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Modules in which fully invariant z-closed submodules are direct summands

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Abstract One of the useful generalization of CS notion is CLS property. If each "z-closed submodule" of a module U is a direct summand of U, then U is stated to be CLS. In this article, we present fully invariant CLS by ignoring all z-closed submodules and taking into only fully invariant z-closed submodules, such a module is referred to as fully invariant CLS module. This new module class is appropriately comprised of the class of CLS and fully invariant extending modules. The CLS module class is closed within direct summands, not under direct sums, as is often known As opposed to CLS modules, the category of modules with previous attributes is demonstrated to be closed under direct sums. In doing so, we get a number of outcomes, some of structural features with examples.

1 Introduction

All throughout this paper, every module is a unitary right R-module and all rings are associative with unitary. In recent years, extending modules and ring theory have played an important role in ring and module theory. Remember that a module U when each of it's submodules is necessary essential in a "direct summand", then U is referred to as a CS module, equivalently, U is CS if every closed submodule of U is a "direct summand, Dung, Huynh, Smith and Wisbauer [1]. There is an ongoing interest in CS modules and some of their generalizations. Several generalizations of the CS property have been explored, see e.g. Enas and Davvaz [2], Kamil and Al-Aeashi [3], Kamil also see [4]. In Akalan, Birkenmeier, and Tercan [5], the authors introduce the relation that sets of submodules:

(i) $V\alpha W$ if and only if there exist $Y \leqslant U$ satisfies $V \leqslant_e Y$ and $W \leqslant_e Y$;

(ii) $V\beta W$ if and only if $V \cap W \leqslant_e V$ and $V \cap W \leqslant_e W$.

It is simple to observe that U is (Goldie) extending if and only if for all submodule V of U, $V\alpha Y$ ($V\beta Y$), Y is a summand of U. A submodule X of U is regarded as z-closed in U if U/X is non-singular. C-closed submodule was studied by Kamil and Davvaz [6], Kamil [7], X is c-closed in U if whenever B/X is singular, then $B = X$, where B is a submodule of U. It is simple to verify that every "z-closed submodule" is c-closed. A module U is referred to as CLS (CCLS) module if all z-closed (c-closed) submodules of U is a summand, view Tercan [8]. In Tercan and Yasar [9], and Yucel [10], the authors used "z-closed submodules" to generalize G-extending and CLS, U is said to be G^z-extending if for every "z-closed submodule" V of U, there is a "direct summand" Y of U such that $V\beta Y$. If for each $f \in End(U)$, V contains $f(V)$, then we assert that V is "fully invariant" in U, when each submodule of U is "fully invariant", U is contacted duo-module, Kamil and Khalid [11], Kamil [12] and [13]. An additional valuable expansion of CS-modules is F.I- extending module, a module U is called F.I-extending if all "fully invariant submodules" of U is essential in a "direct summand", see, Birkenmeier, Muller and Rizvi [14]. In this paper, we search a submodule condition including z-closed property on the set of "fully invariant submodules". We call a submodule V of a module U is "fully invariant z-closed" (for short F.I-z-closed) submodule of U if V is "fully invariant and z-closed" in U, and U is called "fully invariant CLS", (for short F.I-CLS) module, if every F.I-z-closed submodule of U is a "direct summand" . In Section 2, we present the relationships between F.I- CLS module, extending, CCLS, CLS, G-extending, G^z-extending and F.I-extending conditions. Furthermore, we derive the fundamental characteristics and structural behavior of the F.I-extending module class. Section 3 is devoted to the characterizations of F.I-CLS modules and we study the decomposition theory of F.I-CLS. We address when a "direct summand of" F.I-CLS is also F.I-CLS.

2 Basic results

In this part, we obtain fundamental features of the F.I-z- closed submodules and we discuss relations between the F.I-CLS condition and several different refinements of extending idea. We begin with the subsequent lemma which is stated in, Birkenmeier, Muller and Rizvi [14].

