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Analysis and simulation of a stochastic epidemic model with general incidence function and vaccination process

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Abstract Deterministic differential equations have been extensively utilized by the mathematical community to model and analyze the propagation of epidemics. However, these equations are constrained by their dependence on initial value selection, rendering their solutions sensitive to these initial conditions. Consequently, this deterministic framework fails to incorporate random fluctuations, parameter variability, data uncertainties, and unpredictable dynamics inherent in real-world scenarios. Stochastic differential equations, therefore, offer a viable alternative, providing a more robust modeling approach that accounts for these stochastic elements. In this study, we examine a stochastic SVIR (Susceptible, Vaccinated, Infected, Recovered) epidemic model characterized by a general nonlinear incidence function. Initially, we establish the existence of a global positive solution for the stochastic model. Subsequently, we demonstrate that the disease is almost surely permanent under the condition of sufficiently small environmental fluctuations. Furthermore, we identify two specific conditions under which the disease disappears exponentially with near certainty. Lastly, we present numerical simulations employing various incidence functions to corroborate our theoretical findings.

1 Introduction

The wide spread of infectious diseases has a significant impact on societies at multiple levels. It affects income distribution, reduces economic growth and consumption, disrupts supply chains, and increases the rates of unemployment and inflation [1, 2, 3, 4]. The presence of the disease also influences tourism and financial markets, as individuals demonstrate a reduced propensity to travel to affected regions, and investors exhibit adverse reactions to the uncertainty and volatility associated with the disease [5, 6]. At the educational level, the precautionary measures implemented by authorities have resulted in the closure of schools, necessitating a shift towards self-directed learning. This transition has adversely impacted the regular continuity of academic instruction and the efficacy of evaluation processes, while also contributing to an elevated rate of school dropout [7, 8].

To comprehend the diseases' dynamics and predict their behaviors, mathematical models are widely used to describe the transmission of infectious diseases [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Many of these models have their origins in previous works [30, 31, 32]. Moreover, there have been a number of studies examining the stochastic processes [33, 34, 35, 36]. In attempting to curtail the spread of infectious diseases, vaccination is regarded as the most successful intervention strategy (see [37, 38, 39, 40, 41, 42]). In [39], the authors considered the following stochastic SVIR epidemic model

$$\begin{cases} dS(t) = [p + \epsilon V(t) - (\mu + a)S(t) - \beta S(t)I(t)]\, dt - \sigma S(t)I(t)dB(t), \\ dV(t) = [-(\mu + \epsilon)V(t) + aS(t)]\, dt, \\ dI(t) = [-(\mu + \gamma + c)I(t) + \beta S(t)I(t)]\, dt + \sigma S(t)I(t)dB(t), \\ dR(t) = [\gamma I(t) - \mu R(t)]\, dt. \end{cases} \tag{1.1}$$

$S(t)$ denotes the number of members who are susceptible to an infection at time t. $I(t)$ denotes the number of infected members at time t. $V(t)$ is the number of members who are vaccinated. $R(t)$ is defined as the number of recovered individuals. The other symbols involved in model (1.1) are described below.

Symbol	**Meaning**
p	A constant input of new members into the population.
β	The disease transmission coefficient between compartments S and I.
μ	The natural death rate of S, I, V and R.
a	The proportional coefficient of vaccinated individuals for the susceptibles.
γ	The recovery rate of infected individuals.
ϵ	The rate at which the vaccinated individuals lose their immunity.
c	The disease-caused death rate of infected individuals.
$B(t)$	A standard Brownian motion with intensity $\sigma^2 > 0$.

