Palestine Journal of Mathematics

Vol 14(2)(2025) , 653–661

Hesitant Fuzzy T-ideals of TM algebra

Alemayehu Girum*, Berhanu Assaye, Bekalu Tarekegn and Beza Lamesgin

Communicated by Madeleine Al Tahan

MSC 2010 Classifications: Primary 03E72, 03G25; Secondary 08A72.

Keywords and phrases: TM algebra, T-ideals of TM algebra, Hesitant fuzzy set , Hesitant fuzzy T-ideals.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Alemayehu Girum*

Abstract: This paper introduces and explores the concepts of hesitant fuzzy ideals, hesitant fuzzy T-ideals, and hesitant fuzzy closed T-ideals in the context of TM algebras. The relationships between these fuzzy ideals and their level subsets are described in detail, and the homomorphic pre-images of these ideals are studied to uncover important properties. These findings extend the classical notions of ideals and T-ideals in algebraic structures, providing a broader framework for handling uncertainty and vagueness in TM algebras.

1 Introduction

Isaki and Imai examined two types of algebras: BCI-algebras and BCK-algebras[4, 5]. In H. S. Kim and J. Neggers explained the notion of d-algebras, which is another useful generalization of BCK-algebras and investigated several relations between d-algebras and oriented diagraphs[15]. Roh,Jun, and Kim introduced a new concept known as a BH-algebra, which is a generalization of BCI and BCK-algebras. A fuzzy set f of a set S is defined as a mapping mapping elements of S to values within the closed interval $[0, 1]$. Zadeh in 1965 pioneered the introduction of a fuzzy set of a set was initially introduced[28]. H. S. Kim and C. B. Kim [7] introduced the concept of a BG-algebra, which generalizes B-algebras. Somjanta et al. [22] introduced and explored various properties of fuzzy UP-subalgebras and fuzzy UP-ideals within UP-algebras. Megalai and Tamilarasi [11] introduced TMA, which generalize BCK, BCH, Q, and BCI-algebras. Chanwit and Prabpayak [19] also presented the concept of homomorphisms in fuzzy TM algebra and established several properties. Mostafa et al. [15] studied the idea of fuzzy TM algebra within TM algebra and investigated their related properties. Also, more generalized and flexible extension of the classical fuzzy set theory, which allows for more expressive modeling of uncertainty we called hesitant fuzzy set which is introduced by Torra see [1, 24, 25, 21]. A hesitant fuzzy sets are usefull applications in decision-making, pattern recognition, and expert systems where uncertainty is inherent and cannot be accurately captured by a single membership value. Also, they provide a more flexible framework for handling compound or complex uncertain information.
Jun and Song [6] introduced the notions of Boolean, prime, ultra, good hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. Iampan[3] introduced a new algebraic structure called a UP-algebra. Muhiuddin, Ghulam, Madeline Al-Tahan, Ahsan Mahboob, Sarka Hoskova-Mayerova, and Saba Al-Kaseasbeh in 2022 introduced linear diophantine fuzzy Set theory applied to BCK/BCI-Algebras[16]. Mechderso in 2023 investigated the concept of fuzzy pseudo-UP ideals of pseudo-UP algebra[10], and Mosrijai et al.[14] introduced the notion of hesitant fuzzy sets on UP-algebras. Madeleine Al-Tahan, Akbar Rezaei, Saba Al-Kaseasbeh, Bijan Davvaz, and Muhammad Riaz in 2023 investegated linear diophantine fuzzy n-fold weak subalgebras of a BE-algebra[2]. It is known that the notions of hesitant fuzzy ideals of UP algebra play an important role in studying the many logical algebras motivated by it, we discuses the application of hesitant fuzzy sets on ideals in TM algebra using the idea in [14].
In this article, we apply the notion of hesitant fuzzy sets by Torra to the ideals in TM-algebra.

Also, we introduce the connection between the level subsets and T-ideals on hesitant fuzzy sets on TM algebra, to detremine the connections between Cartesian product and hesitant fuzzy T-ideals of TM algebra, to find the basic properties of the homomorphic pre-image of hesitant fuzzy T-ideals of TM algebra are established.

2 Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper.

Definition 2.1. [12] A TM-algebra $(\Bbbk, \star, 0)$ is a non-empty set $\Bbbk$ with a constant "0" and a binary operation " $\star$ " satisfying the following axioms:
i) $\rho \star 0 = \rho$
ii) $(\rho \star \varphi) \star (\rho \star \varrho) = \varrho \star \varphi$, for all $\rho, \varphi, \varrho \in \Bbbk$.

Definition 2.2. [12] Let A be a non-empty subset of a TM -algebra $\Bbbk$. Then A is called a TM-subalgebra of $\Bbbk$ if $\rho \star \varphi \in$ A, for all $\rho, \varphi \in$ A.

Example 2.3. [11] Let $\mathbb{Z}$ be the set of all integers. Let " $\star$ " be a binary operation defined by $\rho \star \varphi = \rho - \varphi$, for all $\rho, \varphi \in \mathbb{Z}$, where $'-'$ is the usual subtraction of integers. Then $(\mathbb{Z}, \star, 0)$ is a TM-algebra. Since $\rho \star 0 = \rho - 0 = \rho$ for all $\rho \in \mathbb{Z}$, and $(\rho \star \varphi) \star (\rho \star \varrho) = (\rho - \varphi) - (\rho - \varrho) = \varrho - \varphi$, for all $\rho, \varphi, \varrho \in Z$. Then $(\mathbb{Z}, \star, 0)$ is a TM-algebras. Moreover, $(n\mathbb{Z}, \star, 0)$ is a TM-subalgebra of $(\mathbb{Z}, \star, 0)$. Where, $n\mathbb{Z} = \{n\rho : \rho \in \mathbb{Z}\}$.