Lemma 2.1. *Fix U be R-module. Then*

(i) Every intersection or sum of "fully invariant submodules" of U again is "fully invariant submodule" of U.

(ii) If $X \leqslant Y \leqslant U$ with X is "fully invariant submodule of Y" and Y is "fully invariant submodule" of U, then X is "fully invariant submodule" of U.

(iii) If $U = \bigoplus_{i=1}^{n} U_i$ and V is a "fully invariant submodule of U", then $V = \bigoplus_{i=1}^{n}(V \cap U_i)$, and $V \cap U_i$ is "fully invariant submodule" of U.

Definition 2.2. Let V be a "fully invariant submodule" of a module U, we say that V is a "fully invariant z-closed submodule" of U (for short F.I-z-closed), if U/V is nonsingular.

Demonstrating that each F.I-z-closed is a "z-closed submodule of U" is simple. When U is duo-module, the converse is true.

Next, we give some properties of this type of submodules.

Proposition 2.3. *Take U be an R-module.*

(i) If $X \leqslant Y$ and X is F.I- z-closed in U, then X is F.I-z-closed in Y.

(ii) If $X \leqslant Y$ and Y/X is F.I-z-closed in U/X, then Y is F.I-z-closed in U.

(iii) If $X \leqslant Y \leqslant U$ and X is F.I-z-closed in Y and Y is F.I-z-closed in U, then X is F.I-z-closed in U.

(iv) If X and Y are both F.I-z-closed submodules of U, then $X \cap Y$ is F.I-z-closed submodule of U.

(v) Let $U = U_1 \oplus U_2$ and let X and Y be F.I-z-closed submodules of U_1 and U_2, respectively, then $X \oplus Y$ is F.I-z-closed in U.

Proof. (i) Take X be F.I-z-closed in U. Since $Y/X \leqslant U/X$ is nonsingular, then X is F.I-z-closed in Y.

(ii) Since Y/X is F.I-z-closed in U/X, to show that Y is "fully invariant submodule" of U, let $f \in End(U)$, then $f(y + X) \in Y/X$, $y \in Y$ and hence $f(y) + X \in Y/X$ implies $f(y) \in Y$, therefore Y is "fully invariant" in U. It is simple to demonstrate that Y is "z-closed submodule" of U.

(iii) For an F.I-z-closed submodule X of Y and Y be an F.I-z-closed submodule of U, then X is "fully invariant in U", by Lemma 2.1. From Proposition 1.2 in Sahib and AL-Bahraany [15], X is z-closed in U. Thus, X is F.I-z-closed in U.

(iv) Let's say X and Y are F.I-z-closed submodules of U. By Lemma 2.1, $X \cap Y$ is "fully invariant" in U, and $X \cap Y$ is z-closed in U, by Sahib and AL-Bahraany (Proposition 1.1, [15]).

(v) Let $U = U_1 \oplus U_2$ and let X and Y be F.I-z-closed submodules of U_1 and U_2, respectively. Then U_1/X and U_2/Y are nonsingular, $(U_1/X) \oplus (U_2/Y)$ is nonsingular, and so $(U_1 \oplus U_2)/(X \oplus$

$Y)$ is nonsingular. By Lemma 2.1, $X \oplus Y$ is "fully invariant submodule of U". Thus, $X \oplus Y$ is F.I-z-closed in U. □

Corollary 2.4. *Let U be an R-module and let $\{Y_\alpha \mid \alpha \in \Lambda\}$ be an independent family of submodules of U and $X_\alpha \leqslant Y_\alpha$, for every $\alpha \in \Lambda$, if X_α is "fully invariant submodule" of Y_α,for every $\alpha \in \Lambda$, then $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is F.I-z-closed in $\bigoplus_{\alpha\in\Lambda} Y_\alpha$.*

There are submodules which are "fully invariant" but not F.I-z-closed, $n\mathbb{Z}$ is "fully invariant submodule" of $\mathbb{Z}$ as $\mathbb{Z}$-module while $n\mathbb{Z}$ is not z-closed in $\mathbb{Z}$.

