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# FIXED POINT OF T−NONEXPANSIVE TYPES MAPPINGS IN CAT(0) SPACES

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**Abstract** In this paper, we prove fixed point theorems for $T-$nonexpansive mappings and the mappings that satisfy condition $(TC)$ in $CAT(0)$ metric spaces.

## 1 Introduction

A metric space $X$ is said to be a $CAT(0)$ space [4] if it is geodesically connected, and if every geodesic triangle in $X$ is at least as "thin" as its comparison triangle in the Euclidean plane. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics [4, 5]. In [4] a very comprehensive exposition on $CAT(0)$ spaces is provided. $CAT(0)$ spaces share several good properties with Hilbert spaces, so it is not that surprising that certain results were originally obtained for Hilbert spaces and counterparts in $CAT(0)$ spaces. Fixed points on $CAT(0)$ spaces, or spaces of globally nonpositive curvature in the sense of Gromov, have been extensively studied in the last years by different authors [1, 2, 8, 9, 12, 15, 16, 17, 19]. One of the useful applications of the fixed point theorem in $CAT(0)$ spaces is in convex optimization [1].
In 2009, Beiranvand et al. [3] defined T-contraction mappings. The key point in $T-$contractions is that mapping like $f(x) = 2x$ (which does not have a Banach contraction property) can be composed with mapping like $Tx = \frac{1}{x}$, which $Tof$ has Banach contraction property. Since by taking $Tx = x$, $T-$contractions and contractions are equivalent, then $T-$contractions are vital improvement. Afterward, some results dealing with other types of fixed point theorems were proved for T-contractions [10, 13, 14, 20, 21, 22]. Similarly, in this work, we introduce $T-$nonexpansive and $T-$Suzuki-generalized nonexpansive mappings that are vital improvements of nonexpansive mappings.
It is natural to extend strong results for the fixed point theorem in $CAT(0)$. Another motivation for this research direction is the application of those results in various convex optimization problems.
The outline of the paper is as follows: In Section 2, we first define the conventions to be held throughout the paper and then define the consequent notions, concepts, and necessary results in the form of lemmas as required in the sequel. In section 3 we prove our main results. Section 4 is devoted to conclusion.

## 2 Some basic notations and definitions

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$ ) is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$ , and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric ) segment joining $x$ and $y$. When it is unique this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if

there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space $(X, d)$ consists of three points $x_1, x_2, x_3 \in X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. The comparison angle of the points $x, y, z$ in the $\triangle(x, y, z)$ at the point $x$ is the interior angle of any comparison triangle $\overline{\triangle}(x, y, z) = \triangle(\overline{x}, \overline{y}, \overline{z})$ at the vertex labelled $\overline{x} \in \mathbb{E}^2$. The comparison angle is denoted $\angle_x(y, z)$ and is well defined provided $x \neq y$ and $x \neq z$.
A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom.
$CAT(0)$ : Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\overline{\triangle}$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leqslant d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

Given two geodesic paths $c : [0, l] \longrightarrow X$ and $c' : [0, l] \longrightarrow X$ in a metric space $X$ originating from the same point $c(0) = c'(0)$, the Alexandrov angle between $c$ and $c'$ is defined to be:

$$\angle(c, c') := \lim_{\epsilon \to 0} \sup_{0 < t, t' < \varepsilon} \angle_{c(0)}(c(t), c'(t')).$$

In this expression, $\angle_{c(0)}(c(t), c'(t'))$ is the comparison angle for the triple of points $(c(0), c(t), c'(t'))$ as defined previously. Given geodesic segments $[x, y]$ and $[x, z]$ we write $\angle([x, y], [x, z])$ to denote the Alexandrov angle $\angle(c, c')$ where $c$ and $c'$ are geodesic paths whose corresponding geodesic segments are $[x, y]$ and $[x, z]$ respectively. Let $X$ be the $CAT(0)$ space. Consider two geodesic segments $[x, y]$ and $[z, w]$ in a metric space $X$, such that $z \in [x, y]$, $z \neq x$ and $z \neq y$. Then if $[x, z]$ and $[z, y]$ are geodesic segments such that $[x, z] \cup [z, y] = [x, y]$ it is easy to prove that $\angle([x, z], [z, w]) + \angle([z, w], [z, y]) \geqslant \pi$. It's mean that one of the angles $\angle([x, z], [z, w])$ or $\angle([z, w], [z, y])$ is greater or equal $\frac{\pi}{2}$.
Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, \{x_n\}).$$

The asymptotic radius $r(x, \{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $x_n$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [8] that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