In $\Bbbk$ we can define a binary operation $\leq$ by $\rho \leq \varphi$ if and only if $\rho \star \varphi = 0$.

Proposition 2.4. *[11] In any TM-algebra* $(\Bbbk, \star, 0)$*, the following properties hold for all* $\rho, \varphi, \varrho \in \Bbbk$.
1) $\rho \star \rho = 0$
2) $(\rho \star \varphi) \star \rho = 0 \star \varphi,$
3) $\rho \star (\rho \star \varphi) = \varphi,$
4) $(\rho \star \varrho) \star (\varphi \star \varrho) \leq \rho \star \varphi,$
5) $(\rho \star \varphi) \star \varrho = (\rho \star \varrho) \star \varphi,$
6) $\rho \star 0 = 0 \Rightarrow \rho = 0,$
7) $\rho \star \varrho \leq \varphi \star \varrho$ *and* $\varrho \star \varphi \leq \varrho \star \rho,$
8) $\rho \star (\rho \star (\rho \star \varphi)) = \rho \star \varphi,$
9) $0 \star (\rho \star \varphi) = \varphi \star \rho = (0 \star \rho) \star (0 \star \varphi),$
10) $(\rho \star (\rho \star \varphi)) \star \varphi = 0,$
11) $\rho \star \varphi = 0$ *and* $\varphi \star \rho = 0$ *imply* $\rho = \varphi.$

Definition 2.5. [15, 13] A non-empty subset I of a TM algebra $\Bbbk$ is called a TM-ideal of $\Bbbk$ if (I_1) $0 \in I$, (I_2) $\rho \star \varrho$ and $\varrho \in I$ imply $\rho \in I$. A non-empty subset I of a TM algebra $\Bbbk$ is called a T-ideal of $\Bbbk$ if it satisfies (I_1) and (I_3) $(\rho \star \varrho) \in I$ whenever $(\rho \star \varphi) \star \varrho \in I$ and $\varphi \in I$. A non-empty subset I of a TM algebra $\Bbbk$ is called closed T-ideal of $\Bbbk$ if it satisfies (I_4) $0 \star \rho \in I$ and (I_5) $(\rho \star \varrho) \in I$ whenever $(\rho \star \varphi) \star \varrho \in I$ and $\varphi \in I$ for all $\rho, \varphi, \varrho \in \Bbbk$.

Definition 2.6. [23] Let $(\Bbbk, \star_\rho, 0_\rho)$ and $(\ell, \star_\varphi, 0_\varphi)$ be two TM-algebras. The direct product $\Bbbk \times \ell$ is also a TM-algebra with the binary operation $\star$ defined as $(\rho_1, \rho_2) \star (\varphi_1, \varphi_2) = (\rho_1 \star_\rho \varphi_1, \rho_2 \star_\varphi \varphi_2)$ for all $(\rho_1, \rho_2), (\varphi_1, \varphi_2) \in \Bbbk \times \ell$ and $0 = (0_\rho, 0_\varphi)$.

Definition 2.7. [9] Let $\Bbbk$ be a reference set, a hesitant fuzzy set $\mathit{\Gamma}$ on $\Bbbk$, denoted by HFS($\Bbbk$) is defined in terms of a function τ that when applied to $\Bbbk$ returns a subset of $[0, 1]$.

Xu and Xia [27] expressed an HFS($\Bbbk$) by the following mathematical expression:

$$\mathit{\Gamma} = \{\langle \rho, \tau(\rho) \rangle / \rho \in \Bbbk\},$$

where $\tau(\rho)$ is a set of some values in $[0, 1]$ denoting the possible membership degrees of the element $\rho \in \Bbbk$ to the set $\mathit{\Gamma}$. For agreements, Xu and Xia [27] called $\tau(\rho)$ a hesitant fuzzy element HFE($\Bbbk$).

Definition 2.8. [24, 26] Let $\tau, \tau_1, \tau_2 \in HFS(\mathbb{k})$. Then
1. Lower bound: $\tau^-(\rho) = \min \tau(\rho)$.
2. Upper bound: $\tau^+(\rho) = \max \tau(\rho)$.
3. Complement: $\tau^c(\rho) = \bigcup_{\gamma \in \tau(\rho)} \{1-\gamma\}$.
4. Involutive: $(\tau^c(\rho))^c = \tau(\rho)$.
5. α-upper bound: $\tau^+_\alpha(\rho) = \{\tau \in \tau(\rho) \mid \tau \geq \alpha\}$
6. α-lower bound: $\tau^-_\alpha(\rho) = \{\tau \in \tau(\rho) \mid \tau \leq \alpha\}$.

α-upper bound and α-lower bounds are a crucial requirement for determining the union and intersection of HFS(X). That is
7. $(\tau_1 \cup \tau_2)(\tau) = \left\{\tau \in (\tau_1(\rho) \cup \tau_2(\rho)) \mid \tau \geq \max\left(\tau_1^-, \tau_2^-\right)\right\} = (\tau_1(\rho) \cup \tau_2(\rho))^+_\alpha$ for $\alpha = \max\left(\tau_1^-(\rho), \tau_2^-(\rho)\right)$.
8. $(\tau_1 \cap \tau_2)(\rho) = \left\{\tau \in (\tau_1(\rho) \cup \tau_2(\rho)) \mid \tau \leq \min\left(\tau_1^+, \tau_2^+\right)\right\} = (\tau_1(\rho) \cup \tau_2(\rho))^-_\alpha$ for $\alpha = \min\left(\tau_1^+(\rho), \tau_2^+(\rho)\right)$.
9. $\tau_1 \otimes \tau_2 = \bigcup_{\substack{\gamma_1 \in \tau_1 \\ \gamma_2 \in \tau_2}} \{\gamma_1 \gamma_2\}$.
10. $\tau_1 \oplus \tau_2 = \bigcup_{\substack{\gamma_1 \in \tau_1 \\ \gamma_2 \in \tau_2}} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$