Definition 2.5. A module U is claimed to be "fully invariant CLS-module" if every "fully invariant z-closed submodule" of U is a "direct summand" (for short F.I-CLS).

We now locate the F.I-CLS condition with regard to a number of the extending conditions known generalizations.

Proposition 2.6. *Given a module U, take consideration subsequence requirements.*

(i) U is extending.

(ii) U is G-extending.

(iii) U is CCLS.

(iv) U is CLS.

(v) U is G^z-extending.

(vi) U is F.I-CLS.

(vii) U is F.I-extending.

Then (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi) and (i)⇒(ii)⇒(vii)⇒(vi). The opposite implications are false, in general.

Proof. (i)⇒(ii): It is obvious what this implies.

(ii)⇒(iii): Let U be a G-extending module and let X be a c-closed submodule of U. There is a "direct summand" D of U such that $X \cap D$ is essential in each X and D. Note that, $D/(X \cap D) \cong (X + D)/X$ is singular. But X is c-closed in U, hence $X \cap D = D$, and so $X = D$. Thus, U is CCLS-module.

(iii)⇒(iv): It is Clear.

(iv)⇒(v): It follows from Tercan et al. [9].

(v)⇒(vi): Let X be an F.I-z-closed submodule of U, then there is a "direct summand" D of U such that $X \cap D \leqslant_e X$ and $X \cap D \leqslant_e D$, then $D/(X \cap D) \cong (D + X)/X \leqslant U/X$ is nonsinguar, which implies that $X = D$. Thus, U is F.I-CLS-module.

(ii)⇒(vii): It follows from Proposition 1.6 in Akalan et al. [5].

(vii)⇒(vi) Let X be an F.I-z-closed submodule of U, then X is essential in a "direct summand" D of U. But X is z-closed in U, therefore $X = D$.

(ii)⇏(i): Give $U = \mathbb{Q} \oplus \mathbb{Z}_p$, as $\mathbb{Z}$-module, p is a prime, U has G-extending property but not CS, see, Akalan et al. [5], Example 3.20.

(iv)⇏(iii): For a field $\mathbb{F}$ and a vector space V with $dim(V_\mathbb{F}) = 2$, take R be the trivial extension of $\mathbb{F}$ with V,

$$R = \begin{bmatrix} \mathbb{F} & V \\ 0 & \mathbb{F} \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} \mid f \in \mathbb{F},\ v \in V \right\}.$$

Then R_R is CLS and not CCLS see Tercan et al. [9], Proposition 2.3. This example also shows that (vi)⇏(vii), because R_R is CLS, R_R is F.I-CLS. Since R is commutative ring, then every ideal is "fully invariant" in R_R but not all ideals are essential in R_R. Thus, R_R not satisfy F.I-extending condition.

(v)⇏(iv): Looking to view of Tercan et al. (Proposition 2.3, [9]).

(vi)⇏(v): Consider a "2-by-2 upper triangular" matrix ring on integers, denoted by R, $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, then R_R is F.I-CLS. But R_R is not G^z-extending because it is not CS and nonsingular. This example also shows that (vii)⇏(ii). □

The diagram in Fig. 1 summarizes the relation between F.I-CLS and some generalizations of extending property.

