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Eight and Sixteen Dimensional Seminormed Hopf Algebras

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Abstract *In this article, first we construct an eight dimensional seminormed algebra* $\mathbb{O}^l$ *and show that its elements preserve the norm relation* $\|XY\| = \|X\|\|Y\|$*. Then we construct a sixteen dimensional associative algebra* $\mathbb{S}^l$*, which is an even subalgebra of* 2^5*-dimensional Clifford algebra* $Cl_{5,0}$ *and show that its elements also preserve the norm relation. We also define the Hopf algebra structure on these algebras and show that the algebra* $\mathbb{O}^l$ *is a* $\mathbb{Z}_2^4/2$*-graded quasialgebra and* $\mathbb{S}^l$ *is a* $\mathbb{Z}_2^5/2$*-graded quasialgebra. Finally we give some applications of these algebras in number theory.*

1 Introduction

In quantum mechanics, the state of a system is described by a mathematical object called a quantum state vector, denoted by ψ*. This state vector encodes all the information about the system's properties and probabilities of various measurement outcomes and this is known as probabilistic interpretation of quantum theory. Jordan attempted to use the algebra of octonions and sedenions to transfer the probabilistic interpretation of quantum theory in eight and sixteen dimensions respectively, known as the exceptional Jordan problem [14]. Dirac [10] noticed that Jordan's attempt to obtain a generalized quantum theory in this manner was not successful because the nonassociative multiplication rule is not compatible with any group of transformations such as Poincare group. The associativity ensures that the Poincare groups operations on spacetime transformations are logically consistent and predictable, supporting the formulation of invariant physical laws and principles like conservation laws. The associativity guarantees that the order in which transformations are applied does not affect the final result. In this article, we construct an eight dimensional semi-normed division associative algebra. Due to associative property it is useful in exceptional Jordan type application to quantum theory. This algebra is different from the algebra of octonions because it is associative and have two real and six imaginary basis elements while the algebra of octonions is nonassociative and has one real and seven imaginary basis elements. Then we define two norms on this algebra and show that these norms satisfies the condition* $\|XY\| = \|X\|\|Y\|$*. Subsequently we construct a 16 dimensional associative algebra which is an even subalgebra of* 32*-dimensional Clifford algebra* $Cl_{5,0}$ *[1] and define a norm on it. This algebra is different from the algebra of sedenion as it is associative and has six real and ten imaginary basis elements, while the algebra of sedenions has one real and fifteen imaginary basis elements.*

Hopf algebras [17] provide the algebraic framework for defining and studying quantum groups, which have applications in mathematical physics [10], theoretical computer science and quantum information theory [15]. Albuquerque and Majid [1] introduced the concept of group graded quasialgebras, which involves reinterpreting certain significant non-associative *algebras as associative algebras in suitable tensor categories, which are algebraic structures on the direct sum of homology and cohomology groups on an H-space in algebraic topology. Linear Gr-category [11] is the category of finite group graded vector spaces, which has applications in tensor categories [4], quantum calculus [18], cohomology of groups and representation theory [16]. Balodi et al. [3] proved that every expression in a G-graded quasialgebra can be reduced to a unique irreducible form and the irreducible words form a basis for the quasialgebra, known as Diamond lemma. In this paper, we also construct the Hopf algebra [5] and group graded quasialgebra [2] structure on these algebras. We also provide some applications of these algebras to construct some new admissible triplets in number theory.*

Outline of the article is as follows: In section 2, we construct an eight-dimensional seminormed algebra $\mathbb{O}^l$, while the sixteen-dimensional seminormed division algebra $\mathbb{S}^l$ is constructed in section 3. The existence of inverse of basis elements and construction of Hopf algebra from eight and sixteen dimensional seminormed algebra is presented in section 4. Group graded quasialgebras, group graded quasialgebras structure of octonion like algebras are discussed in section 5. In section 6, we give a new series of examples of multiplicative pairs which is obtained from eight and sixteen dimensional seminormed division algebras.

Throughout this article, K denotes the field of characteristic zero, $K^* = K - \{0\}$, $\mathbb{O}^l$ is the eight dimensional seminormed algebra, $\mathbb{S}^l$ is the sixteen dimensional seminormed algebra.

2 Eight dimensional seminormed algebra $\mathbb{O}^l$

The eight dimensional seminormed division algebra is an algebra over $\mathbb{R}$ with basis $\{1 = u_0,\ u_1,\ u_2,\ u_3, u_4,\ u_5,\ u_6,\ u_7\}$ and complete multiplication table is given by:

$*$	*1*	u_1	u_2	u_3	u_4	u_5	u_6	u_7
1	*1*	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	u_1	-1	u_3	$-u_2$	$-u_5$	u_4	u_7	$-u_6$
u_2	u_2	$-u_3$	-1	u_1	u_6	u_7	$-u_4$	$-u_5$
u_3	u_3	u_2	$-u_1$	-1	u_7	$-u_6$	u_5	$-u_4$
u_4	u_4	u_5	$-u_6$	u_7	-1	$-u_1$	u_2	$-u_3$
u_5	u_5	$-u_4$	u_7	u_6	u_1	-1	$-u_3$	$-u_2$
u_6	u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	-1	$-u_1$
u_7	u_7	$-u_6$	$-u_5$	$-u_4$	$-u_3$	$-u_2$	$-u_1$	1

The above multiplication table is constructed by the following rules: Consider the algebra A over $\mathbb{R}$ generated by four elements $e_0,\ e_1,\ e_2,\ e_3$ with product given by the following relations:

$$e_i^2 = e_ie_i = 1 \text{ and } e_ie_j = -e_je_i\,,\ \text{for } i \neq j \in \{0,\ 1,\ 2,\ 3\}.$$

Thus there are sixteen basis elements in the algebra A, which are given as $\{1,\ e_0,\ e_1,\ e_2,\ e_3,\ e_0e_1,\ e_2e_0,\ e_0e_3,\ e_1e_2,\ e_3e_1,\ e_2e_3,\ e_0e_1e_2,\ e_0e_1e_3,\ e_0e_2e_3,\ e_1e_2e_3,\ e_0e_1e_2e_3\}$. An eight dimensional seminormed division algebra $\mathbb{O}^l$ is a subalgebra of A with basis $\{1,\ e_0e_1,\ e_2e_0, e_1e_2,\ e_0e_3,\ e_1e_3,\ e_2e_3,\ e_0e_1e_2e_3\}$. For simplicity writing $\{1,\ e_0e_1,\ e_2e_0,\ e_1e_2,\ e_0e_3,\ e_1e_3,\ e_2e_3,\ e_0e_1e_2e_3\}$ as $\{1 = u_0,\ u_1,\ u_2,\ u_3, u_4,\ u_5,\ u_6,\ u_7\}$ satisfying $u_i^2 = 1$, for $i = 0,7$ and $u_i^2 = -1$ for $i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$ which implies that there are two real and six imaginary basis elements, unlike the algebra of octonions [13], which has one real and seven imaginary basis elements. It is clear from multiplication table of $\mathbb{O}^l$ that it is associative and noncommutative. An element $X \in \mathbb{O}^l$ is written as the linear sum of all basis elements of $\mathbb{O}^l$, i.e.

$$X = x_ou_0 + x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 + x_5u_5 + x_6u_6 + x_7u_7.$$

Conjugate of an element $X \in \mathbb{O}^l$, denoted by $X^\dagger$, is defined by changing the sign of coefficients of imaginary basis elements i.e.

$$X^\dagger = x_ou_0 - x_1u_1 - x_2u_2 - x_3u_3 - x_4u_4 - x_5u_5 - x_6u_6 + x_7u_7$$