Definition 2.9. [8] Let $(\tau_i)_{i\in I} \subset HFS(\mathbb{k})$ be an indexed collection of finite hesitant fuzzy sets. Then
(i) the union of $(\tau_i)_{i\in I}$, denoted by $\bigcup_{i\in I} \tau_i$, is a hesitant fuzzy set in $\mathbb{k}$ defined as follows: for each $\rho \in \mathbb{k}$,

$$\left(\bigcup_{i\in I} \tau_i\right)(x) = \bigcup_{\gamma_i \in \tau_i(x)} \bigvee_{i\in I} \gamma_i,$$

(ii) the intersection of $(\tau_i)_{i\in I}$, denoted by $\bigcap_{i\in I} \tau_i$, is a hesitant fuzzy set in $\mathbb{k}$ defined as follows: for each $\rho \in \mathbb{k}$,

$$\left(\bigcap_{i\in I} \tau_i\right)(\rho) = \bigcup_{\gamma_i \in \tau_i(\rho)} \bigwedge_{i\in I} \gamma_i.$$

Definition 2.10. [8] Let $\mathbb{k}$ and ℓ be a nonempty sets, let $\tau_{\mathbb{k}} \in HFS(\mathbb{k})$ and $\tau_\ell \in HFS(\ell)$ and let $f: \mathbb{k} \to \ell$ be a mapping. Then
(i) the image of $\tau_{\mathbb{k}}$ under f, denoted by $f(\tau_{\mathbb{k}})$, is a hesitant fuzzy set in ℓ defined as follows: for each $\varphi \in \ell$,

$$f(\tau_{\mathbb{k}})(\varphi) = \begin{cases} \bigcup_{\rho \in f^{-1}(\varphi)} \tau_{\mathbb{k}}(\varphi) & \text{if } f^{-1}(\varphi) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

(ii) the preimage of τ_ℓ under f, denoted by $f^{-1}(\tau_\ell)$, is a hesitant fuzzy set in ℓ defined as follows: for each $\rho \in \mathbb{k}$,

$$f^{-1}(\tau_\ell)(\rho) = \tau_\ell \circ f(\rho).$$

3 Hesitant Fuzzy T-ideals of TM algebra

This section, we introduce the concepts of hesitant fuzzy TM-ideal of TM algebras and some interesting properties are provided. Let $\mathbb{k}$ and ℓ denote a TM-algebras and $2^{[0,1]}$ is denoted the power set of $[0,1]$ unless otherwise specfied throughout this and the following section.

Definition 3.1. [6] Let $\mathbb{k}$ be a TM-algebra. Given a non-empty subset A of $\mathbb{k}$, a hesitant fuzzy set

$$\Gamma_{\mathbb{k}} := \{(\rho, \tau_{\mathbb{k}}(\rho)) \mid \rho \in \mathbb{k}\}$$

on $\mathbb{k}$ satisfying the following condition:

$$\tau_{\mathbb{k}}(\rho) = \emptyset \text{ for all } \rho \notin A$$

is called a hesitant fuzzy set related to A (briefly, A-hesitant fuzzy set) on $\mathbb{k}$, and is represented by $\Gamma_A := \{(\rho, \tau_A(\rho)) \mid \rho \in \mathbb{k}\}$, where τ_A is a mapping from $\mathbb{k}$ to $2^{[0,1]}$ with $\tau_A(\rho) = \emptyset$ for all $\rho \notin A$.

Definition 3.2. A subset I of a TM-algebra $\Bbbk$ is called hesitant fuzzy ideal of $\Bbbk$ if it satisfies the following condition:

$$(\forall \rho, \varphi \in \Bbbk)\begin{pmatrix} I(0) \supseteq I(\rho) \\ I(\rho) \supseteq I(\rho \star \varphi) \cap I(\varphi) \end{pmatrix} \tag{3.1}$$

Definition 3.3. A subset I of a TM-algebra $\Bbbk$ is called hesitant fuzzy TM-ideal of $\Bbbk$ if it satisfies the following condition:

$$(\forall \rho, \varphi, \varrho \in \Bbbk)\begin{pmatrix} I(0) \supseteq I(\rho) \\ I(\rho \star \varphi) \supseteq I(\rho \star \varrho) \cap I(\varrho \star \varphi) \end{pmatrix} \tag{3.2}$$

Example 3.4. Let $\Bbbk = \{0, 1, 2, 3\}$ with the following Cayley table:

$\star$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Then, $(\Bbbk, \star, 0)$ is a TM algebra. We define the hesitant fuzzy set $\Gamma_{\Bbbk} := \{(\rho, \tau_{\Bbbk}(\rho)) \mid \rho \in \Bbbk\}$ on $\Bbbk$ as follows:
$I(0) = [0, 1], I(1) = \{0.1\}, I(2) = \emptyset, I(3) = \{0.2, 0.3\}$. Then I is a hesitant fuzzy TM-ideal of $\Bbbk$.