**Lemma 2.1.** [7] *If $K$ is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in $K$, then the asymptotic center of $\{x_n\}$ is in $K$.*

**Lemma 2.2.** [6] *Let $(X, d)$ be a $CAT(0)$ space.*
*(i) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y).$$

*We use the notation $(1 - t)x \oplus ty$ for the unique point $z$ satisfying latest equalities.*
*(ii) For $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d((1 - t)x \oplus ty, z) \leqslant (1 - t)d(x, z) + td(y, z).$$

**Lemma 2.3.** [11] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a $CAT(0)$ space $X$ and let $\{\alpha_n\} \subseteq [0, 1)$ such that $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$ . Suppose that $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)x_n$ and $d(y_{n+1}, y_n) \leqslant d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} d(y_n, x_n) = 0$.*

**Definition 2.4.** [3] Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is said sequentially convergent if for every sequence $\{y_n\}$ such that $\{Ty_n\}$ be convergent, then $\{y_n\}$ also is convergence.

**Definition 2.5.** [3] Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is said subsequentially convergent if for every sequence $\{y_n\}$ such that $\{Ty_n\}$ is convergent, then $\{y_n\}$ has a convergent subsequence.

**Proposition 2.6.** [3] *If $(X,d)$ is a compact metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

**Definition 2.7.** Let $(X,d)$ be a metric space. Let $\mathcal{T}$ be the set of all functions $T : X \longrightarrow X$ are satisfying the following conditions:

(i) $T$ is subsequentially convergent;

(ii) $T$ is one-to-one;

(iii) $T$ is onto;

(iv) $T$ is continuous.

**Definition 2.8.** Let $K$ be a nonempty subset of a metric space $X$. A mapping $f : K \longrightarrow K$ is said to $T-$nonexpansive if for some $T \in \mathcal{T}$ we have

$$d(Tfx, Tfy) \leqslant d(Tx, Ty).$$

for all $x, y \in K$.

**Definition 2.9.** Let $K$ be a nonempty subset of a metric space $X$. A mapping $f : K \longrightarrow K$ is said to satisfy condition $(TC)$ (Suzuki-generalized $T-$nonexpansive) if for some $T \in \mathcal{T}$ we have

$$\frac{1}{2}d(Tx, Tfx) \leqslant d(Tx, Ty) \ \ implies \ \ d(Tfx, Tfy) \leqslant d(Tx, Ty).$$

for all $x, y \in K$.

**Definition 2.10.** Let $K$ be a nonempty subset of a metric space $X$. For a mapping $f : K \longrightarrow K$ and $\epsilon > 0$ the $\epsilon-$set of $(T, f)$ is the set

$$F_\epsilon(T, f) = \{x \in X : \ d(Tx, Tfx) < \epsilon\}.$$

## 3 MAIN RESULTS

**Lemma 3.1.** *Let $K$ be a bounded subset of a $CAT(0)$ space $X$, suppose $f : K \longrightarrow K$ is $T-$nonexpansive mappings for some $T \in \mathcal{T}$, and suppose $x, y \in F_\epsilon(T, f)$ with $d(Tx, Ty) = r$. Let $m \in [x, y] \cap K$. Then $Tf(m) \in F_{\phi(\epsilon)}(T, f)$, where $\phi(\epsilon) = \sqrt{\epsilon^2 + 2r\epsilon}$.*

*Proof.* Let $m \in K$ be the point of $[x, y]$ such that $d(Tx, Tm) = \alpha d(Tx, Ty)$. We have,

$$\begin{aligned} d(Tx, Tfm)) &\leqslant d(Tx, Tfx) + d(Tfx, Tfm) \\ &\leqslant d(Tx, Tfx) + d(Tx, Tm) \\ &\leqslant \epsilon + \alpha r. \end{aligned}$$

Similarly we can prove that

$$d(Ty, Tfm)) \leqslant \epsilon + (1 - \alpha)r.$$

At least one of the angels $\angle_{Tm}(Tfm, Tx)$ and $\angle_{Tm}(Tfm, Ty)$ is greater than or equal $\frac{\pi}{2}$. If $\angle_{Tm}(Tfm, Tx) \geqslant \frac{\pi}{2}$ then in the comparison triangle $\overline{\triangle}(Tm, Tx, Tfm)$ the angle at $\overline{Tm}$ is also greater than or equal $\frac{\pi}{2}$. By the law of cosines we have,

$$(\epsilon + \alpha r)^2 \geqslant d(Tx, Tfm)^2 \geqslant (\alpha r)^2 + d(Tm, Tfm)^2.$$