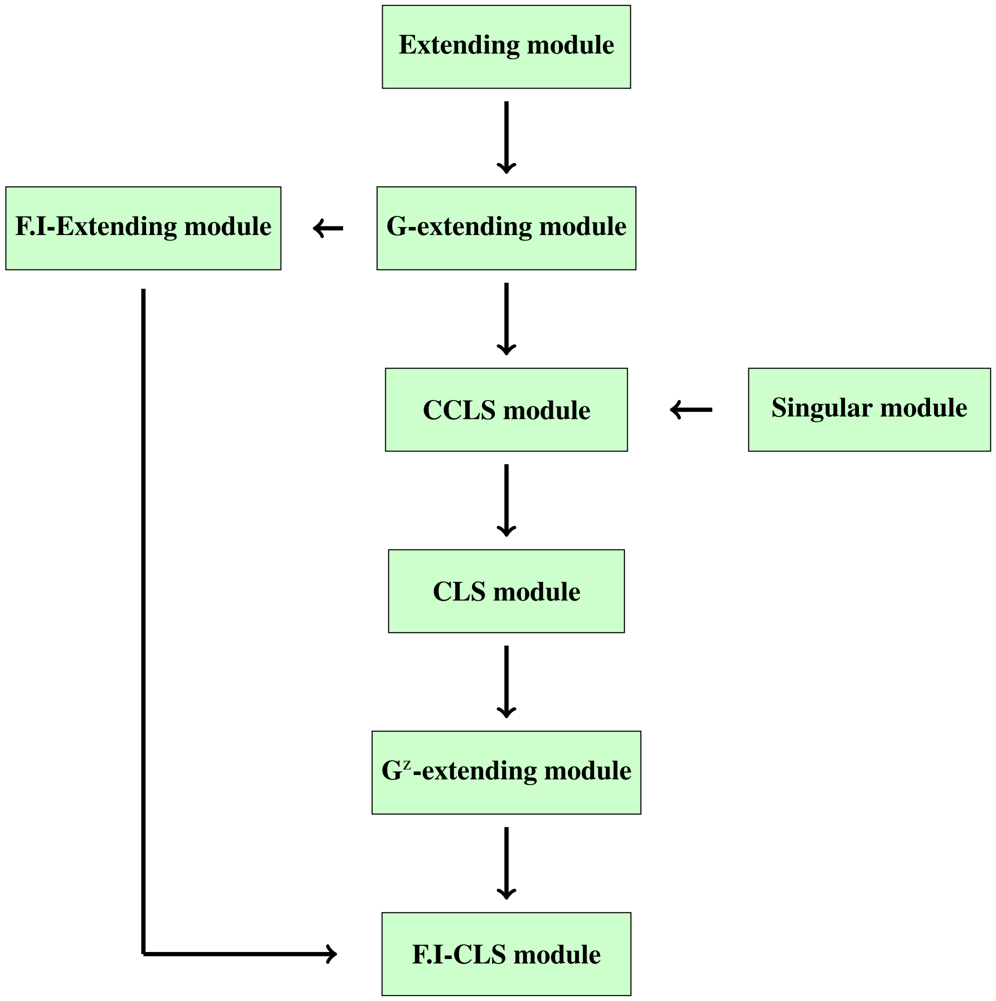

Figure 1. The relation between F.I-CLS and some generalizations of extending property

Proposition 2.7. *Let U be a module.*

(i) If U is a duo module, then U is CLS if and only if U is F.I-CLS.

(ii) If U is a nonsingular, then U is F.I-extending if and only if U is F.I-CLS.

(iii) If U is a complement bounded, then U is CLS if and only if U is F.I-CLS module.

Proof. (i) It is easy to check.

(ii) Assume that U is an F.I- CLS and let X is a “fully invariant submodule" of U, let Y be a closure of X, then $X \leqslant_e Y$. Since U is nonsingular, then Y is “z-closed submodule" of U. It is simple to demonstrate that Y is “fully invariant submodule" of U. But U is F.I-CLS, therefore Y is a “direct summand of" U. Thus, U is F.I-extending. The converse is in Proposition 2.6.

(iii) Let X be a z-closed in U. Since U is complement bounded, then X is an essential extension of a “fully invariant submodule" B of U, Akalan et al. [5]. One can easily prove that X is “fully invariant" in U. But U is F.I-CLS, therefore X is a “direct summand of" U. Thus, U is CLS. □

This section concludes with the following illustration.

Example 2.8. There exists a module that is not F.I-CLS module. Let V be a simple domain which is not division ring. Take $R = \begin{bmatrix} V & V \oplus V \\ 0 & V \end{bmatrix}$, by Akalan et al. ([5], Example 4.11, R_R is not F.I-extending. Since R_R is nonsingular, then R_R is not F.I-CLS, by Proposition 2.7.