The analysis of (1.1) consists of examining the conditions under which the disease persists or disappears. If $R_0 = \frac{p\beta(\mu+\epsilon)}{\mu(\mu+\gamma+c)(\mu+\epsilon+a)} \leq 1$ and $\sigma^2 < (\gamma + c)\mu^2 p^{-2}$, then the equilibrium state $(S_0, V_0, I_0, R_0) = \big(\frac{p(\mu+\epsilon)}{\mu(\mu+\epsilon+a)}, \frac{ap}{\mu(\mu+\epsilon+a)}, 0, 0\big)$ of (1.1) is stochastically asymptotically stable in the large. Per contra, when $R_0 > 1$, the solution of (1.1) oscillates around the state $(S_*, V_*, I_*, R_*) = \big(\frac{\mu+\gamma+c}{\beta}, \frac{a(\mu+\gamma+c)}{\beta(\mu+\epsilon)}, \frac{p}{(\mu+\gamma+c)(1-R_0{}^{-1})}, \frac{\gamma}{\mu}I_1\big)$. This implies that the system (1.1) tends towards the persistence case (see [39]).
The incidence function βSI in (1.1) is bilinear in accordance with the mass-action principle. Several authors have mentioned that it can have a nonlinear mathematical shape. We cite as examples, the saturated incidence $\frac{\beta SI}{1+mI}$, where m^{-1} is the saturation coefficient [9]. In [10], the authors considered $\frac{\beta SI}{1+k_1 I+k_2 I^2}$ as an incidence function, provided that $k_1 > -2\sqrt{k_2}$ to keep the quantity $1+k_1 I+k_2 I^2$ positive. Two other types of incidence functions have been used in [11, 12] to highlight the effect of media coverage on the disease dynamics. The first one is $\beta SIe^{-\alpha I}$, where the parameter $\alpha > 0$ reflects the impact of media coverage on contact transmission. The second is written as $\big(\lambda_1 - \frac{\lambda_2 I}{\lambda_3+I}\big)SI$, with $\lambda_i > 0$, $i = 1, 2, 3$.

To improve the analysis of the model (1.1), we assume that the incidence function has a general form $\frac{\beta SI}{\Phi(I)}$, where Φ is a continuous and derivable function satisfying: $\Phi(0) = 1$, $\Phi(I) \geq 1$ and $0 \leq \Phi^{'}(I) \leq \eta$, such that η is a positive constant that can depend on p and μ. We note that the aforementioned incidence functions have the same characteristics as $\frac{\beta SI}{\Phi(I)}$.
In this paper, we will determine threshold conditions for the following stochastic SVIR epidemic model (1.1) with the incidence function $\frac{\beta SI}{\Phi(I)}$:

$$\begin{cases} dS(t) = \Big[p - \frac{\beta S(t)I(t)}{\Phi(I(t))} - (\mu + a)S(t) + \epsilon V(t)\Big]\, dt - \frac{\sigma S(t)I(t)}{\Phi(I(t))}dB(t), \\ dI(t) = \Big[\frac{\beta S(t)I(t)}{\Phi(I(t))} - (\mu + \gamma + c)I(t)\Big]\, dt + \frac{\sigma S(t)I(t)}{\Phi(I(t))}dB(t), \\ dV(t) = [aS(t) - (\mu + \epsilon)V(t)]\, dt, \\ dR(t) = [\gamma I(t) - \mu R(t)]\, dt. \end{cases} \tag{1.2}$$

For the convenience of the readers, we denote

$$\begin{aligned}\mathbb{R}_+^4 &= \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i > 0, i = 1, 2, 3, 4\right\},\\ \langle h(t)\rangle &= \frac{1}{t}\int_0^t h(s)ds,\\ R_0^S &= R_0\left[1 - \frac{\sigma^2}{2\beta}S_0\right].\end{aligned}$$

The rest of the paper is organized as follows: Section 2 focuses on verifying the biological relevance of model (1.2) by proving that it admits a singular positive solution. In Section 3, if $R_0^S > 1$, we prove that the disease will persist almost surely (abbreviated as a.s.). In Section 4, we demonstrate that the disease undergoes exponential extinction with probability one when the threshold R_0^S is less than 1, contingent upon appropriate conditions pertaining to stochastic perturbations. To confirm the analytical findings, numerical simulations are performed with different incidence functions in Section 5. The paper ends with the conclusion section.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). A positive invariant set for the system (1.2) is defined by

$$\Delta = \left\{\Big(S(t), V(t), I(t), R(t)\Big) \in \mathbb{R}_+^4 : \; S(t) + V(t) + I(t) + R(t) \leq \frac{p}{\mu} \quad \text{for all } t \geq 0\right\}.$$

Henceforth, we assume that: $(S(0), V(0), I(0), R(0)) \in \Delta$.
The following theorem concerns the existence of a global positive solution for the system (1.2). Since the proof is almost identical to that in [45], we omit it here.

Theorem 2.1. *System (1.2) has a unique solution belonging to Δ with probability one.*

Now, we present two lemmas that will be used in the following sections.

