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Anisotropic singular elliptic problems with natural growth terms

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Abstract In this paper, we prove the existence of finite energy and bounded solutions for nonlinear anisotropic elliptic problem whose prototype is

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left[|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u\right] + u\sum_{i=1}^{N}|\partial_{x_i} u|^{p_i} = fh(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary, $1 < p_i$ for all $i = 1, \ldots, N$ and satisfies $1 < \overline{p} < N$, the singular term h is a continuous real function that could blow up at the origin. We show that the presence of lower-order terms has a regularizing effect on the solutions for a nonnegative data $f \in L^\theta(\Omega)$ where $\theta > \frac{N}{\overline{p}}$ or $\theta = 1$.

1 Introduction and preliminaries

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded smooth domain, and denote $\partial_{x_i} u = \frac{\partial u}{\partial x_i}$ for all $i = 1, \ldots, N$. In this paper, we deal with the problem of the form

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left[a_i(x, \nabla u)\right] + \sum_{i=1}^{N} g_i(x, u, \nabla u) = fh(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the exponents $p_1, p_2, \cdots, p_N$ are restricted as follows:

$$\begin{cases} p^- = \min_{1\leq i\leq N}\{p_i\},\ p^+ = \max_{1\leq i\leq N}\{p_i\}, \\ 1 < \overline{p} < N, \qquad \overline{p} = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}\right)^{-1}, \qquad \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}. \end{cases} \tag{1.2}$$

Here, we suppose that $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $(\forall i = 1, \ldots, N)$ are Carathéodory functions such that

$$a_i(x, \xi) \cdot \xi_i \geq \alpha|\xi_i|^{p_i}, \ \ \forall i = 1, \ldots, N, \tag{1.3}$$

$$|a_i(x, \xi_i)| \leq k(x) + \beta|\xi_i|^{p_i-1}, \ \ \forall i = 1, \ldots, N, \tag{1.4}$$

$$\left[a_i(x, \xi) - a_i(x, \eta)\right] \cdot (\xi_i - \eta_i) > 0, \quad \xi_i \neq \eta_i, \ \ \forall i = 1, \ldots, N, \tag{1.5}$$

for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $\alpha, \beta > 0$ and $0 \leq k \in L^{p_i'}(\Omega)$. The nonlinear terms $g_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $(\forall i = 1, \ldots, N)$ are Carathéodory functions and satisfying

$$g_i(x, s, \xi)\text{sign}(s) \geq 0, \ \ \forall i = 1, \ldots, N, \tag{1.6}$$

$$|g_i(x, s, \xi)| \leq l(|s|)|\xi_i|^{p_i}, \ \ \forall i = 1, \ldots, N, \tag{1.7}$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, $l \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ is an increasing function such that $l(\rho) > c_0 > 0$ for $|\rho|$ sufficiently large. The datum f is assumed a nonegative function and belongs to some Lebesgue spaces. Let us emphasize that the sign condition (1.6) satisfied by g enables to derive a priori estimates from the equation; however, if it is not met the problem may not even have a solution. Finally, the function $h :]0, \infty[\to]0, \infty[$ is continuous and satisfies the condition

$$\exists\, c > 0,\ \exists\, \gamma \in (0, 1] \text{ such that } h(s) \leq \frac{c}{s^\gamma} \quad \forall s > 0. \tag{1.8}$$

Let also underline that in the case $h(0) = \lim_{s \to 0^+} h(s)$ is finite, the singular term h becomes continuous and bounded.

The motivation for this work comes from the application of anisotropic equations in several fields. For example, they offer a mathematical models for representing fluid dynamics when conductivity varies along different directions [17]. Furthermore problem (1.1) appears in calculus of variations when we write the Euler-Lagrange equations of appropriate functionals.

Our aim is to study the existence of energy solutions to (1.1). Specifically, we focus on the regularizing effect on the solution of (1.1) in the presence of the natural growth term involving g and possibly singular term h. Moreover there is no necessity to impose additional assumptions on p_i (see Theorem 2.3 in [22]).

Problem (1.1) in the isotropic case, i.e. $p_i = p$ for any i has been extensively studied by many authors, we refer to some papers which mostly influenced us [6, 20, 21, 2]. In our recent paper [22], we studied the problem (1.1) in the anisotropic and elliptic case when the natural growth term does not appear, that is for problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left[\frac{|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u}{(1+u)^\theta}\right] = \frac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \gamma < 1$, $\theta \geq 0$ and $p_i \geq 1, i = 1, \cdots, N$. We have established in terms of the summability of the datum f and on the values of γ and θ some existence and regularity results. In the case $f \in L^1(\Omega)$ and $\theta = 0$ we have proved the existence of at least one distributional solution $u \in W_0^{1,\overrightarrow{q}}(\Omega)$ for (1.9), for every $q_i = \frac{p_i N(\overline{p}-1+\gamma)}{\overline{p}(N-1+\gamma)}$ under the assumption $\frac{\overline{p}(N+\gamma-1)}{N(\overline{p}+\gamma-1)} < p_i < \frac{\overline{p}(N+\gamma-1)}{\overline{p}(N+\gamma-1)-N(\overline{p}+\gamma-1)}$, for any $i = 1, \cdots, N$. Furthermore, if $f \in L^m(\Omega)$ with $m > \frac{N}{\overline{p}}$ then the solution u belongs to $W_0^{1,\overrightarrow{p}}(\Omega \cap L^\infty(\Omega)$. In the non-singular case i.e, $h \equiv 1$, Agnese Di Castro in [4] treated the existence of solution to problem (1.1), it was shown that if $f \in L^1(\Omega)$ and g_i satisfying the conditions (1.6)-(1.7) and there exists $\mu > 0$ such that $|g_i(x, s, \xi)| \geq \mu |\xi_i|^{p_i}$, for any $i = 1, \cdots N$. Then finite energy solutions exist for problem (1.1).

In work [12], the authors obtained existence of $L^\infty-$solutions to the problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(a_i(x, u, \nabla u)) = \sum_{i=1}^{N} b_i(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$.
Here $Au = -\sum_{i=1}^{N} \partial_{x_i}(a_i(x, u, \nabla u))$ is a Leray-Lions operator defined on $W_0^{1,\overrightarrow{p}}(\Omega)$ and $b_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, \ldots, N$ are Carathéodory functions and satisfying a.e. $x \in \Omega$, $\forall (\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$ the following condition

$$|b_i(x, s, \xi)| \leq \beta_1 |\xi_i|^{\frac{p_i(\overline{p}^\star - 1)}{\overline{p}^\star}} + \beta_2 |s|^{-\gamma_i} + \beta_3 |s|^{\overline{p}^\star - 1} + \beta_4,$$

where β_l, $l = 1, \ldots, 4$ are positive constants, $\overline{p}^\star$ the Sobolev conjugate of the harmonic mean $\overline{p}$ with $p_i < \overline{p}^\star$ and $0 < \gamma_i < 1$, for all $i = 1, \ldots, N$. For anisotropic elliptic and parabolic equations, we recommend consulting the following references [11, 10, 9].

In the study of problem (1.1), the main difficulty comes from the nonlinear term h which possibly blows up on the set $\{u = 0\}$. To overcome this problem, we approximate our problem by another one defined through of truncations, whose existence of solution is guaranteed by Schauder's fixed point theorem.

We point out that, crucial tools applicable in the isotropic case cannot be applied to the anisotropic setting (for example the strong maximum principle, see [14]), which is another complication in the study of the problem (1.1). In the next section, We briefly recall some facts on the anisotropic Sobolev space and we give some of their properties.