Similarly, if $\angle_{Tm}(Tfm, Ty) \geqslant \frac{\pi}{2}$ we have,

$$(\epsilon + (1-\alpha)r)^2 \geqslant d(Ty, Tfm)^2 \geqslant (1-\alpha r)^2 + d(Tm, Tfm)^2.$$

Therefore

$$\begin{aligned} d(Tm, Tfm)^2 &\leqslant \max\{\epsilon^2 + 2\alpha r\epsilon, \epsilon^2 + 2(1-\alpha)r\epsilon\} \\ &\leqslant \epsilon^2 + 2r\epsilon. \end{aligned}$$

□

**Theorem 3.2.** *Let $K$ be a nonempty subset of a $CAT(0)$ space $X$. Also, suppose*
*(i) $K$ is bounded closed convex subset of $X$,*
*(ii) $T : K \longrightarrow K$ and $T \in \mathcal{T}$;*
*(iii) $f : K \to K$ is an $T-$nonexpansive such that;*

$$\inf\{d(Tx, Tfx) : \ x \in K\} = 0.$$

*Then $f$ has a fixed point in $K$.*

*Proof.* Let $x_0 \in X$ be fixed. Define

$$\rho_0 = \inf\{\rho > 0 : \ \inf\{d(Tx, Tfx) : \ Tx \in B(Tx_0, \rho) \cap K\} = 0\},$$

and we have $\rho_0 < \infty$. If $\rho_0 = 0$ there exists sequence $\{x_n\}$ and $\rho_n$ such that $Tx_n \in B(Tx_0, \rho_n) \cap K$, $\rho_n \to 0$ and $d(Tx_n, Tfx_n) \to 0$. Since $T$ is subsequentially convergent, by passing to subsequence if necessary, $x_n$ has a subsequence convergent to $x_0$. Without losing the generality of the proof, assume that $x_n \to x_0$. Since $\{x_n\} \subseteq K$ then $x_0 \in K$. By the continuity of $T$ we have $d(Tx_0, Tfx_n) \to 0$. Therefore, $Tfx_n \to Tx_0$ as $n \to \infty$. Since $T$ is subsequentially convergent, by passing to subsequence if necessary, $fx_n$ has a subsequence convergent to $x_0$. Without losing the generality of the proof, assume that $fx_n \to x_0$. On the other hand, we have

$$d(Tfx_n, Tfx_0) \leqslant d(Tx_n, Tx_0).$$

Then we have $Tfx_n \to Tfx_0$. Since $T$ is subsequentially convergent, by passing to subsequence if necessary, $fx_n$ has a subsequence convergent to $fx_0$. Without losing the generality of the proof, assume that $fx_n \to fx_0$. So we have $x_0 = fx_0$.
Now we suppose that $\rho_0 > 0$. We choose $\{Tx_n\} \subseteq K$ such that

$$d(Tx_n, Tfx_n) \to 0, \qquad d(Tx_0, Tx_n) \to \rho_0, \qquad as \quad n \to \infty.$$

If $\{Tx_n\}$ has a convergent subsequence then similarly we can prove that $f$ has a fixed point. So we suppose there exist $\epsilon > 0$ and subsequences $\{u_k\}$ and $\{v_k\}$ of $\{x_n\}$ such that $d(Tu_k, Tv_k) \geqslant \epsilon$. By passing to a subsequence if necessary, we may also suppose $d(Tu_k, Tv_k) \leqslant \rho_0 + \frac{1}{k}$. So for any $k \in \mathbb{N}$ we have

$$\epsilon \leqslant d(Tu_k, Tv_k) \leqslant \rho_0 + \frac{1}{k}. \tag{3.1}$$

Let us to consider geodesic triangle $\triangle(Tx_0, Tu_k, Tv_k)$. Suppose that $z_k$ is the midpoint of the segment $[Tu_k, Tv_k]$. So there exist $m_k \in [u_k, v_k]$ such that $Tm_k = z_k$. Also suppose that $\overline{Tm_k}$ is the point corresponding to $Tm_k$ on the comparison triangle $\overline{\triangle}(Tx_0, Tu_k, Tv_k)$. Then by $CAT(0)$ inequality we have

$$d(Tx_0, Tm_k) \leqslant d(\overline{Tx_0}, \overline{Tm_k}) \leqslant \sqrt{(\rho_0 + \frac{1}{k})^2 - (\frac{\epsilon}{2})}. \tag{3.2}$$