Lemma 2.2. *The class of susceptible individuals satisfies: $S(t) = S_0 + H_0(t) + G(t)$. H_0 and G are defined as follows:*

$$\begin{aligned}H_0(t) = &\left[\frac{p}{\mu} - S(0) - V(0) - I(0)\right]\frac{1 - (\epsilon + a)}{\epsilon + a}e^{-\mu t} + V(0)\frac{1 - a}{a}e^{-(\mu+\epsilon)t}\\ &+ \left[\frac{S(0) + V(0) + I(0) - \frac{p}{\mu}}{\epsilon + a} + \frac{ap}{\mu(\mu + \epsilon + a)} - \frac{V(0)}{a}\right]e^{-(\mu+\epsilon+a)t}\\ =&Xe^{-\mu t} + Ye^{-(\mu+\epsilon)t} + Ze^{-(\mu+\epsilon+a)t},\end{aligned}$$

and

$$\begin{aligned}G(t) = &- I(t) - (c + \gamma)\int_0^t I(s)e^{-\mu(t-s)}ds + a\int_0^t I(s)e^{-(\mu+\epsilon+a)(t-s)}ds\\ &+ a(c + \gamma)\int_0^t e^{-(\mu+\epsilon+a)(t-s)}\int_0^s I(u)e^{-\mu(s-u)}duds.\end{aligned}$$

Proof. From system (1.2), we have:

$$\begin{aligned}S(t) + V(t) + I(t) + R(t) \quad = \quad &\frac{p}{\mu} + \left[S(0) + V(0) + I(0) + R(0) - \frac{p}{\mu}\right]e^{-\mu t}\\ &-\alpha\int_0^t e^{-\mu(t-s)}I(s)ds. \qquad (2.1)\end{aligned}$$

By taking an integration of system (1.2), we get

$$V(t) = V(0)e^{-(\mu+\epsilon)t} + a\int_0^t S(s)e^{-(\mu+\epsilon)(t-s)}ds, \tag{2.2}$$

and

$$R(t) = R(0)e^{-\mu t} + \gamma\int_0^t I(s)e^{-\mu(t-s)}ds. \tag{2.3}$$

Injecting (2.2) and (2.3) into (2.1) gives

$$S(t) = \frac{p}{\mu} - (\alpha+\gamma)\int_0^t I(s)e^{-\mu(t-s)}ds - a\int_0^t S(s)e^{-(\mu+\epsilon)(t-s)}ds - I(t) - H_1(t), \tag{2.4}$$

where

$$H_1(t) = \left[\frac{p}{\mu} - S(0) - V(0) - I(0)\right]e^{-\mu t} + V(0)e^{-(\mu+\epsilon)t}.$$

Now, we shall give the explicit expression of $\int_0^t S(s)e^{-(\mu+\epsilon)(t-s)}ds$.
One can see that

$$\begin{aligned} d\left[e^{at}\int_0^t S(s)e^{(\mu+\epsilon)s}ds\right] &= e^{at}\left[e^{(\mu+\epsilon)t}S(t) + a\int_0^t S(s)e^{(\mu+\epsilon)s}ds\right]dt \\ &= \left[e^{at}e^{(\mu+\epsilon)t}\frac{p}{\mu} - I(t)e^{at}e^{(\mu+\epsilon)t} - H_1(t)e^{at}e^{(\mu+\epsilon)t} - (c+\gamma)e^{at}e^{(\mu+\epsilon)t}\int_0^t I(s)e^{-\mu(t-s)}ds\right]dt. \end{aligned}$$

Integrating the last equality leads to

$$\begin{aligned} \int_0^t S(s)e^{(\mu+\epsilon)s}ds =& \frac{p}{\mu}\int_0^t e^{-a(t-s)}e^{(\mu+\epsilon)s}ds - \int_0^t I(s)e^{-a(t-s)}e^{(\mu+\epsilon)s}ds - \int_0^t H_1(s)e^{-a(t-s)}e^{(\mu+\epsilon)s}ds \\ &- (c+\gamma)\int_0^t e^{-a(t-s)}e^{(\mu+\epsilon)s}\int_0^s e^{-\mu(s-u)}I(u)duds. \end{aligned}$$