1.2 Preliminaries and definition

Let $\overrightarrow{p} = (p_1, p_2, \dots, p_N) \in \mathbb{R}^N$. The anisotropic Sobolev spaces naturally serve as the functional framework for problem (1.1) are $W^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$, which are defined as follows

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{z \in W^{1,1}(\Omega) :\ \partial_{x_i} z \in L^{p_i}(\Omega),\ \ \forall i = 1, \dots, N\right\},$$
$$W_0^{1,\overrightarrow{p}}(\Omega) = \left\{z \in W_0^{1,1}(\Omega) :\ \partial_{x_i} z \in L^{p_i}(\Omega),\ \ \forall i = 1, \dots, N\right\}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ can also be defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|z\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} \|\partial_{x_i} z\|_{L^{p_i}(\Omega)},$$

endowed with this norm $\|\cdot\|_{1,\overrightarrow{p}}$, $W_0^{1,\overrightarrow{p}}(\Omega)$ is a separable and reflexive Banach space.

The theory concerning anisotropic Sobolev spaces was developed in [23, 13, 15, 16]. In particular, under the assumption $\overline{p} < N$, the authors in [23] proved the following continuous embedding

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^\tau(\Omega), \quad \forall \tau \in [1, \overline{p}^\star],$$

additionally, this embedding is compact for $\tau < \overline{p}^\star$. Furthermore, in reference [23], the following Sobolev type inequality is also proved

Lemma 1.1. *There exists positive constants $\mathcal{S}_1$ and $\mathcal{S}_2$, which depend only on Ω, such that*

$$\|z\|_{L^\tau(\Omega)} \le \mathcal{S}_1 \prod_{i=1}^{N} \|\partial_{x_i} z\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \ \ \forall \tau \in [1, \overline{p}^*], \ \ \forall z \in W_0^{1,\overrightarrow{p}}(\Omega), \tag{1.10}$$

$$\|z\|_{L^{\overline{p}^*}(\Omega)}^{p_+} \le \mathcal{S}_2 \sum_{i=1}^{N} \|\partial_{x_i} z\|_{L^{p_i}(\Omega)}^{p_i}, \ \ \forall z \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{1.11}$$

We can replace the geometric mean on the right-hand side of (1.10) with an arithmetic mean, This is justified by the fact that

$$\prod_{i=1}^{N} {b_i}^{1/N} \le \frac{1}{N} \sum_{i=1}^{N} b_i \quad \text{for all } b_i \ge 0, i = 1, \cdots, N.$$

The above inequality can be used to establish the following result

$$\|z\|_{L^q(\Omega)} \le \frac{\mathcal{S}_3}{N} \sum_{i=1}^{N} \|\partial_{x_i} z\|_{L^{p_i}(\Omega)}, \quad \forall \tau \in [1, \overline{p}^*], \ \forall z \in W_0^{1,\overrightarrow{p}}(\Omega), \tag{1.12}$$

where $\mathcal{S}_3$ is positive constant. Consequently, when $\overline{p} < N$, a continuous embedding exists from the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for all $q \in [1, \overline{p}^\star]$. Moreover, for each $i = 1, \dots, N$, there exists a positive constant $\mathcal{S}_i > 0$ (see [5, Lemma 1.1]) such that the following inequality holds:

$$\|z\|_{L^{p_i}(\Omega)} \le \mathcal{S}_i \|\partial_{x_i} z\|_{1,\overrightarrow{p}}, \quad \forall z \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{1.13}$$

The proof of the L^∞-estimate that we will present is founded on a technical lemma of functional analysis.

Lemma 1.2 (See[18]). *Let M_1, k_0, ϱ, ρ be real positive numbers, with $\rho > 1$. Let $\Theta : (0, \infty) \to (0, \infty)$ be a non increasing function such that*

$$\Theta(h) \leq \frac{M_1}{(h-k)^{\varrho}}[\Theta(k)]^{\rho}, \quad \forall h > k \geq k_0.$$

Then there exists $\overline{k} > 0$ such that $\Theta(\overline{k}) = 0$.

Now we present the definition of weak solution to problem (1.1).

Definition 1.3. Let f be a nonegative function in $L^{\theta}(\Omega)$, where $\theta \geq 1$. A function u belongs to $W_0^{1,\overrightarrow{p}}(\Omega)$ is said to be a weak solution for problem (1.1), if $a_i(x, \nabla u) \in L^1_{loc}(\Omega)$, $g_i(x, u, \nabla u) \in L^1_{loc}(\Omega)$ for any $i = 1, \ldots, N$ and u satisfies

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \nabla u)\partial_{x_i}\varphi dx + \sum_{i=1}^{N} \int_{\Omega} g_i(x, u, \nabla u)\varphi dx = \int_{\Omega} f h(u)\varphi dx \tag{1.14}$$

for every $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^{\infty}(\Omega)$.

2 Approximating problems

Let f_n be the sequence of bounded functions in Ω ($f_n \geq 0$) that converges to $f > 0$ in $L^1(\Omega)$. This sequence satisfies the inequalities $f_n \leq n$ and $f_n \leq f$ for every $n \in \mathbb{N}$ (for instance, $f_n = T_n(f)$). Consider the approximation problems defined as follows

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left[a_i(x, \nabla u_n)\right] + \sum_{i=1}^{N} g_i^n(x, u_n, \nabla u_n) = f_n h_n(u_n) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$g_i^n(x, s, \xi) = \frac{g_i(x, s, \xi)}{1 + \frac{1}{n}|g_i(x, s, \xi)|}, \quad \forall i = 1, \ldots, N, \quad \forall n \in \mathbb{N},$$

and

$$h_n(s) = \begin{cases} T_n(h(s)) & \text{for } s \geq 0, \\ \min\{n, h(0)\} & \text{otherwise.} \end{cases}$$

Notice that, for all $i = 1, \ldots, N$

$$|g_i^n(x, s, \xi)| \leq |g_i(x, s, \xi)|, \quad |g_i^n(x, s, \xi)| \leq n.$$

Lemma 2.1. *Suppose that assumptions (1.3)-(1.8) hold true. Then, the problem (2.1) has a non-negative weak solution $u_n \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^{\infty}(\Omega)$ in the sense*

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \nabla u_n)\partial_{x_i}\varphi dx + \sum_{i=1}^{N} \int_{\Omega} g_i^n(x, u_n, \nabla u_n)\varphi dx = \int_{\Omega} f_n h_n(u_n)\varphi dx, \tag{2.2}$$

for all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. The lemma's proof will be carried out by employing Schauder's fixed point argument. Let $n \in \mathbb{N}^*$ be fixed. We define a map $\mathcal{P}$ as

$$\begin{aligned} \mathcal{P} : \quad L^{\overline{p}}(\Omega) &\longrightarrow L^{\overline{p}}(\Omega) \\ v &\longmapsto S(v) = w, \end{aligned}$$

where w is the unique solution of the following problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left[a_i(x, \nabla w)\right] + \sum_{i=1}^{N} g_i^n(x, w, \nabla w) = f_n h_n(|v|) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The map $\mathcal{P}$ is well defined because the existence of a unique weak solution $w \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ for the problem (2.3) is guaranteed by [4]. Furthermore, a positive constant c_n exists that is independent of v and w such that

$$\|w\|_{L^\infty(\Omega)} \leq c_n. \tag{2.4}$$

We claim that w is nonnegative; indeed, we choose $\varphi = -w^- e^{-tw}$ as a test function in (2.1), with $t > 0$ to be chosen later and $w^- = -\min\{w, 0\}$ we obtain

$$\begin{aligned} -\sum_{i=1}^N \int_\Omega a_i(x, \nabla w)\partial_{x_i} w^- e^{-tw} dx + t \sum_{i=1}^N \int_\Omega a_i(x, \nabla w)\partial_{x_i} w w^- e^{-tw} dx \\ -\sum_{i=1}^N \int_\Omega g_i^n(x, w, \nabla w) w^- e^{-tw} dx = -\int_\Omega f_n h_n(|v|) w^- e^{-tw} \leq 0. \end{aligned}$$

Using (1.3), (1.7) and the fact that $-g_i^n(x, w, \nabla w) > -l(c_n)|\partial_{x_i} w|^{p_i}$, we have

$$\alpha \sum_{i=1}^N \int_\Omega |\partial_{x_i} w^-|^{p_i} e^{-tw} dx + \sum_{i=1}^N \int_\Omega |\partial_{x_i} w|^{p_i} w^- e^{-tw}(\alpha t - l(c_n)) dx \leq 0.$$

Choosing $t > \frac{l(c_n)}{\alpha}$ in the previous inequality, we get $w \geq 0$ almost everywhere in Ω.