Clearly $d(Tx_0, Tm_k) \leqslant \rho^* < \rho_0$ for $k$ sufficiently large. On the other hand, by Lemma 3.1, we have

$$d(Tm_k, Tfm_k) \leqslant \sqrt{\epsilon^2 + 2r\epsilon}.$$

That's mean, $d(Tm_k, Tfm_k) \to 0$ as $k \to \infty$. This contradicts the definition of $\rho_0$. □

**Example 3.3.** Let $X = \mathbb{R}$, $K = [\frac{1}{9}, 1]$. Let us define $f : K \longrightarrow K$ and $T : K \longrightarrow K$ as following

$$f(x) = \sqrt{x}, \qquad T(x) = \frac{1}{9x}.$$

It is easy to prove that $T \in \mathcal{T}$ and $f$ is $T-$nonexpanisive mapping. Also, it is easy to prove that $f$ is not nonexpansive mappings. Then all conditions of Theorem 3.2 are satisfied for $f$. $x = 1$ is a fixed point of $f$.

**Lemma 3.4.** *Let $X$ be a metric space, $T : X \longrightarrow X$ and $T \in \mathcal{T}$. Also suppose $\{x_n\}$ is a bounded sequence of $X$. Then*
*(i) $r(\{x_n\})$ is the asymptotic radius of $\{x_n\}$ if and only if $r(\{Tx_n\})$ is the asymptotic radius of $\{Tx_n\}$,*
*(ii) $A(\{x_n\})$ is the asymptotic center of $\{x_n\}$ if and only if $r(\{Tx_n\})$ is the asymptotic center of $\{Tx_n\}$.*

*Proof.* $(i)$ Let $\{x_{n_k}\}$ be a convergent sequence of $\{x_n\}$. By the continuity of $T$, $\{Tx_n\}$ is bounded and $Tx_{n_k}$ is convergent. If $\lim_{k\to\infty} x_{n_k} = x^*$, then $\lim_{k\to\infty} Tx_{n_k} = Tx^*$. On the other hand, if $Tx_n$ is convergent to $Tx^*$, since $T$ is subsequentially convergent, $\{x_n\}$ has a convergent subsequence to $x^*$. So we have

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n) \Longleftrightarrow r(Tx, \{Tx_n\}) = \limsup_{n\to\infty} d(Tx, Tx_n), \tag{3.3}$$

and the proof of $(i)$ is complete.
$(ii)$ It is easily proved by $(i)$ and the definition of asymptotic center. □

**Lemma 3.5.** *Let $K$ be a nonempty subset of a $CAT(0)$ space $X$ and suppose $f : K \longrightarrow K$ satisfies condition $(TC)$ for some $T \in \mathcal{T}$. Then for $x, y \in K$, the following hold:*
*(i) $d(Tfx, Tf^2x) \leqslant d(Tx, Tfx)$.*
*(ii) Either $\frac{1}{2}d(Tx, Tfx) \leqslant d(Tx, Ty)$ or $\frac{1}{2}d(Tfx, Tf^2x) \leqslant d(Tx, y)$ holds.*
*(iii) Either $d(Tfx, Tfy) \leqslant d(Tx, Ty)$ or $d(Tf^2x, Tfy) \leqslant d(Tfx, Ty)$ holds.*

*Proof.* $(i)$ Follows from $\frac{1}{2}d(Tx, Tfx) \leqslant d(Tx, Tfx)$ and the $(TC)$ property of $f$.
$(ii)$ Assume that

$$\frac{1}{2}d(Tx, Tfx) > d(Tx, Ty) \quad and \quad \frac{1}{2}d(Tfx, Tf^2x) > d(Tfx, fy).$$

Then we have by $(i)$,

$$\begin{aligned} d(Tx, Tfx) &\leqslant d(Tx, Ty) + d(Tfx, Ty) \\ &< \frac{1}{2}d(Tx, Tfx) + \frac{1}{2}d(Tfx, Tf^2x) \\ &\leqslant \frac{1}{2}d(Tx, Tfx) + \frac{1}{2}d(Tx, Tfx) \\ &\leqslant d(Tx, Tfx). \end{aligned}$$

This is a contradiction. Therefore we obtain the desired result. $(iii)$ Follows from $(ii)$. □