Multiplying the previous equality by $e^{-(\mu+\epsilon)t}$, we get

$$\begin{aligned} \int_0^t S(s)e^{-(\mu+\epsilon)(t-s)}ds =& \frac{p}{\mu}\int_0^t e^{-(\mu+\epsilon+a)(t-s)}ds - \int_0^t I(s)e^{-(\mu+\epsilon+a)(t-s)}ds - \int_0^t H_1(s)e^{-(\mu+\epsilon+a)(t-s)}ds \\ &- (c+\gamma)\int_0^t e^{-(\mu+\epsilon+a)(t-s)}\int_0^s e^{-\mu(s-u)}I(u)duds \\ =& \frac{p}{\mu(\mu+\epsilon+a)}\left(1 - e^{-(\mu+\epsilon+a)t}\right) - \int_0^t I(s)e^{-(\mu+\epsilon+a)(t-s)}ds \\ &- \int_0^t H_1(s)e^{-(\mu+\epsilon+a)(t-s)}ds - (c+\gamma)\int_0^t e^{-(\mu+\epsilon+a)(t-s)}\int_0^s e^{-\mu(s-u)}I(u)duds. \end{aligned} \tag{2.5}$$

One can see that

$$\begin{aligned} \int_0^t H_1(s)e^{-(\mu+\epsilon+a)(t-s)}ds =& \left[\frac{\frac{p}{\mu} - S(0) - V(0) - I(0)}{\epsilon+a}\right]e^{-\mu t} + \frac{V(0)}{a}e^{-(\mu+\epsilon)t} \\ &- \left[\frac{\frac{p}{\mu} - S(0) - V(0) - I(0)}{\epsilon+a} + \frac{V(0)}{a}\right]e^{-(\mu+\epsilon+a)t}. \end{aligned} \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we will find the seeked formula. □

Lemma 2.3. *The temporal average of susceptible individuals satisfies*

$$\langle S(t)\rangle = S_0 - \frac{(\mu+\epsilon)(\mu+c+\gamma)}{\mu(\mu+\epsilon+a)}\langle I(t)\rangle - \varphi(t),$$

where

$$\varphi(t) = \frac{(\mu+\epsilon)^2\left(S(t)+V(t)+I(t)+R(t)-S(0)-V(0)-I(0)-R(0)\right)-\mu(\mu+\epsilon)(V(t)-V(0))}{\mu(\mu+\epsilon)(\mu+\epsilon+a)t}$$
$$-\frac{(\mu+\epsilon)^2(R(t)-R(0))}{\mu(\mu+\epsilon)(\mu+\epsilon+a)t}.$$

Proof. From system (1.2), we get

$$\frac{S(t)+V(t)+I(t)+R(t)-S(0)-V(0)-I(0)-R(0)}{t} = p-\mu\langle S(t)\rangle-\mu\langle V(t)\rangle-\mu\langle R(t)\rangle - (\mu+c)\langle I(t)\rangle, \tag{2.7}$$

$$\frac{V(t)-V(0)}{t} = a\langle S(t)\rangle-(\mu+\epsilon)\langle V(t)\rangle, \tag{2.8}$$

and

$$\frac{R(t)-R(0)}{t} = \gamma\langle I(t)\rangle-\mu\langle R(t)\rangle. \tag{2.9}$$

Then the desired result is obtained by injecting (2.8) and (2.9) into (2.7). □

3 Persistence in mean

Definition 3.1. The disease is said to be persistent in mean if: $\liminf_{t\to\infty} \langle I(t)\rangle \geq 0$ a.s.

Theorem 3.2. *If* $R_0^S > 1$, *then the disease is persistent in mean, that is,*

$$\lim_{t\to\infty} inf \ \langle I(t)\rangle \geq \frac{\mu(\mu+\gamma+c)(R_0^s-1)}{\eta p+\beta(\mu+\gamma+c)+\sigma^2\varpi} \quad a.s.,$$

where

$$\varpi = S_0\frac{a(\mu+c+\gamma)}{\mu(\mu+\epsilon+a)}+4\left(\frac{p}{\mu}+\frac{p(c+\gamma)^2}{\mu^2}+\frac{pa^2}{\mu(\mu+\epsilon+a)^2}+\frac{pa^2(c+\gamma)^2}{\mu^3(\mu+\epsilon+a)^2}\right).$$