3 Characterizations of F.I-CLS modules

In this part, We give several examples of situations that are similar to weak F.I- extending , Yasar et al. [10] modules in. We handle "direct summand of" F.I-CLS modules. It is widely acknowledged that each "direct summand of" CLS-module again CLS but in case of F.I-CLS, whether the "direct summand" of F.I.-CLS is F.I.-CLS is unknown. In this direction, we put some conditional "direct summand" properties on the module to give some affirmative answers to the inquiry. Also, characterizations of F.I-CLS are given in this section.

Proposition 3.1. *A module U is F.I- CLS if and only if every F.I-z-closed submodule of U is essential in a "direct summand of" U.*

Proof. The proof is routine. □

Proposition 3.2. *A module U is F.I-CLS if and only if for each submodule V of U, $V \alpha Y$, for some summand Y of U.*

Proposition 3.3. *An R-module U is F.I-CLS if and only if for every F.I-z-closed submodule V of U, there is a decomposition $U = U_1 \oplus U_2$ such that $V \leqslant U_1$ and U_2 is a complement of V in U.*

Proof. Let V be an F.I-z-closed submodule of U, then there is a decomposition $U = U_1 \oplus U_2$ such that $V \leqslant U_1$, U_2 is a complement of V in U, hence $V \oplus U_2 \leqslant_e U$, and $(U_1 \oplus U_2)/(V \oplus U_2) \cong U_1/V$ is singular. But V is "z-closed submodule" of U, therefore $V = U_1$. Thus, U is F.I-CLS. The reverse implication is clear. □

Proposition 3.4. *A module U is F.I-CLS if and only if for every F.I-z-closed submodule V of U, there is a complement W of V in U such that each homomorphism $f : V \oplus W \longrightarrow U$ able to lifts to a homomorphism $g : U \longrightarrow U$.*

Proof. This equivalency results directly from Smith and Tercan (Lemma 2, [16]). □

Proposition 3.5. *A module U is F.I-CLS if and only if for each F.I-z-closed submodule V of U, there exists $e^2 = e \in End(U)$ such that $V \leqslant_e eU$.*

Theorem 3.6. *For a module U, the statements that follows are equivalent.*

(i) U is F.I-CLS.

(ii) For every F.I-z-closed submodule V of U, there is a decomposition $U = U_1 \oplus U_2$ so that $V \leqslant U_1$ and $U_2 \oplus V \leqslant_e U$.

(iii) For every F.I-z-closed submodule V of U, there is a decomposition $U/V = (U_1/V) \oplus (K/V)$ such that U_1 is a "direct summand" of U and $K \leqslant_e U$.

Proof. (i)⇒(ii): By Proposition 3.3.

(ii)⇒(iii): Put V be F.I-z-closed submodule of U. By (ii), there is a decomposition $U = U_1 \oplus U_2$ such that $V \leqslant U_1$ and $U_2 \oplus V \leqslant_e U$. It is simple to verify $U/V = (U_1/V) \oplus (U_2 + V)/V$. Take $K = U_2 + V$, we get the result.

(iii)⇒(i): Let V be an F.I-z-closed submodule of U. By (iii), $U/V = (U_1/V) \oplus (K/V)$, U_1 is a summand of U and $U_1/(U_1 \cap K) \cong (U_1 + K)/K = U/K$ is singular. One can easily show that $U_1 \cap K = V$, hence U_1/V is singular, but V is z-closed in U, then $U_1 = V$. Hence, U is F.I-CLS. □

Proposition 3.7. *Every F.I-z-closed submodule of F.I-CLS is F.I-CLS.*

Proof. Let U be F.I-CLS, V be F.I-z-closed in U and X be F.I-z-closed in V. By Proposition 2.3, X is F.I-z-closed in U, but U is F.I-CLS, then X is "direct summand of" U, so X is "direct summand" of V. Thus, V is F.I-CLS. □