Let us consider w as a test function in the weak formulation of (2.3). Using (1.3), (1.8), and the fact that $f_n \leq n$, we obtain

$$\alpha \sum_{i=1}^N \int_\Omega |\partial_{x_i} w|^{p_i} + \sum_{i=1}^N g_i^n(x, w, \nabla w) w dx \leq n^{1-\gamma} \int_\Omega w dx.$$

By assumption (1.7), one has

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} w|^{p_i} \leq \frac{n^{1-\gamma}}{\alpha} \int_\Omega w dx.$$

Applying Hölder's inequality, we can further estimate the right-hand side as follows

$$\sum_{i=1}^N \|\partial_{x_i} w\|_{L^{p_i}(\Omega)}^{p_i} \leq \frac{n^{1-\gamma}}{\alpha} |\Omega|^{\frac{1}{(\overline{p}^*)'}} \left(\int_\Omega w^{\overline{p}^*} dx \right)^{\frac{1}{\overline{p}^*}}. \tag{2.5}$$

From the inequality (1.11), there exists a positive constant $\mathcal{S}_2$, such that

$$\|w\|_{L^{\overline{p}^*}(\Omega)}^{p^+} \leq \frac{n^{1-\gamma}}{\mathcal{S}_2^{-1}\alpha} |\Omega|^{\frac{1}{(\overline{p}^*)'}} \|w\|_{L^{\overline{p}^*}(\Omega)}.$$

This implies that

$$\|w\|_{L^{\overline{p}^*}(\Omega)} \leq C_n, \tag{2.6}$$

where $C_n = \frac{n^{1-\gamma}}{\mathcal{S}_2^{-1}\alpha} |\Omega|^{\frac{1}{(\overline{p}^*)'}} \|w\|_{L^{\overline{p}^*}(\Omega)}$. Since $\overline{p} \leq \overline{p}^*$, then

$$\|w\|_{L^{\overline{p}}(\Omega)} \leq C_n. \tag{2.7}$$

Thus, equation (2.7) implies that the ball $B(0, C_n) \subset L^{\overline{p}}(\Omega)$, is invariant under the map $\mathcal{P}$.

Now, we will prove the continuity of the map $\mathcal{P}$. Let $v \in L^{\overline{p}}(\Omega)$ and let (v_k) be a sequence of functions converges to v in $L^{\overline{p}}(\Omega)$. We denote $w_k = \mathcal{P}(v_k)$ and $w = \mathcal{P}(v)$. To prove that $w_k \longrightarrow w$ in $L^{\overline{p}}(\Omega)$, it suffices to prove that $w_k \longrightarrow w$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ because the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{\overline{p}}(\Omega)$ is compact. In fact, we show that for any subsequence of (w_k), we can extract further subsequence that converges to w.

Let consider $\Phi_\rho(s) = se^{\rho s^2}$ ($\rho > 0$) which satisfies

$$\mu\Phi_\rho'(s) - \nu|\Phi_\rho(s)| \geq \frac{\mu}{2}, \quad \forall s \in \mathbb{R}, \quad \forall \mu, \nu > 0, \quad \forall \rho \geq \frac{\nu^2}{4\mu^2}. \tag{2.8}$$

Let us now consider $\varphi = \Phi_\rho(z_k)$ as a test function in the weak formulation of (2.3) where $z_k = w_k - w$, we have

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} z_k \Phi_\rho'(z_k) = &-\sum_{i=1}^{N} \int_\Omega g_i^n(x, w_k, \nabla w_k)\Phi_\rho(z_k)dx \\ &+ \int_\Omega f_n h_n(|v_k|)\Phi_\rho(z_k)dx.\end{aligned} \tag{2.9}$$

Adding up (1.7) (since $|g_i^n(x, w_k, \nabla w_k)| \leq |g_i(x, w_k, \nabla w_k)|$), (2.4) and (1.3) gives

$$\begin{aligned}-\sum_{i=1}^{N} \int_\Omega g_i^n(x, w_k, \nabla w_k)\Phi_\rho(z_k)dx &\leq \sum_{i=1}^{N} \int_\Omega l(|w_k|)|\partial_{x_i} w_k|^{p_i}|\Phi_\rho(z_k)|dx \\ &\leq l(c_n)\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} w_k|^{p_i}|\Phi_\rho(z_k)|dx \\ &\leq \frac{l(c_n)}{\alpha}\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} w_k|\Phi_\rho(z_k)|dx,\end{aligned}$$

this gives

$$\begin{aligned}-\sum_{i=1}^{N} \int_\Omega g_i^n(x, w_k, \nabla w_k)\Phi_\rho(z_k)dx \leq &\frac{l(c_n)}{\alpha}\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} z_k|\Phi_\rho(z_k)|dx \\ &+ \frac{l(c_n)}{\alpha}\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} w|\Phi_\rho(z_k)|dx.\end{aligned} \tag{2.10}$$

Thanks to (1.4) and (2.5), the sequence $(a_i(x, \nabla w_k))_k$ is bounded in $L^{p_i'}(\Omega)$ for all $i = 1, \dots, N$. Then, since $\partial_{x_i} w|\Phi_\rho(z_k)|$ strongly converges to zero in $L^{p_i}(\Omega)$ as $k \to \infty$ for all $i = 1, \dots, N$, one has

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} w|\Phi_\rho(z_k)|dx = 0. \tag{2.11}$$

Moreover, since $v_k \longrightarrow v$ in $L^{\overline{p}}(\Omega)$ as $k \to \infty$, we can extract a subsequence such that

$$v_k \xrightarrow{k\to\infty} v \text{ a.e. in } \Omega. \tag{2.12}$$

Using (1.8) we have

$$\begin{aligned}|f_n h_n(|v_k|)\Phi_\rho(z_k)| &\leq c_1 n^{1-\gamma}\|z_k\|_{L^\infty(\Omega)} e^{\rho\|z_k\|^2_{L^\infty(\Omega)}} \\ &\leq c_1 n^{1-\gamma} c_n e^{\rho c_n^2} \in L^1(\Omega) \ \ \forall k \in \mathbb{N}.\end{aligned} \tag{2.13}$$

Then, from (2.12) and (2.13), we can apply the dominated convergence theorem to conclude that

$$\lim_{k\to\infty} \int_\Omega f_n h_n(|v_k|)\Phi_\rho(z_k)dx = 0. \tag{2.14}$$

Therefore, by combining (2.9), (2.10), (2.11) and (2.14), we obtain

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} \nabla z_k \Phi_\rho'(z_k) \\ &\quad\leq \frac{l(c_n)}{\alpha}\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla w_k)\partial_{x_i} z_k|\Phi_\rho(z_k)|dx + r(k),\end{aligned}$$

where $\lim_{k\to\infty} r(k) = 0$. Hence,

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \nabla w_k)\partial_{x_i} z_k \left(\Phi'_\rho(z_k) - \frac{l(c_n)}{\alpha} |\Phi_\rho(z_k)| \right) dx \leq r(k).$$

Thus, by (4.2) with $\rho = \frac{[l(c_n)]^2}{4\alpha^2}$, we have

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \nabla w_k)\partial_{x_i} z_k dx \leq r(k),$$

Then,

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \left(a_i(x, \nabla w_k) - a_i(x, \nabla w)\right) \left(\partial_{x_i} w_k - \partial_{x_i} w\right) dx \\ &\quad \leq -\sum_{i=1}^{N} \int_{\Omega} a_i(x, \nabla w)\partial_{x_i} z_k dx + r(k). \end{aligned} \tag{2.15}$$

It follows from (1.4) that the sequence $a_i(x, \nabla w)$ is bounded in $L^{p'_i}(\Omega)$ and by (2.5) we have z_k is weakly converges to 0 in $W_0^{1,\overrightarrow{p}}(\Omega)$. From (2.15) and (1.5) we deduce that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_{\Omega} \left(a_i(x, \nabla w_k) - a_i(x, \nabla w)\right) \left(\partial_{x_i} w_k - \partial_{x_i} w\right) dx = 0.$$

We can then use the same arguments as in [8, Lemma 2.4] to prove that up to subsequences w_k strongly converges to w in $W_0^{1,\overrightarrow{p}}(\Omega)$. This establishes the continuity of $\mathcal{P}$.