Proof. By use of Itô formula and Lemma 2.2, we obtain

$$\frac{1}{t}\ln\frac{I(t)}{I(0)} = \beta\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle-(\mu+\gamma+c)-\frac{\sigma^2}{2}\left\langle\left(\frac{S(t)}{\Phi(I(t))}\right)^2\right\rangle+\frac{M(t)}{t} \tag{3.1}$$

$$\begin{aligned}
&\geq \beta\langle S(t)\rangle-\frac{\sigma^2}{2}\left\langle\left[S_0+(H_0(t)+G(t))\right]^2\right\rangle-(\mu+\gamma+c)\\
&\quad-\beta\left\langle\frac{S(t)[\Phi(I(t))-\Phi(0)]}{\Phi(I(t))}\right\rangle+\frac{M(t)}{t}\\
&\geq \beta S_0-(\mu+\gamma+c)-\frac{\sigma^2S_0^2}{2}-\eta\frac{p}{\mu}\langle I(t)\rangle+K(t)+\frac{M(t)}{t},
\end{aligned} \tag{3.2}$$

where

$$K(t) = \left(\beta-\sigma^2S_0\right)\left(\langle H_0(t)+\langle G(t)\rangle\right)-\sigma^2\left[\langle H_0^2(t)\rangle+\langle G^2(t)\rangle\right],$$

and

$$M(t) = \int_0^t \frac{\sigma S(u)}{\Phi(I(u))}dB(u).$$

The quadratic variation of $M(t)$ is

$$\langle M(t), M(t)\rangle = \int_0^t \left[\frac{\sigma S(u)}{\Phi(I(t))}\right]^2 du \le \sigma^2 \left(\frac{p}{\mu}\right)^2 t.$$

By the strong law of large numbers for martingales [46], we have the limit:

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \qquad \text{a.s.}$$

On the other hand, we have

$$\langle H_0(t)\rangle = X\frac{1-e^{-\mu t}}{\mu t} + Y\frac{1-e^{-(\mu+\epsilon)t}}{(\mu+\epsilon)t} + Z\frac{1-e^{-(\mu+\epsilon+a)t}}{(\mu+\epsilon+a)t}, \tag{3.3}$$

$$\langle H_0^2(t)\rangle \le \frac{3X^2}{2\mu}\left(\frac{1-e^{-2\mu t}}{t}\right) + \frac{3Y^2}{2(\mu+\epsilon)}\left(\frac{1-e^{-2(\mu+\epsilon)t}}{t}\right) + \frac{3Z^2}{2(\mu+\epsilon+a)}\left(\frac{1-e^{-2(\mu+\epsilon+a)t}}{t}\right), \tag{3.4}$$

$$\langle\, G(t)\, \rangle \ge -\frac{\mu+c+\gamma}{\mu}\langle I(t)\rangle, \tag{3.5}$$

$$\langle\, G(t)\, \rangle \le \frac{a(\mu+c+\gamma)}{\mu(\mu+\epsilon+a)}\langle I(t)\rangle, \tag{3.6}$$

$$\begin{aligned}
\langle\, G^2(t)\rangle \le\, & 4\langle I^2(t)\rangle + 4(c+\gamma)^2\left\langle\left(\int_0^t I(s)e^{-\mu(t-s)}ds\right)^2\right\rangle + 4a^2\left\langle\left(\int_0^t I(s)e^{-(\mu+\epsilon+a)(t-s)}ds\right)^2\right\rangle \\
& + 4a^2(c+\gamma)^2\left\langle\left(\int_0^t e^{-(\mu+\epsilon+a)(t-s)}\int_0^s I(r)e^{-\mu(s-r)}drds\right)^2\right\rangle \\
\le\, & 4\frac{p}{\mu}\langle I(t)\rangle + \frac{4p(c+\gamma)^2}{\mu^2}\left\langle\int_0^t I(s)e^{-\mu(t-s)}ds\right\rangle + \frac{4pa^2}{\mu(\mu+\epsilon+a)}\left\langle\int_0^t I(s)e^{-(\mu+\epsilon+a)(t-s)}ds\right\rangle \\
& + \frac{4pa^2(c+\gamma)^2}{\mu^2(\mu+\epsilon+a)}\left\langle\int_0^t e^{-(\mu+\epsilon+a)(t-s)}\int_0^s I(r)e^{-\mu(s-r)}drds\right\rangle \\
\le\, & 4\left[\frac{p}{\mu} + \frac{p(c+\gamma)^2}{\mu^3} + \frac{pa^2}{\mu(\mu+\epsilon+a)^2} + \frac{pa^2(c+\gamma)^2}{\mu^3(\mu+\epsilon+a)^2}\right]\langle I(t)\rangle.
\end{aligned} \tag{3.7}$$