Let $X,\ Y \in \mathbb{O}^l$, where

$$X = x_o u_0 + x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 + x_5 u_5 + x_6 u_6 + x_7 u_7 \text{ and}$$
$$Y = y_o u_0 + y_1 u_1 + y_2 u_2 + y_3 u_3 + y_4 u_4 + y_5 u_5 + y_6 u_6 + y_7 u_7.$$

$$\begin{aligned}
\text{Then} \quad Z = XY &= (x_o u_0 + x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 + x_5 u_5 + x_6 u_6 + x_7 u_7) \\
&\quad (y_o u_0 + y_1 u_1 + y_2 u_2 + y_3 u_3 + y_4 u_4 + y_5 u_5 + y_6 u_6 + y_7 u_7) \\
&= (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 - x_5 y_5 - x_6 y_6 + x_7 y_7) u_0 \\
&\quad + (x_1 y_0 + x_0 y_1 - x_3 y_2 + x_2 y_3 + x_5 y_4 - x_4 y_5 - x_7 y_6 - x_6 y_7) u_1 \\
&\quad + (x_2 y_0 + x_3 y_1 + x_0 y_2 - x_1 y_3 - x_6 y_4 - x_7 y_5 + x_4 y_6 - x_5 y_7) u_2 \\
&\quad + (x_3 y_0 - x_2 y_1 + x_1 y_2 + x_0 y_3 - x_7 y_4 + x_6 y_5 - x_5 y_6 - x_4 y_7) u_3 \\
&\quad + (x_4 y_0 - x_5 y_1 + x_6 y_2 - x_7 y_3 + x_0 y_4 + x_1 y_5 - x_2 y_6 - x_3 y_7) u_4 \\
&\quad + (x_5 y_0 + x_4 y_1 - x_7 y_2 - x_6 y_3 - x_1 y_4 + x_0 y_5 + x_3 y_6 - x_2 y_7) u_5 \\
&\quad + (x_6 y_0 - x_7 y_1 - x_4 y_2 + x_5 y_3 + x_2 y_4 - x_3 y_5 + x_0 y_6 - x_1 y_7) u_6 \\
&\quad + (x_7 y_0 + x_6 y_1 + x_5 y_2 + x_4 y_3 + x_3 y_4 + x_2 y_5 + x_1 y_6 + x_0 y_7) u_7 \\
&= z_o u_0 + z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4 + z_5 u_5 + z_6 u_6 + z_7 u_7.
\end{aligned}$$

Also, the product $Z = XY$ can be written in the matrix form as $Z_r = M_x Y_r$, where M_x is the 8×8 *matrix obtained by taking all the coefficients from left side in the above multiplication given by:*

$$M_x = \begin{bmatrix}
x_0 & -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & x_7 \\
x_1 & x_0 & -x_3 & x_2 & x_5 & -x_4 & -x_7 & -x_6 \\
x_2 & x_3 & x_0 & -x_1 & -x_6 & -x_7 & +x_4 & -x_5 \\
x_3 & -x_2 & x_1 & x_0 & -x_7 & x_6 & -x_5 & -x_4 \\
x_4 & -x_5 & x_6 & -x_7 & x_0 & x_1 & -x_2 & -x_3 \\
x_5 & x_4 & -x_7 & -x_6 & -x_1 & +x_0 & +x_3 & -x_2 \\
x_6 & -x_7 & -x_4 & +x_5 & +x_2 & -x_3 & +x_0 & -x_1 \\
x_7 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & x_0
\end{bmatrix}.$$

Also Y_r and Z_r are the real coefficients of X and Y written in matrix form as:

$$Y_r = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{bmatrix}^T,$$

$$Z_r = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \end{bmatrix}^T.$$

Now we define two seminorms $\|X\|_1$ and $\|X\|_2$ on $\mathbb{O}^l$ as follows:

$$\|X\|_1^2 = x_0 x_0 + x_1 x_1 + x_2 x_2 + x_3 x_3 + x_4 x_4 + x_5 x_5 + x_6 x_6 + x_7 x_7 + x_7 x_0 - x_6 x_1 - x_5 x_2 - x_4 x_3 - x_3 x_4 - x_2 x_5 - x_1 x_6 + x_0 x_7 = \textstyle\sum_{i=0}^{7} x_i^2 - 2\sum_{i=1}^{3} x_1 x_{7-i} + 2x_0 x_7,$$

$$\|X\|_2^2 = x_0 x_0 + x_1 x_1 + x_2 x_2 + x_3 x_3 + x_4 x_4 + x_5 x_5 + x_6 x_6 + x_7 x_7 - x_7 x_0 + x_6 x_1 + x_5 x_2 + x_4 x_3 + x_3 x_4 + x_2 x_5 + x_1 x_6 - x_0 x_7 = \textstyle\sum_{i=0}^{7} x_i^2 + 2\sum_{i=1}^{3} x_1 x_{7-i} - 2x_0 x_7.$$

The above $\|X\|_1$ and $\|X\|_2$ are seminorms but not norms as $\|1 - e_7\|_1 = 0$ but $1 - e_7 \neq 0$, similarly $\|1 + e_7\|_2 = 0$ but $1 + e_7 \neq 0$. In Theorem 1, we prove that $\mathbb{O}^l$ is a seminormed algebra and Theorem 2 proves that $XX^\dagger$ is commutative in $\mathbb{O}^l$.

Theorem 2.1. *An element $X \in \mathbb{O}^l$ has inverse in $\mathbb{O}^l$ if $\|X\|_1 \neq 0$ and $\|X\|_2 \neq 0$.*

Proof. Let $X^{-1} = Y$ exists in $\mathbb{O}^l$. Then $XY = 1$ or in the matrix form, as defined above, it can be written as

$$M_x Y_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Hence inverse of $X \in \mathbb{O}^l$ exists only if the matrix M_x is non-singular. Also M_x is non-singular if and only if its all eigenvalues are nonzero. We calculate the eigenvalues of M_x using the following Matlab code:

$$\begin{array}{l}
\text{syms } x_0\ \ x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5\ \ x_6\ \ x_7\\
M_x = [x_0\ \ -x_1\ \ -x_2\ \ -x_3\ \ -x_4\ \ -x_5\ \ -x_6\ \ x_7;\\
\quad x_1\ \ x_0\ \ -x_3\ \ x_2\ \ x_5\ \ -x_4\ \ -x_7\ \ -x_6;\\
\quad x_2\ \ x_3\ \ x_0\ \ -x_1\ \ -x_6\ \ -x_7\ \ x_4\ \ -x_5;\\
\quad x_3\ \ -x_2\ \ x_1\ \ x_0\ \ -x_7\ \ x_6\ \ -x_5\ \ -x_4;\\
\quad x_4\ \ -x_5\ \ x_6\ \ -x_7\ \ x_0\ \ x_1\ \ -x_2\ \ -x_3;\\
\quad x_5\ \ x_4\ \ -x_7\ \ -x_6\ \ -x_1\ \ +x_0\ \ +x_3\ \ -x_2;\\
\quad x_6\ \ -x_7\ \ -x_4\ \ +x_5\ \ +x_2\ \ -x_3\ \ +x_0\ \ -x_1;\\
\quad x_7\ \ x_6\ \ x_5\ \ x_4\ \ x_3\ \ x_2\ \ x_1\ \ x_0]\\
\text{eig}(M_x),
\end{array}$$