Definition 3.5. A subset I of a TM-algebra $\Bbbk$ is called hesitant fuzzy T-ideal of $\Bbbk$ if it satisfies the following condition:

$$(\forall \rho, \varphi, \varrho \in \Bbbk)\begin{pmatrix} I(0) \supseteq I(\rho) \\ I(\rho \star \varrho) \supseteq I((\rho \star \varphi) \star \varrho) \cap I(\varphi) \end{pmatrix} \tag{3.3}$$

Example 3.6. Let $\Bbbk = \{0, 1, 2, 3\}$ with the following Cayley table:

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then, $(\Bbbk, \star, 0)$ is a TM algebra. We define the hesitant fuzzy set $\Gamma_{\Bbbk} := \{(\rho, \tau_{\Bbbk}(\rho)) \mid \rho \in \Bbbk\}$ on $\Bbbk$ as follows:
$I(0) = [0, 1], I(1) = \{0.7\}, I(2) = I(3) = \{0.3\}$. It is easly verify that I is a hesitant fuzzy T-ideal of $\Bbbk$.

Theorem 3.7. *A hesitant fuzzy set Γ in a TM algebra $\Bbbk$ is ahesitant fuzzy T-ideal if and only if it is a hesitant fuzzy ideal of $\Bbbk$.*

Proof. Let I be a hesitant fuzzy T-ideal of $\Bbbk$. Then $I(\rho \star \varrho) \supseteq I((\rho \star \varphi) \star \varrho) \cap I(\varphi), \forall \rho, \varphi, \varrho \in \Bbbk$.
Assuming $\varrho = 0$ we have $I(\rho) \supseteq I((\rho \star \varphi) \cap I(\varphi)$.
Also, $I(0) \supseteq I(\rho)$.
Hence, I is a hesitant fuzzy ideal of $\Bbbk$.
Conversely, suppose that I is a hesitant fuzzy ideal of $\Bbbk$. Then $I(\rho) \supseteq I((\rho \star \varphi) \cap I(\varphi), \forall \rho, \varphi \in \Bbbk$.
It follows that for all $\rho, \varphi, \varrho \in \Bbbk$ we have $I(\rho \star \varrho) \supseteq I((\rho \star \varphi) \star \varrho) \cap I(\varphi)$. This complate the proof. □

Theorem 3.8. *A hesitant fuzzy ideal of a TM algebra $\Bbbk$ is order reversing.*

Proof. Let $\rho, \varphi, \varrho \in \Bbbk$ be such that $\rho \leq \varrho$. Then $\rho \star \varrho = 0$, and so $I(\rho) \supseteq I(\rho \star \varphi) \cap I(\varphi) = I(0) \cap I(\varphi) = I(\varphi)$.
The proof is completed. □

Definition 3.9. The characteristic hesitant fuzzy set of a subset A of a TM algebra $\Bbbk$ is defined to be

$$\tau_{\Bbbk_A}(\rho) = \begin{cases} [0,1] & \text{if } \rho \in A \\ \emptyset & \text{otherwise} \end{cases}$$

Lemma 3.10. *The constant of a TM algebra $\Bbbk$ is a non-empty subset A of $\Bbbk$ if and only if $\tau_{\Bbbk_A}(0) \supseteq \tau_{\Bbbk_A}(\rho)$*

Proof. If $0 \in A$, then $\tau_A(0) = [0,1]$. Thus $\tau_A(0) = [0,1] \supseteq \tau_A(\rho)$, for all $\rho \in \Bbbk$.
Coversely, assume that $\tau_A(0) \supseteq \tau_A(\rho)$, for all $\rho \in \Bbbk$. Since A is a non-empty subset of A, we have there exists $t \in A$ for some $t \in \Bbbk$. Then $\tau_A(0) \supseteq \tau_A(t) = [0,1]$. So, $\tau_A(0) = [0,1]$.
Hence, $0 \in A$. □

Theorem 3.11. *A non-empty subset I of $\Bbbk$ is a TM-ideal of $\Bbbk$ if and only if the characteristic hesitant fuzzy set is a hesitant fuzzy TM-ideal of $\Bbbk$.*

Proof. Assume that I is a TM-ideal of $\Bbbk$. Since $0 \in I$ it follows from Lemma (3.10) that $\tau_I(0) = [0,1] \supseteq \tau_I(\rho)$ for all $\rho \in \Bbbk$. Next, let $\rho, \varrho \in \Bbbk$.
Case 1: If $\rho, \varrho \in I$, then $\tau_I(\rho) = [0,1]$ and $\tau_I(\varrho) = [0,1]$.
Hence, $\tau_I(\rho) = [0,1] \supseteq \tau_I(\rho \star \varrho) = \tau_I(\rho \star \varrho) \cap \tau_I(\varrho)$.
Case 2: If $\rho \in I$ and $\varrho \notin I$, then $\tau_I(\rho) = [0,1]$ and $\tau_I(\varrho) = \emptyset$.
Hence, $\tau_I(\rho) = [0,1] \supseteq \emptyset = \tau_I(\rho \star \varrho) \cap \tau_I(\varrho)$.
Case 3: If $\rho \notin I$ and $\varrho \in I$, which is similar to case(2).
Case 4: If $\rho \notin I$ and $\varrho \notin I$, then $\tau_I(\rho) = \emptyset$ and $\tau_I(\varrho) = \emptyset$.
Hence, $\tau_I(\rho) \supseteq \emptyset = \tau_I(\rho \star \varrho) \cap \tau_I(\varrho)$.
Hence, τ_I is an hesitant fuzzy ideal of $\Bbbk$.
Conversely, assume that τ_I is an hesitant fuzzy ideal of $\Bbbk$. Since $\tau_I(0) \supseteq \tau_I(\rho)$ for all $\rho \in \Bbbk$, it follows from Lemma (3.10) that $0 \in I$. Let $\rho, \varrho \in \Bbbk$ be such that $\rho \star \varrho \in I$ and $\varrho \in I$. Then, $\tau_I(\rho \star \varrho) = \tau_I(\varrho) = [0,1]$. Thus, $\tau_I(\rho) \supseteq \tau_I(\rho \star \varrho) \cap \tau_I(\varrho)$, so $\tau_I(\rho) = [0,1]$. Hence, $\rho \in I$ and so, I is an ideal of X. □