From (3.3) and (3.4), it is easy to see that

$$\lim_{t\to\infty}\langle H_0(t)\rangle = \lim_{t\to\infty}\langle H_0^2(t)\rangle = 0.$$

Injecting inequalities (3.5), (3.6) and (3.7) into (3.2), we get the desired result. □

4 Stochastic extinction

In this section, we establish conditions that guarantee the extinction of the disease.

Theorem 4.1. *Let us consider the two following assumptions:*

$$\textbf{(A)} \quad \frac{\beta^2}{2\sigma^2} < \mu+\gamma+c.$$

$$\textbf{(B)} \quad S_0 \le \frac{\beta}{\sigma^2} \quad \text{and} \quad R_0^s < 1.$$

Then

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+c) < 0 \quad \text{a.s.,} \quad \text{if } \textbf{(A)} \text{ holds.}$$

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le (\mu+\gamma+c)(R_0^S - 1) < 0 \quad \text{a.s.,} \quad \text{if } \textbf{(B)} \text{ holds.}$$

Proof. From (3.1), we get

$$\begin{aligned}\frac{1}{t}\ln\frac{I(t)}{I(0)} &\leq \frac{\beta^2}{2\sigma^2}-(\mu+\gamma+c)-\frac{\sigma^2}{2}\left[\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle-\frac{\beta^2}{\sigma^2}\right]^2+\frac{M(t)}{t}\\ &\leq \frac{\beta^2}{2\sigma^2}-(\mu+\gamma+c)+\frac{M(t)}{t}.\end{aligned}$$

If $\frac{\beta^2}{2\sigma^2}<\mu+\gamma+c$, then

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}<0 \quad \text{a.s.}$$

On the other hand, returning to (3.1) and bearing in mind Lemma 2.3, yields

$$\begin{aligned}\frac{1}{t}\ln\frac{I(t)}{I(0)} &\leq \beta\left[\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle+\varphi(t)\right]-(\mu+\gamma+c)-\frac{\sigma^2}{2}\left[\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle+\varphi(t)\right]^2-\Psi(t)+\frac{M(t)}{t}\\ &=-\frac{\sigma^2}{2}\left[\left(\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle+\varphi(t)\right)-\frac{\beta}{\sigma^2}\right]^2+\frac{\beta^2}{2\sigma^2}-(\mu+\gamma+c)-\Psi(t)+\frac{M(t)}{t}.\end{aligned} \tag{4.1}$$

where

$$\Psi(t)=\beta\varphi(t)-\frac{\sigma^2}{2}\left[2\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle\varphi(t)+\varphi^2(t)\right].$$

In view of the assumption $S_0\leq\frac{\beta}{\sigma^2}$ and Lemma 2.3, one has

$$\left\langle\frac{S(t)}{\Phi(I(t))}\right\rangle+\varphi(t)\leq\langle S(t)\rangle+\varphi(t)\leq S_0\leq\frac{\beta}{\sigma^2}.$$

Returning to (4.1), it follows that

$$\begin{aligned}\frac{1}{t}\ln\frac{I(t)}{I(0)} &\leq -\frac{\sigma^2}{2}\left[\frac{\beta}{\sigma^2}-S_0\right]^2+\frac{\beta^2}{2\sigma^2}-(\mu+\gamma+c)-\Psi(t)+\frac{M(t)}{t}\\ &=\beta S_0-\frac{\sigma^2}{2}S_0^2-(\mu+\gamma+c)-\Psi(t)+\frac{M(t)}{t}\\ &=(\mu+\gamma+c)\left[R_0^S-1\right]-\Psi(t)+\frac{M(t)}{t}.\end{aligned}$$