which are given by

$$\lambda_0 = \lambda_1 = x_0 + x_7 + i\sqrt{-x_1^2 + 2x_1x_6 - x_2^2 + 2x_2x_5 - x_3^2 + 2x_3x_4 - x_4^2 - x_5^2 - x_6^2}$$
$$\lambda_2 = \lambda_3 = x_0 + x_7 - i\sqrt{-x_1^2 + 2x_1x_6 - x_2^2 + 2x_2x_5 - x_3^2 + 2x_3x_4 - x_4^2 - x_5^2 - x_6^2}$$
$$\lambda_4 = \lambda_5 = x_0 - x_7 + i\sqrt{-x_1^2 - 2x_1x_6 - x_2^2 - 2x_2x_5 - x_3^2 - 2x_3x_4 - x_4^2 - x_5^2 - x_6^2}$$
$$\lambda_6 = \lambda_7 = x_0 - x_7 - i\sqrt{-x_1^2 - 2x_1x_6 - x_2^2 - 2x_2x_5 - x_3^2 - 2x_3x_4 - x_4^2 - x_5^2 - x_6^2}$$

and the magnitude of eigenvalues is given by:

$$|\lambda_0|^2 = |\lambda_1|^2 = |\lambda_2|^2 = |\lambda_3|^2 = \sum_{i=0}^{7} x_i^2 - 2\sum_{i=1}^{3} x_1x_{7-i} + 2x_0x_7 = \|X\|_1,$$

$$|\lambda_4|^2 = |\lambda_5|^2 = |\lambda_6|^2 = |\lambda_7|^2 = \sum_{i=0}^{7} x_i^2 + 2\sum_{i=1}^{3} x_1x_{7-i} - 2x_0x_7 = \|X\|_2.$$

It is given that $\|X\|_1 \neq 0$ and $\|X\|_2 \neq 0$. Thus the magnitude of all eigenvalues are nonzero. Hence the inverse of $X \in \mathbb{O}^l$ exists. □

Theorem 2.2. *Let $X \in \mathbb{O}^l$. Then $XX^\dagger = X^\dagger X$ and $(XX^\dagger)Y = Y(XX^\dagger)$, for all $Y \in \mathbb{O}^l$.*

Proof. Let $X \in \mathbb{O}^l$. Then

$$\begin{aligned}
X &= x_ou_0 + x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 + x_5u_5 + x_6u_6 + x_7u_7,\\
X^\dagger &= x_ou_0 - x_1u_1 - x_2u_2 - x_3u_3 - x_4u_4 - x_5u_5 - x_6u_6 + x_7u_7.\\
\text{and } XX^\dagger &= (x_0x_0 + x_1x_1 + x_2x_2 + x_3x_3 + x_4x_4 + x_5x_5 + x_6x_6 + x_7x_7)u_0\\
&\quad + (x_1x_0 - x_0x_1 + x_3x_2 - x_2x_3 - x_5x_4 + x_4x_5 + x_7x_6 - x_6x_7)u_1\\
&\quad + (x_2x_0 - x_3x_1 - x_0x_2 + x_1x_3 + x_6x_4 + x_7x_5 - x_4x_6 - x_5x_7)u_2\\
&\quad + (x_3x_0 + x_2x_1 - x_1x_2 - x_0x_3 + x_7x_4 - x_6x_5 + x_5x_6 - x_4x_7)u_3\\
&\quad + (x_4x_0 + x_5x_1 - x_6x_2 + x_7x_3 - x_0x_4 - x_1x_5 + x_2x_6 - x_3x_7)u_4\\
&\quad + (x_5x_0 - x_4x_1 + x_7x_2 + x_6x_3 + x_1x_4 - x_0x_5 - x_3x_6 - x_2x_7)u_5\\
&\quad + (x_6x_0 + x_7x_1 + x_4x_2 - x_5x_3 - x_2x_4 + x_3x_5 - x_0x_6 - x_1x_7)u_6\\
&\quad + (x_7x_0 - x_6x_1 - x_5x_2 - x_4x_3 - x_3x_4 - x_2x_5 - x_1x_6 + x_0x_7)u_7.
\end{aligned}$$

We note that all the coefficients of imaginary basis elements are zero. Therefore,

$$\begin{aligned}
XX^\dagger &= (x_0x_0 + x_1x_1 + x_2x_2 + x_3x_3 + x_4x_4 + x_5x_5 + x_6x_6 + x_7x_7)\\
&\quad + (x_7x_0 - x_6x_1 - x_5x_2 - x_4x_3 - x_3x_4 - x_2x_5 - x_1x_6 + x_0x_7)u_7 = X^\dagger X.
\end{aligned}$$

It is clear from the multiplication table that u_0 and u_7 commute with all basis elements. Hence $(XX^\dagger)Y = Y(XX^\dagger)$. □

Now we show that $\|\cdot\|_1$ satisfies the relation $\|X \cdot Y\|_1 = \|X\|_1 \cdot \|Y\|_1$, which is useful in number theory for discovering admissible triplets [14].

Proposition 2.3. *The norm $\|\cdot\|_1$ satisfies $\|X \cdot Y\|_1 = \|X\|_1 \cdot \|Y\|_1$, for all $X,\ Y \in \mathbb{O}^l$.*

Proof. By definition of $\|\cdot\|_1$

$$\|X\|_1^2 = x_0x_0 + x_1x_1 + x_2x_2 + x_3x_3 + x_4x_4 + x_5x_5 + x_6x_6 + x_7x_7 + x_7x_0 \\ -x_6x_1 - x_5x_2 - x_4x_3 - x_3x_4 - x_2x_5 - x_1x_6 + x_0x_7. \\ = \sum_{i=0}^{7} x_i^2 - 2\sum_{i=1}^{3} x_1x_{7-i} + 2x_0x_7 = XX^\dagger.$$

Therefore

$$\|XY\|_1 = \sqrt{XY(XY)^\dagger} = \sqrt{XYY^\dagger X^\dagger} = \sqrt{X\|Y\|_1^2X^\dagger} = \sqrt{XX^\dagger\|Y\|_1^2} \\ = \sqrt{\|X\|_1^2\|Y\|_1^2} = \|X\|_1\|Y\|_1.$$

□

3 The 16-dimensional algebra $\mathbb{S}^l$

The sixteen dimensional seminormed algebra is an algebra over $\mathbb{R}$ with basis $\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$ and complete multiplication table is given by Table 1, which is obtained by the following rules: Consider the algebra A over $\mathbb{R}$ generated by $\{e_0, e_1, e_2, e_3, e_4\}$, with product given by the following relations:

$$e_i^2 = e_ie_i = 1 \text{ and } e_ie_j = -e_je_i\,,\ \text{for } i \neq j \in \{0,\ 1,\ 2,\ 3,\ 4\}.$$

The sixteen dimensional seminormed division algebra $\mathbb{S}^l$ is an even subalgebra of A, i.e.

$$\mathbb{S}^l = span\{1,\ e_0e_1,\ e_0e_2,\ e_0e_3,\ e_1e_2,\ e_3e_1,\ e_2e_3,\ e_0e_1e_2e_3,\ e_0e_4, \\ e_1e_4,\ e_2e_4,\ e_3e_4,\ e_0e_2e_1e_4,\ e_0e_1e_3e_4,\ e_0e_3e_2e_4,\ e_1e_2e_3e_4\}.$$

The algebra $\mathbb{S}^l$ has six $\{1,\ e_0e_1e_2e_3,\ e_oe_2e_1e_4, e_oe_1e_3e_4,\ e_oe_3e_2e_4,\ e_1e_2e_3e_4\}$- real basis elements and ten $\{e_oe_1,\ e_oe_2,\ e_oe_3,\ e_1e_2,\ e_3e_1, e_2e_3,\ e_oe_4,\ e_1e_4,\ e_2e_4,\ e_3e_4\}$ imaginary basis elements, while the algebra of sedenion has one real and fifteen imaginary basis elements. For simplicity writing $\{1,\ e_0e_1,\ e_0e_2,\ e_0e_3,\ e_1e_2,\ e_3e_1,\ e_2e_3,\ e_0e_1e_2e_3,\ e_0e_4,\ e_1e_4, e_2e_4,\ e_3e_4,\ e_0e_2e_1e_4,\ e_0e_1e_3e_4,\ e_0e_3e_2e_4,\ e_1e_2e_3e_4\}$ as $\{1 = u_0,\ u_1,\ u_2,\ u_3, u_4,\ u_5,\ u_6,\ u_7, u_8,\ u_9,\ u_{10}, u_{11},\ u_{12},\ u_{13},\ u_{14},\ u_{15}\}$ in the multiplication table of $\mathbb{S}^l$.