Using equations (2.5) and (2.6), we can deduce that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} w|^{p_i} dx = \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} \mathcal{P}(v)|^{p_i} dx \leq C_n, \quad \forall v \in L^{\overline{p}}(\Omega).$$

By Sobolev embedding, $\mathcal{P}(L^{\overline{p}}(\Omega))$ can be shown to be compact in $L^{\overline{p}}(\Omega)$. As a result, by Schauder's fixed point theorem on $\mathcal{P}$, we establish the existence of a nonnegative fixed point $u_n \in W_0^{1,\overrightarrow{p}}(\Omega)$. This fixed point is identified as a weak solution to (2.3). Furthermore, for a fixed n we have u_n belongs to $L^\infty(\Omega)$ (by [7, Theorem 4.2]) because the right-hand side of (2.1) is in $L^\infty(\Omega)$ and this concludes the proof. □

3 Existence result for $\theta > \frac{N}{\overline{p}}$

In this section we prove the existence of nonnegative weak solutions to problem (1.1) when the datum f is an element of $L^\theta(\Omega)$, with $\theta > \frac{N}{\overline{p}}$.

Theorem 3.1. *Assume* (1.3)-(1.8) *with*

$$f \in L^\theta(\Omega), \quad \theta > \frac{N}{\overline{p}}.$$

Then problem (1.1) *has at least a weak solution* u *such that* $u \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2. 1) In the isotropic case (i.e., $p_i = p$ for every $i = 1, \dots, N$), Theorem 3.1 surpasses the results in [24, Theorem 3.3].
2) Theorem (3.1) improves Theorem 3.2 in [22].

3.1 A priori estimates when $\theta > \frac{N}{\overline{p}}$

In this step of the proof, we want to establish some a priori estimates in $L^\infty(\Omega)$ and in $W_0^{1,\overrightarrow{p}}(\Omega)$ for the sequence of approximate solutions $(u_n)_n$, As pointed out, by these estimates, we deduce that u_n converges up to subsequences, to a function u which is the sought solution. Moreover, we prove the boundedness of g_i^n in $L^1(\Omega)$ and $f_n h_n$ in $L^1_{loc}(\Omega)$.

In the following, we denote by C a constant (independent of n) that may change from one line to another.

Lemma 3.3. *Assume that the assumptions of Theorem 3.1 hold. Then for every solution u_n of (2.1), there exists a positive constant C independent of n such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \tag{3.1}$$

$$\|u_n\|_{1,\overrightarrow{p}} \leq C. \tag{3.2}$$

Proof. Le us take $G_k(u_n)$ as a test function in (2.1), we have

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla u_n)\partial_{x_i} G_k(u_n)dx + \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)G_k(u_n)dx$$
$$= \int_\Omega f_n h_n(u_n)G_k(u_n)dx.$$

which, using (1.3) and (1.6) implies that

$$\begin{aligned}\|\partial_{x_i} G_k(u_n)\|^{p_i}_{L^{p_i}(\Omega)} &\leq \int_{A_k^n} f_n h_n(u_n)G_k(u_n)dx \\ &\leq \frac{1}{\alpha}\frac{c}{k^\gamma}\int_{A_k^n} fG_k(u_n)dx, \quad \forall i = 1,\ldots,N,\end{aligned}$$

where $A_k^n = \{u_n > k\}$, $k \geq 1$. The previous inequality yield to

$$\prod_{i=1}^N \|\partial_{x_i} G_k(u_n)\|^{\frac{1}{N}}_{L^{p_i}(\Omega)} \leq Ck^{-\frac{\gamma}{\overline{p}}}\left(\int_{A_k^n} fG_k(u_n)dx\right)^{\frac{1}{\overline{p}}}.$$

Hence we can apply inequality (1.10) with $\tau = \overline{p}^*$, $\overline{p} < N$, on the left-hand side and using Hölder's inequality with exponent $\overline{p}^*$ in the right one, we obtain

$$\|G_k(u_n)\|_{L^{\overline{p}^*}(\Omega)} \leq Ck^{-\frac{\gamma}{\overline{p}}}\|f\|^{\frac{1}{\overline{p}}}_{L^{\overline{p}^{*\prime}}(A_k^n)}\|G_k u_n)\|^{\frac{1}{\overline{p}}}_{L^{\overline{p}^*}(A_k^n)}.$$

Hence

$$\int_\Omega G_k(u_n)^{\overline{p}^*}dx \leq Ck^{-\frac{\gamma\overline{p}^*}{\overline{p}-1}}\|f\|^{\frac{\overline{p}^*}{\overline{p}-1}}_{L^{\overline{p}^{*\prime}}(A_k^n)}. \tag{3.3}$$

Recalling that $1 - \frac{1}{\overline{p}} > 0$ (since $\overline{p} > 1$) and considering the fact that $f \in L^\theta(\Omega)$ where $\theta > \frac{N}{\overline{p}} \geq \overline{p}^{*\prime}$, this allows to apply Hölder's inequality with exponents $\frac{\theta}{\overline{p}^{*\prime}}$, deducing

$$\begin{aligned}\|f\|^{\frac{\overline{p}^*}{\overline{p}-1}}_{L^{\overline{p}^{*\prime}}(A_k^n)} &\leq \|f\|^{\frac{\overline{p}^*}{\overline{p}-1}}_{L^\theta(A_k^n)}|A_k^n|^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}} \\ &\leq C|A_k^n|^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}}.\end{aligned} \tag{3.4}$$

which thanks to (3.3) and (3.4), implies

$$\int_\Omega G_k(u_n)^{\overline{p}^*}dx \leq Ck^{-\frac{\gamma\overline{p}^*}{\overline{p}-1}}|A_k^n|^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}}$$

Notice that, for every $r > k \geq k_0$ one has $G_k(u_n) \geq r - k$ on the set $\{u_n > r\}$, we arrive at

$$\begin{aligned}(r-k)^{\overline{p}^*}|A_r^n| &\leq \int_{A_r^n} G_k(u_n)^{\overline{p}^*}\,dx \\ &\leq Ck_0^{-\frac{\gamma\overline{p}^*}{\overline{p}-1}}|A_k^n|^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}} \\ &\leq C|A_k^n|^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}},\end{aligned}$$

this allows to deduce that

$$\Theta_n(r) \leq \frac{C}{(r-k)^{\overline{p}^*}}\Theta_n(k)^{\frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta}}, \quad \forall r > k \geq k_0,$$

where $\Theta_n(k) = |A_k^n|$. Thus, since $\theta > \frac{N}{\overline{p}}$ and by Lemma 1.2, applied to

$$\varrho = \overline{p}^*, \quad \text{and} \quad \rho = \frac{1}{\overline{p}-1}\frac{\theta(\overline{p}^*-1)-\overline{p}^*}{\theta} > 1,$$

there exists a positive constant $\overline{k}$ achieves $\Theta_n(\overline{k}) = 0$. By the fact that $|\Theta_n(k)| \leq |\Omega|$ (see the proof of Lemma A.1 of [18]), there exists a positive constant ω independent of n such that $\overline{k} \leq \omega$, satisfying

$$\Theta_n(\omega) = 0. \tag{3.5}$$

Hence (3.5) yields to (3.1).