Proposition 3.8. *If the module $U = U_1 \oplus U_2$ has F.I-CLS property and U_1 is "fully invariant z-closed" in U then both U_1 and U_2 have F.I.CLS property.*

Proof. It is clear that U_1 is F.I-CLS. Let V be F.I-z-closed submodule of U_2. As U_1 is "fully invariant" in U, $Hom(U_1, U_2) = 0$, hence $U_1 \oplus V$ is "fully invariant submodule" of U. Now, $U/(U_1 \oplus V) = (U_1 \oplus U_2)/(U_1 \oplus V) \cong U_2/V$ is nonsingular, so $U_1 \oplus V$ is F.I-z-closed in U. But U is F.I-CLS, therefore $U_1 \oplus V$ is "direct summand of" U, hence V is a "direct summand of" U_2. □

Theorem 3.9. *A module $U = U_1 \oplus U_2$, U_1 is F.I-CLS if and only if there exists a "direct summand" W of U with $U_2 \leqslant W$, $W \cap C = 0$ and $W \oplus C \leqslant_e U$, for every "fully invariant z-closed submodule" C of U_1.*

Proof. Take U_1 be F.I-CLS and C be a "fully invariant z-closed submodule" of U_1, there is a "direct summand" L of U_1 such that $L \cap C = 0$, $L \oplus C \leqslant_e U_1$, by characterization 3.6. Clearly $L \oplus U_2$ is a "direct summand of" U, $(L \oplus U_2) \cap C = 0$ and $(L \oplus U_2) \oplus C \leqslant_e U$.

Conversely, Suppose that U_1 possesses the mentioned asset. Let S be a F.I-z-closed submodule of U_1, there exists a "direct summand" W of U such that $U_2 \leqslant W$, $W \cap S = 0$ and $W \oplus S \leqslant_e U$. Note that $W = W \cap (U_1 \oplus U_2) = U_2 \oplus (W \cap U_1)$, by modularity, so $W \cap U_1 \leqslant_\oplus W \leqslant_\oplus U$ and hence $W \cap U_1 \leqslant_\oplus U_1$, $S \cap (W \cap U_1) = 0$ and $S \oplus (W \cap U_1) = U_1 \cap (S \oplus W)$ which is essential in U_1. □

Theorem 3.10. *Any direct sum of modules having F.I-CLS property again is F.I-CLS module.*

Proof. For U_λ $(\lambda \in \Lambda)$ be a nonempty set of modules that have F.I-CLS and let $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$. Let S be a "fully invariant z-closed submodule" of U, then $S = S \cap U = S \cap (\bigoplus_{\lambda\in\Lambda} U_\lambda) = \bigoplus_{\lambda\in\Lambda}(S \cap U_\lambda)$ and each $S \cap U$ is "fully invariant", by 2.1. Note that $U_\lambda/(S \cap U_\lambda) \cong (S + U_\lambda)/S \leqslant U/S$ is nonsingular, hence $S \cap U_\lambda$ is "fully invariant z-closed submodule" of U_λ, for every $\lambda \in \Lambda$. Since U_λ is F.I-CLS, for every $\lambda \in \Lambda$, it follows that S is a "direct summand of" U. □

Corollary 3.11. *If U is a direct sum of CLS modules, then U is F.I-CLS.*

Example 3.12. The $\mathbb{Z}$-module $U = \mathbb{Z} \oplus Z_2$, whereas $Z_2 = \{a/b \mid a, b \in \mathbb{Z}$ and b is odd$\}$, both $\mathbb{Z}$ and Z_2 are CLS, hence U is F.I-CLS, while U is not CLS, see Tercan [8].

Proposition 3.13. *The quotient of F.I-CLS is F.I-CLS.*

Proof. Fix S is a submodule of F.I-CLS module U, let V/S be a "fully invariant z-closed submodule" of U/S, then V is "fully invariant z-closed submodule" of U, by Proposition 2.3. Since U is F.I-CLS, it follows that V is a "direct summand of" U. One can easily show that V/S is a summand of U/S. □

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