Proposition 3.1. *The algebra $\mathbb{S}^l$ is closed under multiplication.*

Proof. Let $S,\ T \in \mathbb{S}^l$. Then S and T can be written as

$$S = s_0 + s_1e_oe_1 + s_2e_oe_2 + s_3e_oe_3 + s_4e_1e_2 + s_5e_3e_1 + s_6e_2e_3 + s_7e_oe_1e_2e_3 + s_8e_oe_4 + s_9e_1e_4 + \\ s_{10}e_2e_4 + s_{11}e_3e_4 + s_{12}e_oe_2e_1e_4 + s_{13}e_oe_1e_3e_4 + s_{14}e_oe_3e_2e_4 + s_{15}e_1e_2e_3e_4,$$

$$T = t_0 + t_1e_oe_1 + t_2e_oe_2 + t_3e_oe_3 + t_4e_1e_2 + t_5e_3e_1 + t_6e_2e_3 + t_7e_oe_1e_2e_3 + t_8e_oe_4 + t_9e_1e_4 + \\ t_{10}e_2e_4 + t_{11}e_3e_4 + t_{12}e_oe_2e_1e_4 + t_{13}e_oe_1e_3e_4 + t_{14}e_oe_3e_2e_4 + t_{15}e_1e_2e_3e_4,$$

where $s_i,\ t_i \in K,\ \forall\ i = 0,\ 1,\ldots,15$. It is evident from the multiplication table that $\mathbb{S}^l$ is closed under multiplication, i.e. there exists $U = ST \in \mathbb{S}^l$ such that

$$U = u_0 + u_1e_oe_1 + u_2e_oe_2 + u_3e_oe_3 + u_4e_1e_2 + u_5e_3e_1 + u_6e_2e_3 + u_7e_oe_1e_2e_3 + u_8e_oe_4 + \\ u_9e_1e_4 + u_{10}e_2e_4 + u_{11}e_3e_4 + u_{12}e_oe_2e_1e_4 + u_{13}e_oe_1e_3e_4 + u_{14}e_oe_3e_2e_4 + u_{15}e_1e_2e_3e_4.$$

□

Clearly every $S \in \mathbb{S}^l$ can be written as dual of $\mathbb{O}^l$, i.e. $S = S_r + S_d\varepsilon$, where $S_r,\ S_d \in \mathbb{O}^l$ and $\varepsilon = -e_1e_2e_3e_4$, such that $\varepsilon^2 = 1$ and $\varepsilon^\dagger = \varepsilon$ († is the reverse operation defined in [8], [9]),

$$S_r = s_0 + s_1e_oe_1 + s_2e_oe_2 + s_3e_oe_3 + s_4e_1e_2 + s_5e_3e_1 + s_6e_2e_3 + s_7e_oe_1e_2e_3,$$

$$S_d = -s_{15} + s_{14}e_oe_1 + s_{13}e_oe_2 + s_{12}e_oe_3 + s_{11}e_1e_2 + s_{10}e_3e_1 + s_9e_2e_3 + s_8e_oe_1e_2e_3.$$

Hence $S_d\varepsilon = s_8e_oe_4 + s_9e_1e_4 + s_{10}e_2e_4 + s_{11}e_3e_4 + s_{12}e_oe_2e_1e_4 + s_{13}e_oe_1e_3e_4 + s_{14}e_oe_3e_2e_4 + s_{15}e_1e_2e_3e_4$.

$\cdot$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
u_1	-1	$-u_4$	u_5	u_2	$-u_3$	u_7	$-u_6$	$-u_9$	u_8	$-u_{12}$	u_{13}	u_{10}	$-u_{11}$	u_{15}	$-u_{14}$
u_2	u_4	-1	$-u_6$	$-u_1$	u_7	u_3	$-u_5$	$-u_{10}$	u_{12}	u_8	$-u_{14}$	$-u_9$	u_{15}	u_{11}	$-u_{13}$
u_3	$-u_5$	u_6	-1	u_7	u_1	$-u_2$	$-u_4$	$-u_{11}$	$-u_{13}$	u_{14}	u_8	u_{15}	u_9	$-u_{10}$	$-u_{12}$
u_4	$-u_2$	u_1	u_7	-1	u_6	$-u_5$	$-u_3$	$-u_{12}$	$-u_{10}$	u_9	u_{15}	u_8	u_{14}	$-u_{13}$	$-u_{11}$
u_5	u_3	u_7	$-u_1$	$-u_6$	-1	u_4	$-u_2$	$-u_{13}$	u_{11}	u_{15}	$-u_9$	$-u_{14}$	u_8	u_{12}	$-u_{10}$
u_6	u_7	$-u_3$	u_2	u_5	$-u_4$	-1	u_1	$-u_{14}$	u_{15}	$-u_{11}$	u_{10}	u_{13}	$-u_{12}$	u_8	$-u_9$
u_7	$-u_6$	$-u_5$	$-u_4$	$-u_3$	$-u_2$	$-u_1$	1	$-u_{15}$	$-u_{14}$	$-u_{13}$	$-u_{12}$	$-u_{11}$	$-u_{10}$	$-u_9$	$-u_8$
u_8	u_9	u_{10}	u_{11}	$-u_{12}$	$-u_{13}$	$-u_{14}$	u_{15}	-1	$-u_1$	$-u_2$	$-u_3$	u_4	u_5	u_6	$-u_7$
u_9	$-u_8$	u_{12}	$-u_{13}$	u_{10}	$-u_{11}$	u_{15}	u_{14}	u_1	-1	$-u_4$	u_5	$-u_2$	u_3	$-u_7$	$-u_6$
u_{10}	$-u_{12}$	$-u_8$	u_{14}	$-u_9$	u_{15}	u_{11}	u_{13}	u_2	u_4	-1	$-u_6$	u_1	$-u_7$	$-u_3$	$-u_5$
u_{11}	u_{13}	$-u_{14}$	$-u_8$	u_{15}	u_9	$-u_{10}$	u_{12}	u_3	$-u_5$	u_6	-1	$-u_7$	$-u_1$	u_2	$-u_4$
u_{12}	u_{10}	$-u_9$	$-u_{15}$	u_8	u_{14}	$-u_{13}$	u_{11}	u_4	$-u_2$	u_1	u_7	1	$-u_6$	u_5	$-u_3$
u_{13}	$-u_{11}$	$-u_{15}$	u_9	$-u_{14}$	u_8	u_{12}	u_{10}	u_5	u_3	u_7	$-u_1$	u_6	1	$-u_4$	$-u_2$
u_{14}	$-u_{15}$	u_{11}	$-u_{10}$	u_{13}	$-u_{12}$	u_8	u_9	u_6	u_7	$-u_3$	u_2	$-u_5$	u_4	1	$-u_1$
u_{15}	u_{14}	u_{13}	u_{12}	$-u_{11}$	$-u_{10}$	$-u_9$	u_8	u_7	$-u_6$	$-u_5$	$-u_4$	u_3	u_2	u_1	1

Multiplication table of the algebra $\mathbb{S}^l$.