Now we prove the a priori estimates in $W_0^{1,\overrightarrow{p}}(\Omega)$ given by (3.2). Let $0 < \gamma \leq 1$. To show the estimate (3.2) we choose $\varphi = u_n$ as a test function in (2.1); we have

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla u_n)\partial_{x_i}u_n dx + \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)u_n dx = \int_\Omega f_n h_n(u_n)u_n dx.$$

Using (1.3), (1.6) and (1.8), one has

$$\alpha\sum_{i=1}^N \int_\Omega |\partial_{x_i}u_n|^{p_i}dx \leq \int_\Omega f u_n^{1-\gamma}dx.$$

Hölder's inequality and assumption $f \in L^\theta(\Omega)$ on the right-hand side then gives

$$\begin{aligned}\alpha\sum_{i=1}^N \|\partial_{x_i}u_n\|_{L^{p_i}(\Omega)}^{p_i} &\leq \|f\|_{L^{\frac{N}{\overline{p}}}(\Omega)}\left(\int_\Omega u_n^{(1-\gamma)\frac{N}{N-\overline{p}}}dx\right)^{\frac{N-\overline{p}}{N}} \\ &\leq \|f\|_{L^\theta(\Omega)}|\Omega|^{\frac{(N-\overline{p})(\overline{p}-1+\gamma)}{N\overline{p}}}\left(\int_\Omega u_n^{\overline{p}^*}dx\right)^{\frac{1-\gamma}{\overline{p}^*}} \\ &\leq C\|u_n\|_{L^{\overline{p}^*}(\Omega)}^{1-\gamma}.\end{aligned}$$

Using Young's inequality on the right-hand side we arrive at

$$\begin{aligned}\alpha\sum_{i=1}^N \|\partial_{x_i}u_n\|_{L^{p_i}(\Omega)}^{p_i} &\leq C\mathcal{S}_2^{1-\gamma}\left(\frac{1-\gamma}{p^+}\right)\varepsilon^{\frac{p^+}{1-\gamma}}\|u_n\|_{L^{\overline{p}^*}(\Omega)}^{p^+} \\ &\quad + C\left(\frac{p^+-1+\gamma}{p^+}\right)\varepsilon^{-\frac{p^+}{p^+-1+\gamma}},\end{aligned}$$

where ε is any positive constant. Inequality (1.11) implies that

$$\begin{aligned}\alpha\sum_{i=1}^N \|\partial_{x_i}u_n\|_{L^{p_i}(\Omega)}^{p_i} &\leq C\mathcal{S}_2^{1-\gamma}\left(\frac{1-\gamma}{p^+}\right)\varepsilon^{\frac{p^+}{1-\gamma}}\sum_{i=1}^N \|\partial_{x_i}u_n\|_{L^{p_i}(\Omega)}^{p_i} \\ &\quad + C\left(\frac{p^+-1+\gamma}{p^+}\right)\varepsilon^{-\frac{p^+}{p^+-1+\gamma}}.\end{aligned}$$

Choosing $\varepsilon > 0$ in the previous inequality, such that

$$C\mathcal{S}_2^{1-\gamma}\left(\frac{1-\gamma}{p^+}\right)\varepsilon^{\frac{p^+}{1-\gamma}} < \frac{\alpha}{2},$$

we get

$$\sum_{i=1}^{N}\|\partial_{x_i}u_n\|_{L^{p_i}(\Omega)}^{p_i} \leq C. \tag{3.6}$$

Hence, by (3.6) the proof of (3.2) is concluded. □

Lemma 3.4. *Assume that the assumptions of Theorem 3.1 hold. Let u_n be a nonnegative solution to problem* (2.2)*. Then*

$$\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)dx \leq C, \tag{3.7}$$

$$\int_\Omega h_n(u_n)f_n\varphi dx \leq C, \quad \forall\varphi\in C_c^1(\Omega). \tag{3.8}$$

Proof. We choose $T_1(u_n)$ as test function in (2.2) obtaining

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\nabla u_n)\partial_{x_i}T_1(u_n)dx + \sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_1(u_n)dx = \int_\Omega f_nh_n(u_n)T_1(u_n)dx. \tag{3.9}$$

Now for $\lambda > 0$, using (1.3) and (1.8) we can write

$$\sum_{i=1}^{N}\int_{\{u_n\geq 1\}} g_i^n(x,u_n,\nabla u_n)T_1(u_n)dx \leq C\int_{\{u_n\leq\lambda\}}u_n^{1-\gamma}f_n + \int_{\{u_n>\lambda\}}h_n(u_n)f_ndx \leq C\left(\lambda^{1-\gamma}+\sup_{s\in[\lambda,\infty)}h(s)\right)\|f\|_{L^1(\Omega)}. \tag{3.10}$$

In the other hand remark that, by (1.7) one has

$$\sum_{i=1}^{N}\int_{\{u_n<1\}} g_i^n(x,u_n,\nabla u_n) \leq C\sum_{i=1}^{n}\int_\Omega|\partial_{x_i}u_n|^{p_i}dx \leq C. \tag{3.11}$$

According to (3.10) and (3.11) it follows that $g_i^n(x,u_n,\nabla u_n)$ is bounded in $L^1(\Omega)$.

Finally, we show that $h_n(u_n)f_n$ is bounded in $L^1_{loc}(\Omega)$, we consider a nonnegative $\varphi\in C_c^1(\Omega)$ as a test function in (2.2), we have

$$\int_\Omega h_n(u_n)f_n\varphi dx = \sum_{i=1}^{N}\int_\Omega a_i(x,\nabla u_n)\partial_{x_i}\varphi dx + \sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)\varphi dx.$$

In view of (1.4) and (3.2), it easy to check that $a_i(x,\nabla u_n)$ is bounded in $L^{p_i'}(\Omega)$ with respect to n, and by the fact that $g_i^n(x,u_n,\nabla u_n)$ is bounded in $L^1(\Omega)$ we conclude that (3.8) holds. This finishes the proof of the Lemma 3.4. □

3.2 Convergence of gradients almost everywhere

Let u_n be a nonegative solutions to (2.1), then according to the Lemma 3.3 there exists a subsequence of u_n (still denoted u_n) and a function u in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\overrightarrow{p}}(\Omega) \quad \text{and a.e. in } \Omega, \tag{3.12}$$

Lemma 3.5. *Assume that the assumptions of Theorem 3.1 hold. Let (u_n) be a no-negative solutions to problem (2.1), then we have up to sub sequence,*

$$\forall k > 0, \qquad T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,\vec{p}}(\Omega) \quad \text{and a.e. in } \Omega. \tag{3.13}$$

Moreover $\partial_{x_i} u_n$ converge to $\partial_{x_i} u$ almost everywhere.

Proof. Let us consider for any $k > 0$ the function $v_{n,k} = T_k(u_n) - T_k(u)$, taking $\Phi_\rho(v_{n,k}) = v_{n,k} e^{\rho v_{n,k}^2}$ (which satisfies (4.2)) as a test function in the weak formulation (2.2), obtaining

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x, \nabla u_n) \partial_{x_i}(v_{n,k}) \Phi'_\rho(v_{n,k}) dx + \sum_{i=1}^N \int_\Omega g_i^n(x, u_n, \nabla u_n) \Phi_\rho(v_{n,k}) dx,\\ = \int_\Omega f_n h_n(u_n) \Phi_\rho(v_{n,k}) dx,\end{aligned} \tag{3.14}$$

The result will derive from applying Lemma 5 in [19] once we establish that for any for any $k > 0$

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega \left[a_i(x, \nabla T_k(u_n)) - a_i(x, T_k(u)\right] \partial_{x_i}(T_k(u_n) - T_k(u)) dx = 0.$$

Step 1 : We will estimate the quantity

$$\sum_{i=1}^N \int_\Omega \left[a_i(x, \nabla T_k(u_n)) - a_i(x, T_k(u)\right] \partial_{x_i}(T_k(u_n) - T_k(u)) dx.$$