Definition 3.12. Given a non-empty subset A of $\Bbbk$, an A-hesitnat fuzzy set $\mathit{\Gamma}_A := \{(\rho, I_A(\rho)) \mid \rho \in \Bbbk\}$ on $\Bbbk$ is called a hesitant fuzzy closed T-ideal of $\Bbbk$ related to A (brifely, A-hesitant fuzzy closed T-ideal of $\Bbbk$) if it satisfies:

$$(\forall \rho, \varphi, \varrho \in A)\begin{pmatrix} I_A(0 \star \rho) \supseteq I_A(\rho) \\ I_A(\rho \star \varrho) \supseteq I_A((\rho \star \varphi) \star \varrho) \cap I_A(\varphi) \end{pmatrix} \tag{3.4}$$

Definition 3.13. Given a non-empty subset A of $\Bbbk$ and let $r_1, r_2 \in 2^{[0,1]}$. A-hesitant fuzzy set $\mathit{\Gamma}_A := \{(\rho, \tau_A(\rho)) \mid \rho \in \Bbbk\}$ be given. Then, the set $\mathbb{U}(\tau_A : r_1) = \{\rho \in \Bbbk \supseteq r_1\}$ and $\mathbb{L}(\tau_A : r_2) = \{\rho \in \Bbbk \subseteq r_2\}$ are called an upper r_1-level of $\tau_A(\rho)$ and lower r_2-level set of $\tau_A(\rho)$ respectively.

Theorem 3.14. *$\mathit{\Gamma}_{\Bbbk} := \{(\rho, \tau_{\Bbbk}(\rho)) \mid \rho \in \Bbbk\}$ be a hesitant fuzzy T-ideal of a TM-algebra $\Bbbk$ if and only if the non-empty upper r_1-level cut $\mathbb{U}(I_A : r_1)$ is a T-ideal of $\Bbbk$, for any $r_1, r_2 \in 2^{[0,1]}$.*

Proof. Suppose that $\mathit{\Gamma}_{\Bbbk} := \{(\rho, I_{\Bbbk}(\rho)) \mid \rho \in \Bbbk\}$ be a hesitant fuzzy T-ideal of a TM-algebra $\Bbbk$. For any $r_1, r_2 \in 2^{[0,1]}$. We define the sets $\mathbb{U}(I_A : r_1) = \{\rho \in \Bbbk \supseteq r_1\}$. Since $\mathbb{U}(I_A : r_1) \neq \emptyset$, for $\rho \in \mathbb{U}(I_A : r_1)$ .Then
$\Rightarrow I_A(\rho) \supseteq r_1$
$\Rightarrow I_A(0) \supseteq I_A(\rho) \supseteq r_1$
$\Rightarrow I_A(0) \supseteq r_1$
$\Rightarrow 0 \in \mathbb{U}(I_A : r_1)$.
Assume that $(\rho \star \varphi) \star \varrho \in \mathbb{U}(I_A : r_1)$ and $\varphi \in \mathbb{U}(I_A : r_1)$. It implies that $(I_A((\rho \star \varphi) \star \varrho)) \supseteq r_1$ and $I_A(\varphi) \supseteq r_1$. Since $I_A(\rho \star \varrho) \supseteq I_A((\rho \star \varphi) \star \varrho) \cap I_A(\varphi) \supseteq r_1 \cap r_1 = r_1$. Thus, $I_A(\rho \star \varrho) \supseteq r_1$.

Therefore, $\mathbb{U}\left(I_A : r_1\right)$ is a T-ideal of $\Bbbk$.
Conversely, suppose that for any $r_1 \in 2^{[0,1]}$ the non-empty subset $\mathbb{U}\left(I_A : r_1\right)$ is a T-ideal of $\Bbbk$. Let $\rho \in \Bbbk$. Then $I_A(\rho) \in 2^{[0,1]}$. Choose $r_1 = I_A(\rho) \in 2^{[0,1]}$. Then $I_A(\rho) \supseteq r_1$. So, $\rho \in \mathbb{U}(I_A, r_1)$ is a T-ideal of $\Bbbk$ and thus $0 \in \mathbb{U}(I_A, r_1)$. Thus, $I_A(0) \supseteq r_1 = I_A(\rho)$. Let $\rho, \varphi, \varrho \in \Bbbk$. Then $I_A(\varphi), I_A((\rho \star \varphi) \star \varrho) \in 2^{[0,1]}$. Now, taking $r_1 = I_A((\rho \star \varphi) \star \varrho) \cap I_A(\varphi) \in 2^{[0,1]}$. So, $I_A((\rho \star \varphi) \star \varrho) \supseteq r_1$ and $I_A(\varphi) \supseteq r_1$. It implies that $\varphi, (\rho \star \varphi) \star \varrho \in \mathbb{U}(I_A, r_1) \neq \emptyset$. Since $\mathbb{U}(I_A, r_1)$ is a T-ideal of $\Bbbk$. Hence, $(\rho \star \varphi) \in \mathbb{U}(I_A, r_1)$. Thus, $I_A(\rho \star \varphi) \supseteq r_1 = I_A((\rho \star \varphi) \star \varrho) \cap I_A(\varphi)$.
Therefore, $\mathit{\Gamma}$ is a hesitant fuzzy T-ideal of a TM algebra $\Bbbk$. □

Theorem 3.15. $\mathit{\Gamma}_{\Bbbk} := \{(\rho, \tau_{\Bbbk}(\rho)) \mid \rho \in \Bbbk\}$ *be a hesitant fuzzy T-ideal of a TM-algebra* $\Bbbk$ *if and only if the non-empty upper* r_1*-level cut* $\mathbb{U}\left(I_A : r_1\right)$ *is a closed T-ideal of* $\Bbbk$*, for any* $r_1, r_2 \in 2^{[0,1]}$.