4 Norm on $\mathbb{S}^l$

In this section, we define a norm on $\mathbb{S}^l$ and show that this norm preserves the condition $\|ST\| = \|S\|\|T\|$ and $S\cdot S^\dagger$ is commutative [i.e. $S\cdot S^\dagger = S^\dagger\cdot S$ and $(S\cdot S^\dagger)\cdot T = T(S\cdot S^\dagger)$], $\forall\ S,\ T\in\mathbb{S}^l$. If $S\in\mathbb{S}^l$, then norm of $S=\sum_{i=0}^{15} s_i^2$, which is equal to the square root of $SS^\dagger$ with non scalar of $SS^\dagger$ set to zero. Let $S = S_r + S_d\varepsilon$. Then $S^\dagger = (S_r+S_d\varepsilon)^\dagger = S_r^\dagger + S_d^\dagger\varepsilon$, since $\varepsilon^\dagger = \varepsilon$. Hence

$$\begin{aligned} SS^\dagger &= (S_r+S_d\varepsilon)(S_r^\dagger+S_d^\dagger\varepsilon)\\ &= (S_rS_r^\dagger + S_dS_d^\dagger) + (S_rS_d^\dagger + S_dS_r^\dagger)\varepsilon\\ &= (\|S_r\|^2+\|S_d\|^2) + (S_rS_d^\dagger + S_dS_r^\dagger)\varepsilon, \end{aligned}$$

which implies that $\|S\|^2 = \|S_r\|^2+\|S_d\|^2$, if S_r and S_d are orthogonal in $\mathbb{O}^l$ i.e. $S_rS_d^\dagger + S_dS_r^\dagger = 0$. Then we define $\mathbb{S}^l$ as

$$\mathbb{S}^l = \{S := S_r + S_d\varepsilon \mid \|S\|^2 = \|S_r\|^2+\|S_d\|^2\}.$$

Example 4.1. Let $S = 1+e_0e_1e_2e_3+e_0e_4+e_1e_2e_3e_4$. Then $S_r = 1+e_0e_1e_2e_3$ and $S_d = -1+e_0e_1e_2e_3$, since $(-1+e_0e_1e_2e_3)\cdot\varepsilon = (-1+e_0e_1e_2e_3)\cdot e_1e_2e_3e_4 = e_0e_4+e_1e_2e_3e_4$, which implies that $S = S_r+S_d\varepsilon$. Hence

$$\begin{aligned} S_r &= S_r^\dagger = 1+e_0e_1e_2e_3,\\ S_d &= S_d^\dagger = -1+e_0e_1e_2e_3, \text{ which implies that}\\ S_rS_d^\dagger + S_dS_r^\dagger &= (-1+e_0e_1e_2e_3 - e_0e_1e_2e_3+1)\\ &\quad +(-1-e_0e_1e_2e_3+e_0e_1e_2e_3+1) = 0. \end{aligned}$$

Now in order to use the condition $(S_rS_d^\dagger + S_dS_r^\dagger)\varepsilon = 0$, we must ensure that it is closed under multiplication.

Lemma 4.2. *The normed algebra $\mathbb{S}^l = \{S := S_r+S_d\varepsilon \mid \|S\|^2 = \|S_r\|^2+\|S_d\|^2\}$ is closed under multiplication.*

Proof. Let $S = S_r+S_d\varepsilon$ and $T = T_r+T_d\varepsilon$ are in $\mathbb{S}^l$. Then $(S_rS_d^\dagger+S_dS_r^\dagger) = 0$ and $(T_rT_d^\dagger+T_dT_r^\dagger) = 0$. Now we prove that $(ST)_r(ST)_d^\dagger + (ST)_d(ST)_r^\dagger = 0$. Here

$$\begin{aligned} ST &= (S_r+S_d\varepsilon)(T_r+T_d\varepsilon)\\ &= (S_rT_r+S_dT_d) + (S_rT_d+S_dT_r)\varepsilon\\ (ST)_r &= (S_rT_r+S_dT_d) \text{ and } (ST)_d = (S_rT_d+S_dT_r)\\ (ST)_r^\dagger &= (S_rT_r+S_dT_d)^\dagger = T_r^\dagger S_r^\dagger + T_d^\dagger S_d^\dagger\\ (ST)_d^\dagger &= (S_rT_d+S_dT_r)^\dagger = T_d^\dagger S_r^\dagger + S_d^\dagger T_r^\dagger. \end{aligned}$$

Therefore,

$$\begin{aligned} (ST)_r(ST)_d^\dagger + (ST)_d(ST)_r^\dagger &= (S_rT_r+S_dT_d)(T_d^\dagger S_r^\dagger + S_d^\dagger T_r^\dagger) + (S_rT_d+S_dT_r)(T_r^\dagger S_r^\dagger + T_d^\dagger S_d^\dagger)\\ &= S_rT_rT_d^\dagger S_r^\dagger + S_rT_rS_d^\dagger T_r^\dagger + S_dT_dT_d^\dagger S_r^\dagger + S_dT_dS_d^\dagger T_r^\dagger + S_rT_dT_r^\dagger S_r^\dagger\\ &\quad + S_rT_dT_d^\dagger S_d^\dagger + S_dT_rT_r^\dagger S_r^\dagger + S_dT_rT_d^\dagger S_d^\dagger\\ &= (S_rT_rT_d^\dagger S_r^\dagger + S_rT_dT_r^\dagger S_r^\dagger) + (S_rT_rS_d^\dagger T_r^\dagger + S_dT_rT_r^\dagger S_r^\dagger)\\ &\quad + (S_dT_dT_d^\dagger S_r^\dagger + S_rT_dT_d^\dagger S_d^\dagger) + (S_dT_dS_d^\dagger T_r^\dagger + S_dT_rT_d^\dagger S_d^\dagger)\\ &= S_rS_r^\dagger(T_rT_d^\dagger + T_dT_r^\dagger) + T_rT_r^\dagger(S_rS_d^\dagger + S_dS_r^\dagger)\\ &\quad + T_dT_d^\dagger(S_dS_r^\dagger + S_rS_d^\dagger) + S_dS_d^\dagger(T_dT_r^\dagger + T_rT_d^\dagger)\\ &= 0, \text{ for } (S_rS_d^\dagger + S_dS_r^\dagger) = 0 \text{ and } (T_rT_d^\dagger + T_dT_r^\dagger) = 0. \end{aligned}$$

Hence normed $\mathbb{S}^l$ is closed under multiplication. □

Definition 4.3. Let G be a group, KG be its group algebra and A,B are two subsets of G. Then (A,B) is called a multiplicative pair if it satisfies

$$\|ab\| = \|a\|\|b\|, \ \forall \ a \in \text{span}(A), \ b \in \ \text{span}(B).$$

The span of A is defined as $span(A) = \sum \alpha_i a_i, \ \forall \alpha_i \in K, \ a_i \in A$ and norm of an element $a = \sum \alpha_i a_i$ is given by $\|a\| = \sum \alpha_i^2$.

As an application, in order to find some admissible triplets on the algebra $\mathbb{S}^l$, it is very important to show that $\|ST\| = \|S\|\|T\|$.