First, observe that in the set $\{u_n > k\}$, one has $\partial_{x_i}(v_{n,k}) = -\partial_{x_i}(T_k(u))$, using (1.4), we deduce that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, \nabla u_n) \partial_{x_i}(v_{n,k}) \Phi'_\rho(v_{n,k}) dx\\ &= \sum_{i=1}^N \int_{\{u_n \le k\}} a_i(x, \nabla u_n) \partial_{x_i}(v_{n,k}) \Phi'_\rho(v_{n,k}) dx\\ &\quad - \sum_{i=1}^N \int_{\{u_n > k\}} a_i(x, \nabla u_n) \partial_{x_i}(T_k(u)) \Phi'_\rho(v_{n,k}) dx\\ &\ge \sum_{i=1}^N \int_\Omega a_i(x, \nabla T_k(u_n)) \partial_{x_i}(v_{n,k}) \Phi'_\rho(v_{n,k}) dx\\ &\quad - \int_{\{u_n > k\}} \left(k(x) + \beta |\partial_{x_i} u_n|^{p_i - 1}\right) |\partial_{x_i}(T_k(u))| |\Phi'_\rho(v_{n,k})| dx.\end{aligned}$$

Combining the previous inequality into (3.14) we get

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, \nabla T_k(u_n) \partial_{x_i}(v_{n,k}) \Phi'_\rho(v_{n,k}) dx\\ &\quad \le -\sum_{i=1}^N \int_\Omega g_i^n(x, u_n, \nabla u_n) \Phi_\rho(v_{n,k}) dx + \int_\Omega f_n h_n(u_n) \Phi_\rho(v_{n,k}) dx\\ &\quad - \int_{\{u_n > k\}} \left(k(x) + \beta |\partial_{x_i} u_n|^{p_i - 1}\right) |\partial_{x_i}(T_k(u))| |\Phi'_\rho(v_{n,k})| dx.\end{aligned} \tag{3.15}$$

Concerning the last term in (3.15), we use the boundedness of the sequence (u_n) in $W_0^{1,\overrightarrow{p}}(\Omega)$, we see that $\left(k(x)+\beta|\partial_{x_i}u_n|^{p_i-1}\right)|\partial_{x_i}(T_k(u))||\Phi'_\rho(v_{n,k})|$ is bounded in $L^{p'_i}(\Omega)$, $i=1,\cdots,N$ with respect to n and by the fact that $\partial_{x_i}(T_k(u))\chi_{\{u_n>k\}} \longrightarrow 0$ strongly in $L^{p_i}(\Omega)$ as $n\to 0$, we conclude

$$\lim_{n\to\infty}\int_{\{u_n>k\}}\left(k(x)+\beta|\partial_{x_i}u_n|^{p_i-1}\right)|\partial_{x_i}(T_k(u))||\Phi'_\rho(v_{n,k})|dx=0.$$

For the first integral of the right hand side of (3.15), observe that $\Phi_\rho(v_{n,k}) = 0$ on the set $\{u_n>k\}$, and by (1.3), (1.7) we can derive

$$-g_i^n(x,u_n,\nabla u_n)\leq l(u_n)\frac{a_i(x,\nabla u_n)\partial_{x_i}u_n}{\alpha},\quad \text{for any } i=1,\cdots,N.$$

Hence, for any $i=1,\cdots N$ (since $\max_{s\in[0,k]}l(s)=l(k)$) one has

$$\begin{aligned}-\int_\Omega g_i^n(x,u_n,\nabla u_n)\Phi_\rho(v_{n,k})dx &\leq \int_{\{u_n\leq k\}} g_i^n(x,u_n,\nabla u_n)|\Phi_\rho(v_{n,k})|dx\\ &\leq \frac{l(k)}{\alpha}\int_\Omega a_i(x,\nabla T_k(u_n))\partial_{x_i}(T_k(u_n))|\Phi_\rho(v_{n,k})|dx.\end{aligned}$$

Notice that $a_i(x,\nabla T_k(u_n))$ is bounded in $L^{p'_i}(\Omega)$ for any $i=1,\cdots,N$, and $\partial_{x_i}(T_k(u))|\Phi_\rho(v_{n,k})|\to 0$ strongly in $L^{p'_i}(\Omega)$ as n goes to ∞, as a result

$$\lim_{n\to\infty}\int_\Omega a_i(x,\nabla T_k(u_n))\partial_{x_i}(T_k(u))|\Phi_\rho(v_{n,k})|dx=0.$$

Now, we can write

$$\begin{aligned}&-\int_\Omega g_i^n(x,u_n,\nabla u_n)dx\\ &\qquad\leq \frac{l(k)}{\alpha}\int_\Omega a_i(x,\nabla T_k(u_n))\partial_{x_i}(v_{n,k})|\Phi_\rho(v_{n,k})|dx+\varepsilon_1(n)\end{aligned}$$

where $\varepsilon_1(n)$ is a quantity that tends to 0 as $n\to 0$. The above estimate combining with (3.15) gives

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,\nabla T_k(u_n)\partial_{x_i}(v_{n,k})\Phi'_\rho(v_{n,k})dx\\ &\leq \frac{l(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,\nabla T_k(u_n))\partial_{x_i}(v_{n,k})|\Phi_\rho(v_{n,k})|dx\\ &+\int_\Omega f_nh_n(u_n)\Phi_\rho(v_{n,k})dx+\varepsilon_2(n).\end{aligned}\tag{3.16}$$

Applying now (4.2) with $\mu=1,\nu=\frac{l(k)}{\alpha}$ and $\rho=\frac{l^2(k)}{4\alpha^2}$ which implies that

$$\sum_{i=1}^N\int_\Omega a_i(x,\nabla T_k(u_n)\partial_{x_i}(v_{n,k})dx\leq 2\int_\Omega f_nh_n(u_n)\Phi_\rho(v_{n,k})dx+\varepsilon_2(n).\tag{3.17}$$

By virtue of (1.5) and (3.12), we can affirm that

$$a_i(x,\nabla T_k(u_n))\partial_{x_i}(T_k(u_n)-T_k(u))\longrightarrow 0,\qquad \text{in } L^1(\Omega),\qquad \forall i=1,\cdots,N.$$

Then by adding and subtracting this quantity into (3.17), we get

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega\left[a_i(x,\nabla T_k(u_n))-a_i(x,T_k(u)\right]\partial_{x_i}(T_k(u_n)-T_k(u))dx\\ &\qquad\leq 2\int_\Omega f_nh_n(u_n)\Phi_\rho(v_{n,k})dx+\varepsilon_2(n).\end{aligned}\tag{3.18}$$

Step 2 : We prove that

$$\lim_{n\to\infty}\int_\Omega f_n h_n(u_n)\Phi_\rho(v_{n,k})dx = 0, \tag{3.19}$$

for any fixed $k > 0$.

If $h(0)$ is finite notice that

$$|h_n(u_n)f_n\Phi_\rho(v_{n,k})| \le Cf\|h\Phi_\rho\|_{L^\infty(\Omega)},$$

since $\Phi_\rho(v_{n,k})$ converges to 0 a.e in Ω, so by Lebesgue's dominated convergence theorem we easily pass to the limit to achieve the result. Otherwise, if $h(0) = +\infty$ using (3.8), (3.12), and Fatou's lemma, we obtain

$$\int_\Omega h(u)f\varphi dx \le C, \tag{3.20}$$

where C is a positive constant independent of n, then we have

$$\int_{\{x\in\Omega,\ u(x)=0\}} h(u)f\varphi\, dx < +\infty,$$

so that, $f\varphi = 0$ a.e. on $\{x \in \Omega,\ u(x) = 0\}$ for all nonnegative $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$, which yield to

$$f \equiv 0 \text{ a.e. on the set } \{x \in \Omega,\ u(x) = 0\}. \tag{3.21}$$

Now, for $\delta > 0$ small enough, using (1.8), we write

$$\int_\Omega f_n h_n(u_n)\Phi_\rho(v_{n,k})dx = \int_{\{u_n\le\delta\}} f_n h_n(u_n)\Phi_\rho(v_{n,k})dx + \sup_{s\in[\delta,\infty)} h(s)\int_{\{u_n>\delta\}} f_n\Phi_\rho(v_{n,k})dx = I^1_{n,\delta} + I^2_{n,\delta}.$$