Since

$$\lim_{t\to\infty}\varphi(t)=0 \qquad \text{a.s.,}$$

then

$$\lim_{t\to\infty}\Psi(t)=0 \qquad \text{a.s.}$$

Assuming $R_0^S<1$, we conclude that

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}<0 \quad \text{a.s.}$$

□

5 Numerical confirmation

In this section, we will combine several common types of incidence functions to check our analytical results. Using the classical high-order discrete method developed by Desmond Higham [47], the corresponding discretization equation of system (1.2) is given by

$$\begin{cases} S_{j+1}=S_j+\Big(p+\epsilon V_j-(\mu+a)S_j+\beta\frac{S_j\,I_j}{\Phi(I_j)}\Big)\Delta t-\sigma\frac{S_j\,I_j}{\Phi(I_j)}\sqrt{\Delta t}\,\pi,\\ V_{j+1}=V_j+\Big(-(\mu+\epsilon)V_j+aS_j\Big)\Delta t,\\ I_{j+1}=I_j+\Big(-(\mu+\gamma+c)I_j+\beta\frac{S_jI_j}{\Phi(I_j)}\Big)\Delta t+\sigma\frac{S_jI_j}{\Phi(I_j)}\sqrt{\Delta t}\,\pi,\\ R_{j+1}=R_j+\Big(\gamma I_j-\mu R_j\Big)\Delta t,\end{cases}$$

where $\Delta t > 0$ is the time increment, and π is a white noise process with intensity σ.
Now, we will plot the figures corresponding to the extinction and persistence of the disease, for the following three cases:

(i) The saturated incidence: $\Phi(I) = 1 + mI$.

(ii) The non-monotone incidence: $\Phi(I) = 1 + k_1 I + k_2 I^2$.

(iii) The exponential media alert incidence: $\Phi(I) = e^{\alpha I}$.

5.1 Persistence of the disease

The initial value and parameters are assumed as follows:
$\big(S(0), V(0), I(0), R(0)\big) = \big(50, 50, 20, 50\big), p = 0.3, \beta = 0.2, \mu = 0.1, a = 0.2, \gamma = 0.1, \epsilon = 0.2, c = 0.1,$
$\sigma = 0.1, m = 1.25, k_1 = k_2 = 0.1$ and $\alpha = 0.2$.
A simple calculation shows that $R_0^S = 1.146 > 1$. According to Theorem 3.2, the disease persists almost surely as it is depicted in Figure 1.

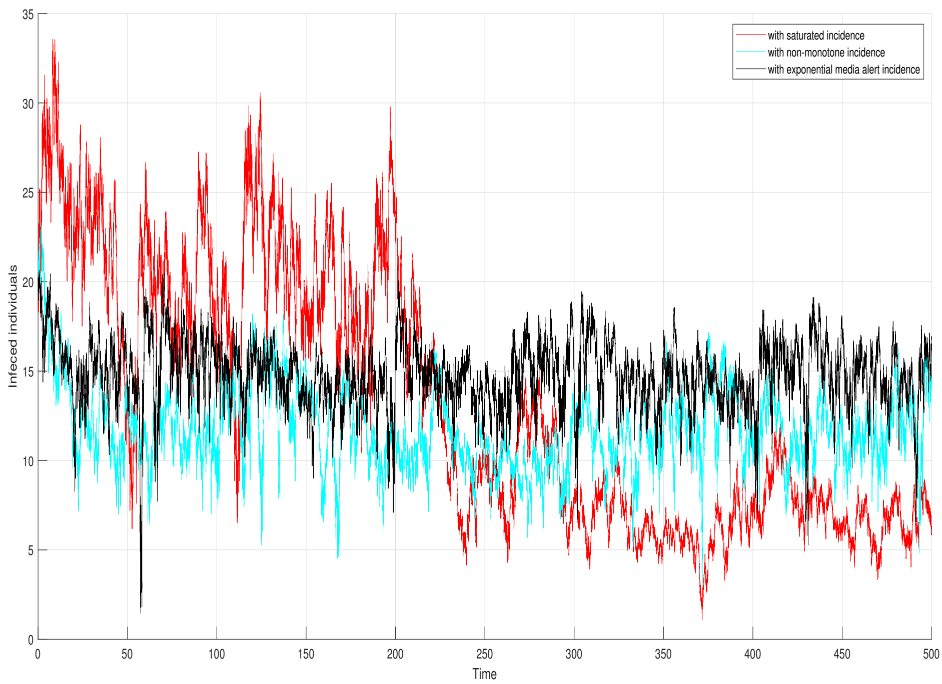

Figure 1. The path of $I(t)$ for the model (1.2), with different incidence functions.