Theorem 4.4. *The norm defined in (1) satisfies $\|ST\| = \|S\|\|T\|$, for all $S,T \in \mathbb{S}^l$.*

Proof. Since every number in $\mathbb{S}^l$ can be written as dual of $\mathbb{O}^l$, then we write S and T as dual of $\mathbb{O}^l$. Let

$$S = s_0 + s_1e_oe_1 + s_2e_oe_2 + s_3e_oe_3 + s_4e_1e_2 + s_5e_3e_1 + s_6e_2e_3 + s_7e_oe_1e_2e_3 + s_8e_oe_4 + s_9e_1e_4 + s_{10}e_2e_4 + s_{11}e_3e_4 + s_{12}e_oe_2e_1e_4 + s_{13}e_oe_1e_3e_4 + s_{14}e_oe_3e_2e_4 + s_{15}e_1e_2e_3e_4,$$

$$T = t_0 + t_1e_oe_1 + t_2e_oe_2 + t_3e_oe_3 + t_4e_1e_2 + t_5e_3e_1 + t_6e_2e_3 + t_7e_oe_1e_2e_3 + t_8e_oe_4 + t_9e_1e_4 + t_{10}e_2e_4 + t_{11}e_3e_4 + t_{12}e_oe_2e_1e_4 + t_{13}e_oe_1e_3e_4 + t_{14}e_oe_3e_2e_4 + t_{15}e_1e_2e_3e_4.$$

Then $S = S_r + S_d\varepsilon$ and $T = T_r + T_d\varepsilon$, where $S_r, \ S_d, \ T_r, \ T_d$ are elements of $\mathbb{O}^l$ given by

$$S_r = s_0 + s_1e_oe_1 + s_2e_oe_2 + s_3e_oe_3 + s_4e_1e_2 + s_5e_3e_1 + s_6e_2e_3 + s_7e_oe_1e_2e_3,$$

$$S_d = -s_{15} + s_{14}e_oe_1 + s_{13}e_oe_2 + s_{12}e_oe_3 + s_{11}e_1e_2 + s_{10}e_3e_1 + s_9e_2e_3 + s_8e_oe_1e_2e_3 \text{ and}$$

$$T_r = t_0 + t_1e_oe_1 + t_2e_oe_2 + t_3e_oe_3 + t_4e_1e_2 + t_5e_3e_1 + t_6e_2e_3 + t_7e_oe_1e_2e_3,$$

$$T_d = -t_{15} + t_{14}e_oe_1 + t_{13}e_oe_2 + t_{12}e_oe_3 + t_{11}e_1e_2 + t_{10}e_3e_1 + t_9e_2e_3 + t_8e_oe_1e_2e_3.$$

Therefore $ST = (S_r + S_d\varepsilon)(T_r + T_d\varepsilon) = (S_rT_r + S_dT_d) + (S_rT_d + S_dT_r)\varepsilon$,

$$(ST)^\dagger = ((S_rT_r + S_dT_d) + (S_rT_d + S_dT_r)\varepsilon)^\dagger = (S_rT_r + S_dT_d)^\dagger + (S_rT_d + S_dT_r)^\dagger\varepsilon$$

$$\begin{aligned}\|ST\|^2 &= (ST)(ST)^\dagger \\ &= \{(S_rT_r + S_dT_d) + (S_rT_d + S_dT_r)\varepsilon\}\{(S_rT_r + S_dT_d)^\dagger + (S_rT_d + S_dT_r)^\dagger\varepsilon\} \\ &= \{(S_rT_r + S_dT_d)(S_rT_r + S_dT_d)^\dagger + (S_rT_d + S_dT_r)(S_rT_d + S_dT_r)^\dagger\} \\ &\quad + \{(S_rT_r + S_dT_d)(S_rT_d + S_dT_r)^\dagger + (S_rT_d + S_dT_r)(S_rT_r + S_dT_d)^\dagger\}\varepsilon.\end{aligned}$$

In the above sum the coefficient of ε is zero by Lemma 2. Hence

$$\begin{aligned}\|ST\|^2 &= \{(S_rT_r + S_dT_d)(S_rT_r + S_dT_d)^\dagger + (S_rT_d + S_dT_r)(S_rT_d + S_dT_r)^\dagger\} \\ &= S_rT_rT_r^\dagger S_r^\dagger + S_rT_rT_d^\dagger S_d^\dagger + S_dT_dT_r^\dagger S_r^\dagger + S_dT_dT_d^\dagger S_d^\dagger + S_rT_dT_d^\dagger S_r^\dagger + S_rT_dT_r^\dagger S_d^\dagger \\ &\quad + T_rS_dT_d^\dagger S_r^\dagger + T_rS_dT_d^\dagger S_d^\dagger.\end{aligned}$$

In the above sum, second and sixth terms are cancelled as $(T_rT_d^\dagger + T_dT_r^\dagger) = 0$, third and seventh terms are cancelled as $(S_rS_d^\dagger + S_dS_r^\dagger) = 0$, which implies that

$$\begin{aligned}\|ST\|^2 &= S_rT_rT_r^\dagger S_r^\dagger + S_dT_dT_d^\dagger S_d^\dagger + S_rT_dT_d^\dagger S_r^\dagger + T_rS_dT_d^\dagger S_d^\dagger \\ &= \|S_r\|^2\|T_r\|^2 + \|S_d\|^2\|T_d\|^2 + \|S_r\|^2\|T_d\|^2 + \|S_d\|^2\|T_r\|^2.\end{aligned}$$

Hence $\|ST\| = \|S\|\|T\|$. □

It is clear from the multiplication table of $\mathbb{S}^l$ that $SS^\dagger$ for all $S \in \mathbb{S}^l$ contains only real basis elements, which are commutative. Hence $SS^\dagger = S^\dagger S$ and $(SS^\dagger)T = T(SS^\dagger)$.

5 Hopf algebra structure on the algebras $\mathbb{O}^l$ and $\mathbb{S}^l$

In this section we construct Hopf algebra structure on $\mathbb{O}^l$. A Hopf algebra is a six-tuple $(H,\ \mu,\ \eta,\ \triangle,\ \varepsilon,\ S)$, where H is a vector space over K, the product $\mu : H \otimes H \to H$ and unit $\eta : H \to k$ satisfy the commutativity of the following diagrams:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "id \otimes \mu"] \arrow[d, "\mu \otimes id"'] & H \otimes H \arrow[d, "\mu"] \\
H \otimes H \arrow[r, "\mu"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
K \otimes H \arrow[r, "\eta \otimes id"] \arrow[dr] & H \otimes H \arrow[d, "\mu"] & K \otimes H \arrow[l, "id \otimes \eta"'] \arrow[dl] \\
& H &
\end{tikzcd}$$

The coproduct $\triangle : H \to H \otimes H$ and counit $\varepsilon : K \to H$ are such that the following diagrams commutes.

$$\begin{tikzcd}
C \arrow[r, "\triangle"] \arrow[d, "\triangle"] & C \otimes C \arrow[d, "\triangle \otimes id"] \\
C \otimes C \arrow[r, "\triangle \otimes id"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & \mathbb{F} \otimes C \\
& C \arrow[ul] \arrow[u, "\triangle"] \arrow[ur] &
\end{tikzcd}$$

The antipode $S : H \to H$, satisfies the commutativity of following diagram:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] & A & A \otimes A \arrow[l, "\mu"'] \\
A \otimes A \arrow[u, "id \otimes S"] & A \arrow[l, "\triangle"'] \arrow[r, "\triangle"] \arrow[u, "\eta \otimes \varepsilon"] & A \otimes A \arrow[u, "S \otimes id"']
\end{tikzcd}$$

Lemma 5.1, Lemma 5.2 gives the existence of inverse of basis elements in $\mathbb{O}^l$, $\mathbb{S}^l$ respectively, which is used in the existence of antipode on $\mathbb{O}^l$ and $\mathbb{S}^l$.