We treat $I^1_{n,\delta}$, by Lebesgue's dominated convergence theorem with respect to n, the crucial result (3.21) and $f \in L^1(\Omega)$ it follows that,

$$\begin{aligned}\limsup_{\delta\to 0^+}\limsup_{n\to\infty} I^1_{n,\delta} &= \limsup_{\delta\to 0^+}\limsup_{n\to\infty}\left(C\delta^{1-\gamma}\int_{\{u_n\le\delta\}} fe^{\rho v^2_{n,k}}dx\right)\\ &= \limsup_{\delta\to 0^+}\left(C\delta^{1-\gamma}\int_{\{u\le\delta\}} f dx\right) = 0. \end{aligned}\tag{3.22}$$

For $I^2_{n,\delta}$, we note that

$$\Phi_\rho(v_{n,k}) \rightharpoonup 0, \quad \text{weakly* in } L^\infty(\Omega), \text{ as } n\to\infty,$$

Recalling that f_n converges to f in $L^1(\Omega)$, thus

$$\lim_{n\to\infty} I^2_{n,\delta} = 0. \tag{3.23}$$

By (3.22) and (3.23) we affirm (3.19), which is sufficient to apply Lemma 5 in [19] to obtain

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,\overrightarrow{p}}(\Omega).$$

The above strong convergence implies, for some subsequence still indexed by n, that

$$\nabla u_n \to \nabla u, \quad \text{a.e. } x \in \Omega. \tag{3.24}$$

Remark 3.6. In Lemma 1.2, we have shown that $g_i(x, u_n, \nabla u_n)$ is bounded in $L^1(\Omega)$, from this fact and by (3.24), we can apply the Fatou's Lemma to conclude that $g_i(x, u, \nabla u) \in L^1(\Omega)$ for any $i = 1, \cdots N.$.

□

3.3 Strong convergence of g_i^n in $L^1(\Omega)$

Lemma 3.7. *Suppose that the hypotheses of Theorem 3.1 are satisfied, let u_n be a weak solution of (1.1). Then*

$$g_i^n(x,u_n,\nabla u_n) \longrightarrow g_i(x,u,\nabla u), \quad \text{in } L^1(\Omega) \text{ as } n\to\infty. \tag{3.25}$$

Proof. Recalling that, g_i is a Caratheodory function. Hence, (3.12) and (3.24) allows to conclude

$$g_i^n(x,u_n,\nabla u_n) \longrightarrow g_i(x,u,\nabla u), \quad \text{a.e. in } \Omega \text{ as } n\to\infty,$$

So, it remains to prove the equi-integrability of the sequence $\{g_i^n(x,u_n,\nabla u_n)\}_n$ for any $i = 1,\cdots,N$. For $k>0$ fixed, let us consider the following function

$$V_{\lambda,k}(s) := \begin{cases} 0, & s\le k, \\ \frac{s-k}{\lambda} & k<s<k+\lambda, \\ 1 & s\ge k+\lambda. \end{cases}$$

Choosing $V_{\lambda,k}(u_n)$ as a test function in (2.2) obtaining

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla u_n)\partial_{x_i}u_n V'_{\lambda,n}(u_n)dx + \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)V_{\lambda,k}(u_n)dx$$
$$= \int_\Omega f_n h_n(u_n)V_{\lambda,n}(u_n)dx,$$

using (1.3), (1.6) and dropping the nonegative first term in the previous estimate, implies (since $V_{\lambda,k}(\cdot)\le 1$)

$$\sum_{i=1}^N \int_{\{u_n>k\}} g_i^n(x,u_n,\nabla u_n)V_{\lambda,k}(u_n)dx \le \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)V_{\lambda,k}(u_n)dx$$
$$\le \sup_{s\in[k,\infty)} h(s)\int_{\{u_n>k\}} f dx,$$

Applying Fatou Lemma (since $V_{\lambda,k}(u_n)\chi_{\{u_n>k\}}$ goes to 1 as $\lambda\to 0$) we obtain

$$\sum_{i=1}^N \int_{\{u_n>k\}} g_i^n(x,u_n,\nabla u_n)dx \le \sup_{s\in[k,\infty)} h(s)\int_{\{u_n>k\}} f dx. \tag{3.26}$$

Since (u_n) is bounded in $L^1(\Omega)$, we see that

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}} \text{meas}\{u_n>k\} = 0.$$

Moreover $f\in L^1(\Omega)$, yielding to

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}} \int_{\{u_n>k\}} f dx = 0. \tag{3.27}$$

On the other hand, for any measurable subset $E\subset\Omega$ and for all $k>0$, we can write for any $i=1,\cdots,N$

$$\int_E g_i^n(x,u_n,\nabla u_n)dx = \int_{E\cap\{u_n>k\}} g_i^n(x,u_n,\nabla u_n)dx \tag{3.28}$$
$$+ l(k)\int_{E\cap\{u_n\le k\}} |\partial_{x_i}T_k(u_n)|^{p_i}dx, \tag{3.29}$$

Since $\partial_{x_i}T_k(u_n)$ strongly converges to $\partial_{x_i}T_k(u)$ in $L^{p_i}(\Omega)$ for all i, the inequality (3.28) combined with (3.26) and (3.27) gives the equi-integrability of the sequence $\{g_i^n(x,u_n,\nabla u_n)\}_n$ for any $i=1,\cdots,N$. By Vitali's Theorem we get the sought result (3.25).

□

3.4 Strong convergence of the singular term in $L^1(\Omega)$

To achieve the aim of this step of the proof, we argue similarly as in [22, Lemma 5.2] and we use the compactness arguments obtained in previous subsections.

Lemma 3.8. *Under the assumptions of Theorem 3.1, let u_n be a weak solution of (1.1). Then*

$$\lim_{n\to+\infty}\int_\Omega h_n(u_n)f_n\varphi dx=\int_\Omega h(u)f\varphi dx, \tag{3.30}$$

for all $\varphi\in W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$.

Proof. If $h(0)$ is finite, we easily obtain (3.30) by Lebesgue's dominated convergence theorem. In the sequel, we deal the case $h(0)=+\infty$. For every fixed $\sigma>0$, we can write

$$\begin{aligned}\int_\Omega f_nh_n(u_n)\varphi dx&=\int_{\{x\in\Omega,\ u_n(x)>\sigma\}}f_nh_n(u_n)\varphi dx\\&\quad+\int_{\{x\in\Omega,\ u_n(x)\le\sigma\}}f_nh_n(u_n)\varphi dx.\end{aligned} \tag{3.31}$$

For the first term on the right-hand of (3.31), one has

$$0\le h_n(u_n)f_n\chi_{\{x\in\Omega,\ u_n(x)>\sigma\}}\varphi\le\sup_{s\in[\sigma,+\infty)}[h(s)]f\varphi\in L^1(\Omega).$$

Applying Lebesgue's dominated convergence theorem with respect to n, and using the fact that that

$$\chi_{\{x\in\Omega,\ u_n(x)>\sigma\}}\longrightarrow\chi_{\{x\in\Omega,\ u(x)\ge\sigma\}},\qquad\text{a.e. in }\Omega,$$

we get

$$\lim_{n\to\infty}\int_{\{x\in\Omega,\ u_n(x)>\sigma\}}f_nh_n(u_n)\varphi dx=\int_{\{x\in\Omega,\ u(x)\ge\sigma\}}h(u)f\varphi dx.$$

Recalling that $h(u)f\varphi\in L^1_{loc}(\Omega)$, and applying again Lebesgue's dominated convergence theorem, with respect to σ, it follow

$$\begin{aligned}\lim_{\sigma\to0^+}\lim_{n\to\infty}\int_{\{x\in\Omega,\ u_n(x)>\sigma\}}f_nh_n(u_n)\varphi dx&=\int_{\{x\in\Omega,\ u(x)\ge0\}}h(u)f\varphi dx\\&=\int_\Omega h(u)f\varphi dx.\end{aligned} \tag{3.32}$$