5.2 Extinction of the disease

Here, we make the following choice:
$\big(S(0), V(0), I(0), R(0)\big) = \big(20, 20, 50, 20\big), p = 0.4, \beta = 0.5, \mu = 0.3, a = 0.2, \gamma = 0.1, \epsilon = 0.2, c = 0.1,$
$\sigma = 0.7, m = 0.5, k_1 = k_2 = 0.1$ and $\alpha = 0.8$. Then, we have

$$\frac{\beta^2}{2\sigma^2} - (\mu + \gamma + c) = -0.2449,\ S_0 - \frac{\beta}{\sigma^2} = -0.0680 \text{ and } R_0^s = 0.5079 < 1,$$

which means that the conditions **(A)** and **(B)** hold. From Theorem 4.1, the disease will die out exponentially almost surely, as shown in Figure 2.

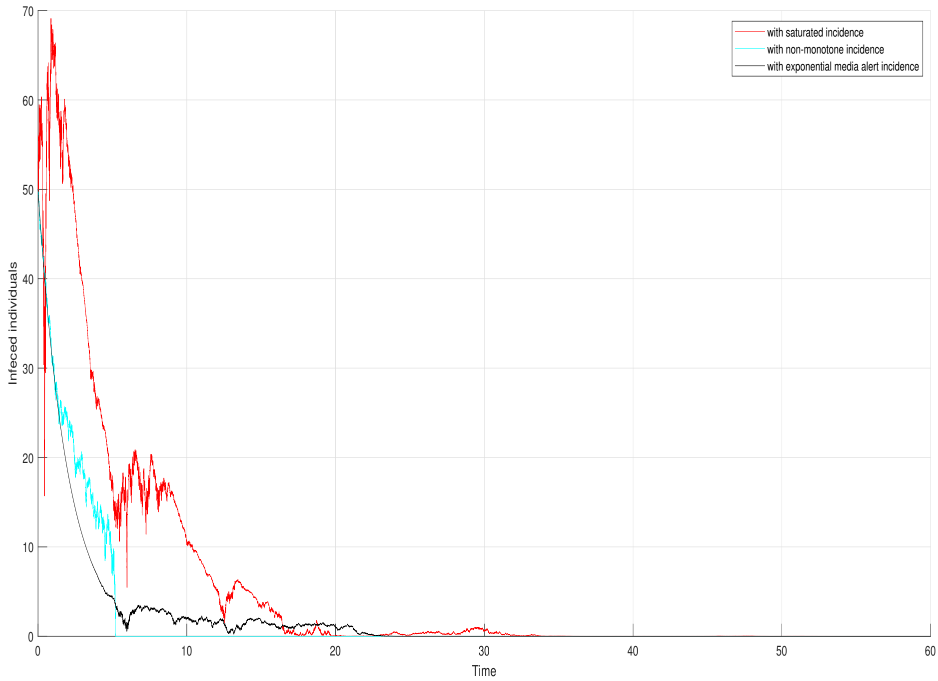

Figure 2. The path of $I(t)$ for the model (1.2), with different incidence functions.

6 Conclusion

By adopting a nonlinear form of the incidence function that generalizes saturation effects, non-monotone behavior, and the influence of media coverage, this research presents an analytical and numerical framework to investigate the dynamics of a stochastic epidemic model. More precisely, we defined the threshold $R^S 0$ for the model (1.2), a pivotal parameter that discriminates between the potential for stochastic extinction or persistence of the disease. Our findings indicate that, subject to additional conditions, if $R^S 0 < 1$, the disease is almost inevitably driven to exponential extinction. On the other hand, if $R_0^S > 1$, the disease is likely to persist. Numerical simulations demonstrate that the intensity of white noise exerts a significant influence on the model's dynamics: with sufficiently low noise intensity, the disease tends towards extinction, whereas higher noise intensity favors persistence. Moreover, the simulations underscore that enhanced media awareness campaigns lead to a rapid decrease in the number of infected individuals.
In pursuit of further refining our study of the stochastic model (1.2), future research efforts will involve introducing an additional noise component to the parameter μ, following the approach utilized by the authors in [13].

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