Lemma 5.1. *All basis elements $u_i,\ i = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ in $\mathbb{O}^l$ have unique inverse $\mathbb{O}^l$.*

Proof. Since for basis elements $u_i,\ i = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, \|u_i\|_1 = 1$ and $\|u_i\|_2 = 1$. Therefore both norms are non-zero for all basis elements u_i, hence inverse of u_i exists, by Theorem 2.1. □

Lemma 5.2. *All basis elements in $\mathbb{S}^l$ have unique inverse $\mathbb{S}^l$.*

Proof. Since all real basis elements $u_i,\ i = 0,\ 7,\ 12,\ 13,\ 14,\ 15$ satisfy $u_i^2 = 1$ and all imaginary basis elements $v_j,\ j = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11$ satisfy $v_i^2 = -1$. Therefore $u_i^{-1} = u_i$ and $v_j^{-1} = v_j$. □

Theorem 5.3. *The algebras $\mathbb{O}^l$ and $\mathbb{S}^l$ are Hopf algebras with suitable coproduct, counit and antipode.*

Proof. The algebra $\mathbb{O}^l$ is closed under multiplication and has basis elements u_i as given in the multiplication table. Define the coproduct $\triangle(x) : \mathbb{O}^l \to \mathbb{O}^l \otimes \mathbb{O}^l$ and counit $\varepsilon : \mathbb{O}^l \to \mathbb{R}$ by

$$\triangle(x) = \triangle(\textstyle\sum a_i u_i) = \sum a_i (u_i \otimes u_i). \qquad \varepsilon(u_i) = 1,$$

If $x = \sum a_i u_i \in \mathbb{O}^l$, then $(id \otimes \triangle) \triangle (x) = (id \otimes \triangle) \triangle (\sum a_i u_i) = (id \otimes \triangle)(\sum a_i (u_i \otimes u_i)) = \sum a_i (u_i \otimes (u_i \otimes u_i)) = a_i ((u_i \otimes u_i) \otimes u_i) = (\triangle \otimes id)(a_i (u_i \otimes u_i)) = (\triangle \otimes id) \triangle (\sum a_i u_i) = (\triangle \otimes id) \triangle (x)$. Therefore $(id \otimes \triangle) \triangle (u_i) = (\triangle \otimes id) \triangle (u_i)$ and $(\varepsilon \otimes id) \triangle (x) = (\varepsilon \otimes id) \triangle (\sum a_i u_i)(\varepsilon \otimes id)(a_i (u_i \otimes u_i)) = a_i (1 \otimes u_i)$. Similarly $(id \otimes \varepsilon) \triangle (x) = (id \otimes \varepsilon) \triangle (\sum a_i u_i) = (id \otimes \varepsilon)(a_i (u_i \otimes u_i)) = a_i (u_i \otimes 1)$. Since $u_1 \otimes 1 = 1 \otimes u_1 \in \mathbb{O}^l$ implies that $(\varepsilon \otimes id)\triangle = (id \otimes \varepsilon)\triangle$ which proves that it is a coalgebra. Define the antipode S as

$$S(x) = S(\sum a_i u_i) = \sum a_i u_i^{-1},$$

for all $x \in \mathbb{O}^l$, where $x = \sum a_i u_i$ and u_i^{-1} (inverse of u_i) exists by Lemma 5.1 . Now we prove that $id * S = S * id$ for basis elements of $\mathbb{O}^l$ and apply linearity to get the result for general elements. For any basis element $u_i \in \mathbb{O}^l$,

$$\begin{aligned} id * S(u_i) &= \mu \circ (id \otimes S) \circ \triangle (u_i) = \mu \circ (id \otimes S)(u_i \otimes u_i) = \mu (u_i \otimes u_i^{-1}) = 1 = \eta \circ \varepsilon (u_i), \\ S * id(u_i) &= \mu \circ (S \otimes id) \circ \triangle (u_i) = \mu \circ (S \otimes id)(u_i \otimes u_i) = \mu (u_i^{-1} \otimes u_i) = 1 = \eta \circ \varepsilon (u_i). \end{aligned}$$

Thus $id * S = S * id = \eta \circ \varepsilon$. For the algebra $\mathbb{S}^l$ define $\triangle$, ε, S same as for the algebra $\mathbb{O}^l$. The existence of inverse of basis elements are given in Lemma 5.2. Hence $\mathbb{O}^l$ and $\mathbb{S}^l$ are Hopf algebras. □

6 $\mathbb{Z}_2^n$-Graded Quasialgebra structure on the algebras $\mathbb{O}^l$ and $\mathbb{S}^l$

Group graded quasialgebra [12] structure on $\mathbb{O}^l$ helps to find some multiplicative pairs on it. Therefore it is useful to construct the group graded quasialgebra structure on $\mathbb{O}^l$. A G-graded vector space means a vector space V which can be written as $V = \oplus_{g \in G} V_g$ and $V_g \cdot V_h \subseteq V_{gh}$, for all $g,\ h \in G$, the elements of V_g are called homogeneous elements of degree g, denoted by $|v|$. A normalized 3-cocycle on G is a map $\phi : G \otimes G \otimes G \to K^*$ satisfying the following conditions:

$$\phi(ab,c,d)\phi(a,b,cd) = \phi(a,b,c)\phi(a,bc,d)\phi(b,c,d)$$
$$\phi(a,e,b) = 1, \ \ \forall \ \ a,b,c,d \in G.$$

A G-graded quasialgebra is a G-graded vector space V, a product map $V \otimes V \to V$ preserving the total degree and associativity in the sense that

$$(u \cdot v) \cdot w = u \cdot (v \cdot w)\phi(|u|,|v|,|w|), \ \forall \ u,v,w \in V.$$

Let $x = (x_1,\ x_2,\ \ldots.\ x_n) \in \mathbb{Z}_2^n$, define a group homomorphism $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ by $f(x) = \sum_{i=1}^n x_i$. Then $Ker(f)$ is a subgroup of $\mathbb{Z}_2^n$, denoted by $\mathbb{Z}_2^n/2$ and called the even subgroup of $\mathbb{Z}_2^n$.

Example 6.1. An even subalgebra of $\mathbb{Z}_2^4$ is $\mathbb{Z}_2^4/2 = \{(0,0,0,0),\ (0,0,1,1),\ (0,1,0,1),\ (0,1,1,0), (1,0,0,1), (1,0,1,0), (1,1,0,0), (1,1,1,1)\}$. Define $\mathbb{Z}_2^4/2$-grading on the basis elements $\{1,\ e_0e_1,\ e_0e_2,\ e_0e_3,\ e_1e_2,\ e_2e_3,\ e_3e_1, e_0e_1e_2e_3\}$ of the algebra $\mathbb{O}^l$ as follows: $|e_ie_je_ke_l| = (a_3,\ a_2,\ a_1,\ a_0)$, where $a_m = 1$ if $m = i,\ j,\ k,\ l$, otherwise $a_m = 0$. Therefore we have

$$\begin{aligned} |1| &= (0,0,0,0); & |e_0e_1| &= (0,0,1,1); \\ |e_0e_2| &= (0,1,0,1); & |e_1e_2| &= (0,1,1,0); \\ |e_0e_3| &= (1,0,0,1); & |e_3e_1| &= (1,0,1,0); \\ |e_2e_3| &= (1,1,0,0); & |e_1e_2e_3e_4| &= (1,\ 1,\ 1,\ 1). \end{aligned}$$

If $u_iu_j = u_k$, then clearly $|u_i| + |u_j| = |u_k|$, for all $u_i,u_j,u_k \in \{1,\ e_0e_1,\ e_0e_2,\ e_0e_3, e_1e_2, e_2e_3, e_3e_1, e_0e_1e_2e_3\}$. Define

$$\phi : \mathbb{Z}_2^4/2 \times \mathbb{Z}_2^4/2 \times \mathbb{Z}_2^4/2 \to K^* \text{ by } \phi(x,y,z) = 1, \text{ for all } x,y,z \in \mathbb{Z}_2^4/2.$$

For $u_i \in \mathbb{O}^l$, $|u_i| \in \mathbb{Z}_2^4/2$, then $(u_iu_j)u_k = u_i(u_ju_k)\phi(|u_i|,|u_j|,|u_k|)$. Hence $\mathbb{O}^l$ is a $\mathbb{Z}_2^4/2$-graded quasialgebra.

