to see that $H(\tau)$ is the largest complete subgraph of $G(\tau)$ as $V(H(\tau)) = \{\{x\} : x \in X\}$ is the largest collection of subsets of X such that no two members intersect. Hence, the clique number of $G(\tau)$ is $|X|$. □

Remark 5.6. Let (X, τ) be a discrete topological space such that $|X| > 2$. Then, the clique number of $G(\tau)$ is the same as the dominating number of $G(\tau)$.

Theorem 5.7. *Let (X, τ) be a discrete topological space. Then, $G(\tau)$ is planar if and only if $|X| \leq 3$.*

Proof. Suppose $G(\tau)$ is planar and $|X| \geq 4$. Let $a,\ b,\ c,\ d$ be distinct points in X. As X is a discrete topological space, $G(\tau)$ contains $K_{3,3}$ as a subgraph as illustrated by $Fig(iv)$. But $K_{3,3}$ is a non-planar graph. This contradicts that $G(\tau)$ is planar. Hence, $|X| \leq 3$. Conversely, if $|X| \leq 3$, then the graphs are as shown in $Fig(ii)$ and $Fig(iii)$, which are planar.



$Fig(iv)$

□

Conclusion

In this paper, we introduced the completely separated topological graph and studied various interrelations among (X, τ) as a topological space and $G(\tau)$ as a graph. We showed that (X, τ) is T_1 if and only if $X\backslash\{p\}$ is a vertex of $G(\tau)$ for all $p \in X$ and X is discrete space if and only if $X\backslash\{p\}$ is a pendant vertex of $G(\tau)$ for all $p \in X$, and p is an isolated point of X if and only if $X\backslash\{p\}$ is a pendant vertex of $G(\tau)$. We also discussed various graph properties like completeness, connectedness, dominating number, clique number, planar, etc. for discrete topological space. For further research in this area, one can explore topics like connectedness, independent set, dominating set, etc. in the case of an arbitrary topological space (X, τ). One can also study the line graph of $G(\tau)$ and the complement graph of $G(\tau)$.

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