Definition 3.16. Let $\{\mathit{\Gamma}_\alpha \mid \alpha \in \Omega\}$ be a family of hesitant fuzzy sets on a reference set $\Bbbk$. We define the hesitant fuzzy set $\bigcap_{\alpha\in\Omega} \mathit{\Gamma}_\alpha = \left(\bigcap_{\alpha\in\Omega} \tau_\alpha\right)$ by $\left(\bigcap_{\alpha\in\Omega} \tau_\alpha\right)(\rho) = \bigcap_{\alpha\in\Omega} \tau_\alpha(\rho)$ and for all $\rho \in \Bbbk$, which is called the hesitant intersection of hesitant fuzzy sets.

Theorem 3.17. *If* $\{\mathcal{I}_\alpha \mid \alpha \in \Omega\}$ *is a family of hesitant fuzzy* T*-ideal of* $\Bbbk$*, then* $\bigcap_{\alpha\in\Omega} \mathcal{I}_\alpha$ *is an hesitant fuzzy T-ideal of* $\Bbbk$.

Proof. Let $\{\mathcal{I}_\alpha \mid \alpha \in \Omega\}$ be a family of hesitant fuzzy T-ideal of $\Bbbk$. Let $\rho \in \Bbbk$. Then

$$\left(\bigcap_{\alpha\in\Omega} \tau_\alpha\right)(0) = \bigcap_{\alpha\in\Omega} \tau_\alpha(0) \supseteq \bigcap_{\alpha\in\Omega} \tau_\alpha(\rho) = \left(\bigcap_{\alpha\in\Omega} \tau_\alpha\right)(\rho)$$

.

Let $\rho, \varphi, \varrho \in \Bbbk$. Then

$$\begin{aligned}
\left(\bigcap_{\alpha\in\Omega} I_\alpha\right)(\rho \star \varphi) &= \bigcap_{\alpha\in\Omega} I_\alpha(\rho \star \varphi) \\
&\supseteq \bigcap_{\alpha\in\Omega} \left(I_\alpha((\rho \star \varphi) \star \varrho) \cap I_\alpha(\varphi)\right) \\
&= \left(\bigcap_{\alpha\in\Omega} I_\alpha((\rho \star \varphi) \star \varrho)\right) \cap \left(\bigcap_{\alpha\in\Omega} I_\alpha(\varphi)\right) \\
&= \left(\bigcap_{\alpha\in\Omega} I_\alpha\right)((\rho \star \varphi) \star \varrho) \cap \left(\bigcap_{\alpha\in\Omega} I_\alpha\right)(\varphi)
\end{aligned}$$

Hence, $\bigcap_{\alpha\in\Omega} \mathcal{I}_\alpha$ is hesitant fuzzy T-ideal of $\Bbbk$. □

Definition 3.18. Let $\mathit{\Gamma}_A := \{(\rho, \tau_A(\rho)) \mid \rho \in \Bbbk\}$ and $\mathit{\Gamma}_B := \{(\varrho, \tau_B(\varrho)) \mid \varrho \in \ell\}$ be a hesitant fuzzy sets on a TM algebra $\Bbbk$ and ℓ respectively. Then the Cartesian product of $\mathit{\Gamma}_A$ and $\mathit{\Gamma}_B$ is denoted by $\mathit{\Gamma}_A \times \mathit{\Gamma}_B$ is defined by $\left(\mathit{\Gamma}_A \times \mathit{\Gamma}_B\right)(\rho, \varrho) = \mathit{\Gamma}_A(\rho) \cap \mathit{\Gamma}_B(\varrho)$ where $\mathit{\Gamma}_A \times \mathit{\Gamma}_B : \Bbbk \times \ell \to 2^{[0,1]}$ for all $\rho \in \Bbbk, \varrho \in \ell$.

Remark 3.19. Let $(\Bbbk, \star, 0_{\Bbbk})$ and $(\ell, \cdot, 0_\ell)$ be two TM-algebras. Then $(\Bbbk \times \ell, \odot, (0_{\Bbbk}, 0_\ell))$ is a TM-algebra defined by $(\rho, \varrho) \odot (\phi, \varphi) = (\rho \star \phi, \varrho \cdot \varphi)$ for all $\rho, \phi \in \Bbbk$ and $\varrho, \varphi \in \ell$.

Theorem 3.20. *Let* $I_A := \{(\rho, I_A(\rho)) \mid \rho \in \Bbbk\}$ *and* $I_B := \{(\varrho, I_B(\varrho)) \mid \varrho \in \ell\}$ *are two hesitant fuzzy T-ideals on a TM-algebra* $\Bbbk$ *and* ℓ *respectively, then the cartesian product* $I = I_A \times I_B$ *is also a hesitant fuzzy T-ideal of* $\Bbbk \times \ell$.