Now we focus on the second term on the right-hand of (3.31). For $\sigma>0$ sufficiently small, let us take $S_\sigma(u_n)\varphi$ as a test function in (2.2) where S_σ is defined by

$$S_\sigma(s):=\begin{cases}1, & s\le\sigma,\\ \frac{2\sigma-s}{\sigma} & \sigma<s<2\sigma,\\ 0 & s\ge2\sigma,\end{cases}$$

and φ is a nonnegative $W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$ function, we obtain

$$\begin{aligned}\int_{\{u_n\le\sigma\}}f_nh_n(u_n)\varphi dx&\le\int_\Omega f_nh_n(u_n)\varphi dx\\&\le\sum_{i=1}^N\int_\Omega a_i(x,\nabla u_n)\partial_{x_i}\varphi S_\sigma(u_n)dx+\sum_{i=1}^N\int_\Omega g_i^n(x,u_n,\nabla u_n)S_\sigma(u_n)\varphi dx\\&\quad-\frac{1}{\sigma}\sum_{i=1}^N\int_{\{x\in\Omega,\ \sigma<u_n(x)<2\sigma\}}a_i(x,\nabla u_n)\varphi\partial_{x_i}u_ndx\\&\le\sum_{i=1}^N\int_\Omega a_i(x,\nabla u_n)\partial_{x_i}\varphi S_\sigma(u_n)dx+\sum_{i=1}^N\int_\Omega g_i^n(x,u_n,\nabla u_n)S_\sigma(u_n)\varphi dx\\&\le I^1_{\sigma,n}+I^2_{\sigma,n}.\end{aligned}$$

By (1.4) and (3.2) one has that $a_i(x, , \nabla u_n)$ is bounded in $L^{p'_i}(\Omega)$, moreover by (3.13), (3.24) and the fact that $S_\sigma \leq 1$, we deduce up a sub-sequence that

$$a_i(x, , \nabla u_n)S_\sigma(u_n) \rightharpoonup a_i(x, \nabla u)S_\sigma(u),$$

weakly in $L^{p'_i}(\Omega)$ as n tends to infinity. This implies that

$$\limsup_{n\to\infty} I^1_{\sigma,n} \leq \sum_{i=1}^N \int_\Omega a_i(x, \nabla u)\partial_i\varphi S_\sigma(u)dx.$$

For $I^2_{\sigma,n}$, using (3.25) result we derive

$$\limsup_{n\to\infty} I^2_{\sigma,n} \leq \sum_{i=1}^N \int_\Omega g_i(x, u, \nabla u)S_\sigma(u)\varphi dx.$$

Thanks to (1.4), and that $\{S_\sigma(u)\}_\sigma$ converges to $\chi_{\{x\in\Omega,\ u(x)=0\}}$ a.e. in Ω as σ tends to 0, applying the Lebesgue Theorem, deducing that

$$\begin{aligned}&\limsup_{\sigma\to0}\limsup_{n\to\infty} \int_{\{x\in\Omega,u_n(x)\leq\sigma\}} f_nh_n(u_n)\varphi dx \\ &\leq \sum_{i=1}^N \int_{\{x\in\Omega,\ u(x)=0\}} a_i(x, \nabla u)\partial_{x_i}\varphi dx + \sum_{i=1}^N \int_{\{x\in\Omega,u(x)=0\}} g_i(x, u, \nabla u)S_\sigma(u)\varphi dx,\end{aligned} \tag{3.33}$$

for all nonnegative $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, and it follows from (1.3), (1.6) and (1.7) that $a_i(x, 0) = 0, g_i(x, 0, 0) = 0$ for almost every $x \in \Omega$ and for any $i = 1, \cdots, N$. It allows to deduce

$$\limsup_{\sigma\to0}\limsup_{n\to\infty} \int_{\{x\in\Omega,u_n(x)\leq\sigma\}} f_nh_n(u_n)\varphi dx = 0. \tag{3.34}$$

By (3.32) and (3.34) we deduce that, for all nonnegative $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega f_nh_n(u_n)\varphi dx = \int_\Omega fh(u)\varphi dx. \tag{3.35}$$

Moreover, in the general case (the function φ has any sign) we can write $\varphi = \varphi^+ - \varphi^-$ with $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = -\min\{\varphi, 0\}$, we conclude that (3.35) holds for every $\varphi \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$. This achieves (3.30). □

Finally, It yields from (3.2) and (1.8) that

$$a_i(x, \nabla u_n) \rightharpoonup a_i(x, \nabla u) \quad \text{weakly in } L^{p'_i}(\Omega), \forall i = 1, \cdots, N. \tag{3.36}$$

According to the last convergence and (3.25), (3.35) we can pass to the the limit in the approximate problem (2.2) to obtain (1.14).

4 Existence result for $\theta = 1$

In the present section, we consider the problem (1.1) where the datum f belongs to $L^1(\Omega)$. Our goal is to prove the existence of finite energy solutions to problem (1.1) without any additional condition on p_i. To obtain an a priori estimate in the energy space $W_0^{1,\overrightarrow{p}}(\Omega)$, we need to assume a kind of coercivity condition on g_i.

Theorem 4.1. *Let $f \in L^1(\Omega)$, assume that hypotheses (1.3)-(1.5) and (1.8) hold true, we also assume for all $i = 1, \cdots N$, g_i be a Caratheodory function satisfying the conditions (1.6)-(1.7) and*

$$|g_i(x, s, \xi)| \geq \mu|\xi_i|^{p_i}, \quad \forall i = 1, \ldots, N, \tag{4.1}$$

where $\mu > 0$, for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$. Then there exists a weak solution u for problem (1.1) in the sens of definition 1.3.

Remark 4.2. The natural growth term satisfying (4.1) provide a supplementary regularity to the solutions for the problem (1.1) with L^1-data (see theorem 3.5 in [1]). Moreover, in this case we do not need to added more assumptions on p_i (see theorem 2.3 in [4]).

Remark 4.3. Notice that we have assumed an additional hypothesis in comparison with Theorem 3.1. Therefore, the process of passing to the limit remains similar to that in Theorem 3.1 when we prove the compactness result.

4.1 A priori estimates when $\theta = 1$

Lemma 4.4. *Assume that the assumptions of Theorem 4.1 hold. let u_n be a nonnegative solution to problem (2.2). Then there exists a positive constant C independent of n such that*

$$\|u_n\|_{1,\overrightarrow{p}} \leq C, \tag{4.2}$$

Proof. We choose $T_k(u_n)$ as test function in (2.2), obtaining

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\partial_{x_i}(T_k(u_n))dx + \sum_{i=1}^{N}\int_{\Omega} g_i^n(x,u_n,\nabla u_n)T_k(u_n)dx \\ = \int_{\Omega} f_n h_n(u_n)T_k(u_n)dx,\end{aligned}$$

which implies by virtue of (1.3) and (1.8)

$$\begin{aligned}\alpha\sum_{i=1}^{N}\int_{\{u_n\leq k\}} |\partial_{x_i}u_n|^{p_i}dx &+ \sum_{i=1}^{N}\int_{\{u_n\leq k\}} g_i^n(x,u_n,\nabla u_n)u_n dx \\ &+ k\sum_{i=1}^{N}\int_{\{u_n> k\}} g_i^n(x,u_n,\nabla u_n)dx \\ &\leq \int_{\{u_n\leq k\}} h_n(u_n)f_n u_n dx + k\int_{\{u_n>k\}} h_n(u_n)f_n dx \\ &\leq 2k^{1-\gamma}\int_{\Omega} f dx \leq Ck^{1-\gamma}.\end{aligned} \tag{4.3}$$

Thanks to (1.6), (4.1) and (4.3) we deduce

$$\alpha\sum_{i=1}^{N}\int_{\{u_n\leq k\}} |\partial_{x_i}u_n|^{p_i}dx + k\mu\sum_{i=1}^{N}\int_{\{u_n>k\}} |\partial_{x_i}u_n|^{p_i}dx \leq C.$$

From the previous estimate it follows

$$\min(\alpha,k\mu)\sum_{i=1}^{N}\int_{\Omega} |\partial_{x_i}u_n|^{p_i}dx \leq C.$$

Consequently, u_n is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ with respect to n. □

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