**Lemma 3.6.** *Let $K$ be a nonempty subset of a $CAT(0)$ space $X$. Suppose $f : K \longrightarrow K$ satisfies condition $(TC)$ for some $T \in \mathcal{T}$. Then*

$$d(Tx, Tfy) \leqslant 3d(Tfx, Tx) + d(Tx, Ty),$$

*holds for all $x, y \in K$.*

*Proof.* By Lemma 3.5, either

$$d(Tfx, Tfy) \leqslant d(Tx, Ty) \ \ or \ \ d(Tf^2x, Tfy) \leqslant d(Tfx, Ty),$$

holds. In the first case, we have

$$\begin{aligned} d(Tx, Tfy) &\leqslant d(Tx, Tfx) + d(Tfx, Tfy) \\ &\leqslant d(Tx, Tfx) + d(Tx, Ty) \\ &\leqslant 3d(Tx, Tfx) + d(Tx, Ty). \end{aligned}$$

In the second case, we have by Lemma 3.5

$$\begin{aligned} d(Tx, Tfy) &\leqslant d(Tx, Tfx) + d(Tfx, Tf^2x) + d(Tf^2x, Tfy) \\ &\leqslant d(Tx, Tfx) + d(Tfx, Tfx) + d(Tf^2x, Tfy) \\ &\leqslant 2d(Tx, Tfx) + d(Tfx, Tfy) + d(Tf^2x, Tfx) \\ &\leqslant 2d(Tx, Tfx) + d(Tfx, Tfy) + d(Tfx, Tx) \\ &\leqslant 3d(Tx, Tfx) + d(Tx, Ty). \end{aligned}$$

Therefore we obtain the desired result in both cases. □

**Lemma 3.7.** *Let $K$ be a nonempty bounded and convex subset of a complete $CAT(0)$ space $X$ and suppose $f : K \longrightarrow K$ satisfies condition $(TC)$ for some $T \in \mathcal{T}$. Define a sequence $\{x_n\}$ by $x_1 \in K$ and*

$$Tx_{n+1} = \alpha_n Tfx_n \oplus (1 - \alpha_n)Tx_n,$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [\frac{1}{2}, 1)$ such that* $\limsup_n \alpha_n < 1$. *Then*

$$\lim_{n\to\infty} d(Tfx_n, Tx_n) = 0.$$

*Proof.* It follows from Lemma 2.2 $(i)$ that

$$\frac{1}{2}d(Tx_n, Tfx_n) \leqslant \alpha_n d(Tx_n, Tfx_n) = d(Tx_n, Tx_{n+1}),$$

for all $n \in \mathbb{N}$. By condition $(TC)$, we have

$$d(Tfx_n, Tfx_{n+1}) \leqslant d(Tx_n, Tx_{n+1}),$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} d(Tfx_n, Tx_n) = 0$ by Lemma 2.3. □

**Theorem 3.8.** *Let $K$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. Suppose $f : K \longrightarrow K$ satisfies condition $(TC)$ for some $T \in \mathcal{T}$. Then $T$ has a fixed point in $K$.*

*Proof.* Let $x_1 \in K$. By Lemma 2.2 and since $T$ is onto, we can define sequence $\{x_n\}$ such that

$$Tx_{n+1} = \frac{1}{2}Tfx_n \oplus \frac{1}{2}Tx_n.$$

By Proposition 7 of [15], there exist $z \in X$ such that $A(\{x_n\}) = \{z\}$. By $(i)$ of Lemma 3.4, $A(\{x_n\}) = \{z\}$ if and only if $A(\{Tx_n\}) = \{Tz\}$. It follows from Lemma 2.1 that $z \in K$ and so $Tz \in K$. By Lemma 3.7 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and by Lemma 3.7, we have

$$d(Tx_n, Tfz) \leqslant 3d(Tfx_n, Tx_n) + d(Tx_n, Tz).$$

Taking lim sup on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty} d(x_n, Tfz) \leqslant \limsup_{n\to\infty} d(Tx_n, Tz).$$

That is

$$r(Tfz, \{Tx_n\}) \leqslant r(Tz, Tx_n).$$

By the uniqueness of asymptotic centers, we have $Tz = Tfz$. since $T$ is one to one we have $fz = z$. This completes the proof. □

## 4 Conclusion

By weakening condition nonexpansivity and considering the family of a certain class of functions, we define so called $T-$nonexpansive and the mappings with condition $(TC)$. For such mappings, some fixed point theorems are also proved. In this way, we obtain a very large class of operators which include not only some of the aforementioned-conditions but even mappings that need not be nonexpansive. Due to the authors' knowledge, fixed points of such operators have not been examined so far.

**Remark 4.1.** It is proposed to investigate the fixed point property for multivalued nonexpansive mappings ([24]) in $CAT(0)$ spaces. Another interesting topic could be the generalization of convergence theorems, as proved in [18] and [23] in $CAT(0)$ spaces.

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