Proof. Let $(\rho, \varrho), (\phi, \varphi) \in \Bbbk \times \ell$. Then by definition of cartesian product we have

$$\begin{aligned}
I((\rho, \varrho) \odot (\phi, \varphi)) &= I((\rho \star \phi, \varrho \cdot \varphi)) \\
&= I_A(\rho \star \phi) \cap I_B(\varrho \cdot \varphi) \\
&\supseteq I_A((\rho \star m) \star \phi) \cap I_A(m) \cap I_B((\varrho \cdot n) \cdot \varphi) \cap I_B(n) \\
&= I_A((\rho \star m) \star \phi) \cap I_B((\varrho \cdot n) \cdot \varphi) \cap I_A(m) \cap I_B(n) \\
&= I((\rho \odot m) \odot \phi), \varrho \odot n) \odot \varphi)) \cap I(m, n) \\
&= I((\rho, \varrho) \odot (m, n) \odot (\phi, \varphi)) \cap I(m, n)
\end{aligned}$$

Again, let $(\rho, \varrho) \in \Bbbk \times \ell$. Then

$$\begin{aligned} I(0_{\Bbbk}, 0_{\ell}) &= I_A(0_{\Bbbk}) \cap I_B(0_{\ell}) \\ &\supseteq I_A(\rho) \cap I_B(\varrho) \\ &= I(\rho, \varrho) \end{aligned}$$

Therefore, the cartesian product $\mathbb{I}_A \times \mathbb{I}_B$ is a hesitant fuzzy T-ideal of $\Bbbk \times \ell$. □

Theorem 3.21. *Let $\mathbb{I}_A := \{(\rho, I_A(\rho)) \mid \rho \in \Bbbk\}$ and $\mathbb{I}_B := \{(\varrho, I_B(\varrho)) \mid \varrho \in \ell\}$ are two hesitant fuzzy closed T-ideals on a TM-algebra $\Bbbk$ and ℓ respectively, then the cartesian product $\mathbb{I}_A \times \mathbb{I}_B$ is also a hesitant fuzzy closed T-ideal of $\Bbbk \times \ell$.*

Definition 3.22. A mapping $f : (\Bbbk, \cdot, 0_{\Bbbk}) \to (\ell, \star, 0_{\ell})$ of a TM-algebras is said to be a hesitant homomorphism if $f(\rho \cdot \varrho) = f(\rho) \star f(\varrho)$ for all $\rho, \varrho \in \Bbbk$.

Definition 3.23. Let f be a mapping from a TM-algebra $\Bbbk$ to ℓ and A be a subset of a TM-algebra $\Bbbk$. If $\Gamma = \{(\rho, \tau_A(\rho)/\rho \in \Bbbk)\}$ is a hesitant fuzzy set on ℓ, then the hesitant fuzzy set $f^{-1}(\Gamma) = \tau_A o f$ in $\Bbbk$ is called the pre-image of Γ under f.

Theorem 3.24. *Let f be a homomorphism from a TM-algebra $\Bbbk$ to ℓ and A be a subset of a TM-algebra $\Bbbk$. If $\Gamma = \{(\rho, \tau_A(\rho)/\rho \in \Bbbk)\}$ is a hesitant fuzzy T-ideals of ℓ, then $f^{-1}(\Gamma) = \tau_A o f$ in $\Bbbk$ is a hesitant fuzzy T-ideal of $\Bbbk$.*

Proof. We know that $\tau_A(f(o_{\Bbbk})) = \tau_A(o_{\ell}) \supseteq \tau_A(\rho)$ for all $\rho \in \Bbbk$.
Let $\rho, \varrho \in \Bbbk$. Then

$$\begin{aligned} (\tau_A o f)(\rho \star \varrho) &= \tau_A(f(\rho \star \varrho)) \\ &= \tau_A(f(\rho) \star f(\varrho)) \\ &\supseteq \tau_A((f(\rho) \star f(\varphi) \star \varrho)) \cap \tau_A(f(\varphi)) \\ &= \tau_A(f((\rho \star \varphi) \star \varrho)) \cap \tau_A(f(\varphi)) \\ &= (\tau_A o f)((\rho \star \varphi) \star \varrho) \cap (\tau_A o f)(\varrho) \end{aligned}$$

Therefore, $f^{-1}(\Gamma)$ is a hesitant fuzzy T-ideal of $\Bbbk$. □

Proposition 3.25. *Let f be a homomorphism from a TM-algebra $\Bbbk$ to ℓ and A be a subset of a TM-algebra $\Bbbk$. If $\Gamma = \{(\rho, \tau_A(\rho)/\rho \in \Bbbk)\}$ is a hesitant fuzzy closed T-ideals of ℓ, then $f^{-1}(\Gamma) = \tau_A o f$ in $\Bbbk$ is a hesitant fuzzy closed T-ideal of $\Bbbk$.*

4 Conclusion

The study of fuzzy algebraic structures has gained significant attention in recent years, particularly in the context of TM (Ternary-Membership) algebras. In classical algebraic structures, ideals and T-ideals play a central role in understanding the properties and behavior of algebraic systems. However, when dealing with uncertainty and vagueness, fuzzy sets and fuzzy ideals become crucial. In this paper, we extend these classical concepts by introducing hesitant fuzzy ideals, hesitant fuzzy T-ideals, and hesitant fuzzy closed T-ideals within the framework of TM-algebras. These new concepts allow for a more generalized approach to the study of algebraic structures in fuzzy environments, addressing situations where elements of the set have degrees of membership that are not fully determined.
This paper introduces and explores the concepts of hesitant fuzzy ideals, hesitant fuzzy T-ideals, and hesitant fuzzy closed T-ideals in the context of TM-algebras. The relationships between these fuzzy ideals and their level subsets are described in detail, and the homomorphic pre-images of these ideals are studied to uncover important properties. These findings extend the classical notions of ideals and T-ideals in algebraic structures, providing a broader framework for handling uncertainty and vagueness in TM-algebras. To extend the results of this paper, future research could focus on intuitionistic hesitant fuzzy sets in the concept of ideals, T-ideals in TM-algebras.