Example 6.2. $\mathbb{Z}_2^5/2$-grading on the basis elements of the algebra $\mathbb{S}^l$ as follows: $|e_ie_je_ke_le_m| = (a_4,\ a_3,\ a_2,\ a_1,\ a_0)$, where $a_n = 1$ if $n = i,\ j,\ k,\ l,\ m$, otherwise $a_n = 0$. Therefore we have

$$|1| = (0,0,0,0,0); \qquad |e_0e_1| = (0,0,0,1,1);$$
$$|e_0e_2| = (0,0,1,0,1); \qquad |e_0e_3| = (0,1,0,0,1);$$
$$|e_1e_2| = (0,0,1,1,0); \qquad |e_3e_1| = (0,1,0,1,0);$$
$$|e_0e_4| = (1,0,0,0,1); \qquad |e_1e_4| = (1,0,0,1,0);$$
$$|e_2e_4| = (1,0,1,0,0); \qquad |e_3e_4| = (1,1,0,0,0);$$
$$|e_2e_3| = (0,1,1,0,0); \qquad |e_0e_1e_2e_3| = (0,1,1,1,1);$$
$$|e_0e_2e_1e_4| = (1,0,1,1,1); \qquad |e_0e_1e_3e_4| = (1,1,0,1,1);$$
$$|e_0e_3e_2e_4| = (1,1,1,0,1); \qquad |e_1e_2e_3e_4| = (1,1,1,1,0),$$

Clearly, for $u_i,\ u_j,\ u_k\ \in\ \{\ 1,\ e_0e_1,\ e_0e_2,\ e_0e_3,\ e_1e_2,\ e_3e_1,\ e_2e_3,\ e_0e_4,\ e_1e_4,\ e_2e_4,\ e_3e_4,$ $e_0e_1e_2e_3,\ e_0e_2e_1e_4,\ e_0e_1e_3e_4,\ e_0e_3e_2e_4,\ e_1e_2e_3e_4,\}$ the above grading satisfies that if $u_iu_j = u_k$, then $|u_i|+|u_j| = |u_k|$. Define $\phi : \mathbb{Z}_2^5/2\times\mathbb{Z}_2^5/2\times\mathbb{Z}_2^5/2 \to K^*$ by $\phi(x,y,z) = 1$ for all $x,y,z \in \mathbb{Z}_2^5/2$. Since $\mathbb{S}^l$ is associative, therefore $\phi = 1$ preserve the condition

$$(u_iu_j)u_k = u_i(u_ju_k)\phi(|u_i|,|u_j|,|u_k|),\ \ \forall\ u_i,\ u_j,\ u_k\ \in\ \mathbb{S}^l.$$

Hence $\mathbb{S}^l$ is a $\mathbb{Z}_2^5$-graded quasialgebra.

7 Application

*In number theory, Hurwitz problem [14] asks for the description of all admissible triples of positive integers [r, s, N], in the sense that there exists a sum of squares formula of the type $(a_1^2+a_2^2+...+a_r^2)(b_1^2+b_2^2+...+b_s^2) = c_1^2+c_2^2+...+c_N^2$, where $a = (a_1,a_2,...,a_r)$ and $b = (b_1,b_2,...,b_s)$. Also each c_k is a linear combination of a_ib_j with coefficients 1 or -1. The identity $(x_1^2+x_2^2+...x_n^2)\cdot(y_1^2+y_2^2+...y_n^2) = (z_1^2+z_2^2+...z_n^2)$ is known as the n-square identity. An admissible triplet of size $[n,\ n,\ n]$ is known as n square identity. Hurwitz found **1, 2, 4, 8-square identities** by using the Yuzvinsky's novel method [19] on the algebra of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ respectively. As an application, we apply the Yuzvinsky's novel method on the algebra $\mathbb{S}^l$ to get some admissible triplets. First we describe the Yuzvinsky method to find square identities. Let V be a $\mathbb{Z}_2^n$ graded algebra. Then an element $u \in V$ can be written as $u = \sum_{x\in\mathbb{Z}_2^n} a_xu_x$, where u_x denotes the x degree element of V. A pair of subsets (A,B) of $\mathbb{Z}_2^n$ is called a multiplicative pair if it satisfies*

$$\|a\|\cdot\|b\| = \|a\cdot b\|,\ \forall\ a = \sum_{x\in A} a_xu_x,\ b = \sum_{y\in B} a_yu_y. \tag{7.1}$$

The idea of Yuzvinsky is that there exists an admissible triplet of size $[r,\ s,\ N]$ corresponding to a multiplicative pair $(A,\ B)$, where $r =$ card(A), $s =$ card(B) and $N =$ card$(A+B)$. This square identity is given by

$$\left(\sum_{x\in A} a_x^2\right)\cdot\left(\sum_{y\in B} b_y^2\right) = \sum_{z\in A+B} c_z^2, \quad where \quad c_z = \sum_{z=x+y} Sign(u_x,u_y)a_xb_y, \tag{7.2}$$

where $A+B = \{a+b \mid a\in A,\ b\in B\}$ and $Sign(u_x,u_y)$ is defined as follows:

$$Sign(u_x,u_y) = (-1)^{\sum_i x_iy_i+\sum_{i<j}x_iy_j},\ for\ all\ x = (x_1,x_2,...x_n), y = (y_1,y_2,...y_n) \in \mathbb{Z}_2^n.$$

Example 7.1. Let $A = B = \{(0,0,0,0,0),\ (0,0,0,1,1),\ (0,0,1,0,1),\ (0,0,1,1,0)\}$ are two subsets of $\mathbb{Z}_2^5/2$. Since by theorem 4.4, A and B satisfies the equation (2). Therefore by applying Yuzvinsky method on A and B, we get an admissible triplet of size $[4,\ 4,\ 4]$ as card$(A) = 4$, card$(B) = 4$ and card$(A+B) = 4$. Hence we obtain the well known square identity given by:

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)\cdot(b_1^2+b_2^2+b_3^2+b_4^2) = {} & (a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2\\ & +(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2\\ & +(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2\\ & +(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2.\end{aligned}$$

Example 7.2. Let $A = \{(0,0,0,0,0),\ (0,0,0,1,1),\ (0,0,1,0,1),\ (0,0,1,1,0)\}$ and $B = \{(0,0,0,0,0),\ (0,0,0,1,1),\ (0,1,0,0,1),\ (0,1,0,1,0)\}$ are two subsets of $\mathbb{Z}_2^5/2$. Clearly A and B satisfy the equation (2), $card(A) = card(B) = 4$ and $card(A+B) = 14$. Now by applying the Yuzvinsky method on A and B we get an admissible triplet of size $[4,\ 4,\ 14]$, which is given as follows:

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)\cdot(b_1^2+b_2^2+b_3^2+b_4^2) = {} & (a_1b_1-a_2b_2)^2+(a_1b_2+a_2b_1)^2+(a_1b_3)^2+ \\ & (a_1b_4)^2+(a_2b_3)^2+(a_2b_4)^2+(a_3b_1)^2+(a_3b_2)^2+(a_3b_3)^2+ \\ & (a_3b_4)^2+(a_4b_1)^2+(a_4b_2)^2+(a_4b_1)^2+(a_4b_2)^2.\end{aligned}$$

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