References

[1] M.Y. Abbasi, A.F. Talee, S.A. Khan, K. Hila, A Hesitant Fuzzy Set Approach to Ideal Theory in Γ-Semigroups, Advances in Fuzzy Systems, **1**(2018), 5738024.

[2] M. Al-Tahan, A. Rezaei, S. Al-Kaseasbeh, S. Davvaz, B., and M.Riaz, Linear Diophantine fuzzy n-fold weak subalgebras of a BE-algebra, Missouri Journal of Mathematical Sciences, **35**(2)(2023), 136-148.

[3] A. Iampan, A new branch of the logical algebra: UP-algebras, J. Algebra Relat. Top., **5**(1)(2017), 35-54.

[4] K. Iseki, On BCI-algebras, Math. Seminar Notes **8** (1980), 125-130.

[5] Y. Jun, On BH-algebras. Scientiae Mathematicae, **1**(3) (1998),347-354.

[6] Y. B. Jun, S.Z. Song, Hesitant fuzzy set theory applied to filters in MTL-algebras, Honam Math. J.,**36**(4)(2014), 813-830.

[7] C. B. Kim, H.S. Kim, On BG-algebras, Demonstratio Mathematica,**41** (3)(2008), 497-506.

[8] J. H. Kim, Y. B. Jun, P. K. Lim, J. G. Lee, K. Hur, The category of hesitant H-fuzzy sets, Infinite Study, (2019).

[9] J. Kim, P. K. Lim, J. G. Lee, K. Hur, Hesitant fuzzy subgroups and subrings, Ann. Fuzzy Math. Inform. **18**(2) (2019), 105-122.

[10] AA. Mechdesro, BA. Alaba BA, and TM. Munie, et al.: Fuzzy pseudo-UP ideal of pseudo-UP algebra. Res. Math. **10**(1) (2023), 2234217.

[11] K. Megalai, A. Tamilarasi, TM-algebra-an introduction, Special Issue on Computer Aided Soft Computing Techniques for Imaging and Biomedical Applications, CASCT (2010).

[12] K. Megalai, A. Tamilarasi, Fuzzy subalgebras and fuzzy T-ideals in TM-algebra, Journal of Mathematics and Statistics,**7** (2) (2011), 107-111.

[13] K. Megalai, A. Tamilarasi, A study on Fuzzy TM- algebras.

[14] P. Mosrijai, W. Kamti, A. Satirad, A. Iampan, Hesitant fuzzy sets on UP-algebras, Konuralp J. Math., **5** (2)(2017), 8-280.

[15] S. M. Mostafa, M. Naby, O. R. Elgendy, Fuzzy TM-ideals of TM- algebras. Journal of American Science, **7(9)** (2011), 17-21.

[16] G. Muhiuddin, M. Al-Tahan, A. Mahboob, S. Hoskova-Mayerova, and S. Al-Kaseasbeh, Linear Diophantine Fuzzy Set Theory Applied to BCK/BCI-Algebras, Mathematics, **10**(12)(2022), 2138.

[17] J. Neggers, S. S. Ahn, H. S. Kim, On Q-algebras, International Journal of Mathematics and Mathematical Sciences, **27**(12)(2001), 749-757.

[18] Z. Pei, L. Yi, A note on operations of hesitant fuzzy sets, International Journal of Computational Intelligence Systems, **8** (2) (2015), 226-239.

[19] C. Prabpayak, Some Homomorphism Properties of TM-algebras, Applied Mechanics and Materials, (866)(2017), 406-409.

[20] P. Sharma, Homomorphism of intuitionistic fuzzy groups, In International Mathematical Forum, **6**(2011), 3169-3178.

[21] R. M. Rodriguez, L. Martinez, V. Torra, Z.S. Xu, F. Herrera, Hesitant fuzzy sets: state of the art and future directions, International journal of intelligent systems, **29**(6)(2014), 495-524.

[22] J. Somjanta, N.Thuekaew, P. Kumpeangkeaw, A.Iampan, Fuzzy sets in UP-algebras, Ann. Fuzzy Math. Inform, **12**(6)(2016), 739-756.

[23] J. Thomas, K. Indhira, V. M. Chandrasekaran, T-normed fuzzy TM-subalgebra of TM-algebras, International Journal of Computational Intelligence Systems, **12**(2) (2019), 706-712.

[24] V. Torra, Hesitant fuzzy sets, International journal of intelligent systems, **25**(6) (2010), 529-539.

[25] V.Torra, Y. Narukawa, On hesitant fuzzy sets and decision, In Proceedings of the IEEE International Conference on Fuzzy Systems, Jeju-do,Republic of Korea, (2009), 1378-1382.

[26] R. Verma, B.D. Sharma, New operations over hesitant fuzzy sets, Fuzzy Information and Engineering. **5** (2013), 129-146.

[27] Z. S. Xu, M. M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inform. Sci. **181** (2011), 2128-2138.

[28] L. A. Zadeh, Fuzzy sets, Information and control,**8** (1965), 338-353.

Author information

Alemayehu Girum*, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: alemayehugirum21@gmail.com

Berhanu Assaye, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: birhanu.assaye290113@gmail.com

Bekalu Tarekegn, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: t.bekalu@yahoo.com

Beza Lamesgin, Department of Mathematics, Debre Markos University, Ethiopia.
E-mail: dbezalem@gmail.com

Received: 2024-11-18
Accepted: 2-25-01-18