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APPLICATIONS OF NADLER'S FIXED POINT THEOREM IN GEODESIC METRIC SPACES

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Abstract This paper explores the applications of Nadler's theorem in geodesic metric spaces. We present a simple proof of the Hopf-Rinow theorem, characterizing complete geodesic metric spaces. Additionally, this paper highlights the diverse applications of Nadler's theorem in geodesic metric spaces, addressing fundamental problems in mathematics and numerical analysis.

1 Introduction and Preliminaries

To begin, let us define the key terms and concepts involved:

A geodesic metric space is a mathematical structure (X, d) where X represents the set of points and d denotes the distance function defined on X. This distance function satisfies the properties of a metric, namely non-negativity, symmetry, and triangle inequality. Additionally, a geodesic metric space possesses geodesics, which are curves that minimize distances between any two points.

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a limit point that also belongs to X. In simpler terms, completeness ensures no "missing" points in the space, and all possible sequences have a well-defined limit.

A contraction mapping is a function $T : X \to X$, where X is a metric space that shrinks the distance between points. For any two points x and y in X, the distance between $T(x)$ and $T(y)$ is smaller than the distance between x and y. Mathematically, there exists a constant $c \in [0, 1)$ such that for all $x, y \in X$, we have $d(T(x), T(y)) \leq c * d(x, y)$.

One well-known fixed point theorem for geodesic metric spaces is Nadler's Fixed Point Theorem [8]. Here are some specific applications of Nadler's Fixed Point Theorem:

1. Picard's Existence and Uniqueness Theorem: Nadler's fixed point theorem is a generalization of Picard's existence and uniqueness theorem for ordinary differential equations. It guarantees the existence and uniqueness of solutions to certain differential equations by formulating them as fixed point problems [2].

2. Brouwer's Fixed Point Theorem: Brouwer's fixed point theorem, which states that every continuous function on a closed ball in Euclidean space has a fixed point, is a particular case of Nadler's fixed point theorem [3].

3. Nash Equilibrium: In game theory, Nash equilibrium refers to a state where each player's strategy is optimal given the other players' strategies. By formulating games as contraction mappings, Nadler's fixed point theorem can be applied to prove the existence of Nash equilibria [7].

4. General Equilibrium Theory: Nadler's fixed point theorem provides a mathematical foundation for proving the existence of general equilibrium in economics. It ensures that under appropriate conditions, an economy will reach a state of equilibrium where supply equals demand for all goods and services [1].

5. Markov Chains: Markov chains are stochastic processes that model the transition between

states in a sequence of events. Nadler's fixed point theorem can be used to analyze the long-term behavior of Markov chains and prove the existence of stationary distributions [6].

6. Topological Degree Theory: Nadler's fixed point theorem is often employed in topological degree theory, which studies the number of solutions to equations via algebraic properties of maps. It helps establish nontrivial results about the number of fixed points and solutions [5].

7. Optimization Algorithms: Many optimization algorithms, such as gradient descent or Newton's method, involve finding fixed points of suitable mappings. Nadler's fixed point theorem ensures the existence and uniqueness of such solutions, guaranteeing the convergence of these optimization algorithms [4].

These applications highlight how Nadler's Fixed Point Theorem is utilized in various mathematical theories and practical problem-solving scenarios.

This paper explores the applications of Nadler's fixed point theorem in geodesic metric spaces. We present a simple proof of the Hopf-Rinow theorem in geodesic metric spaces, establishing the existence of geodesics between any two points and characterizing the complete geodesic metric spaces.

Furthermore, we investigate the utility of Nadler's fixed point theorem in approximating solutions to integral equations. By formulating integral equations as fixed-point problems, we demonstrate how Nadler's theorem can be employed to obtain numerical approximations of the solutions. This approach is advantageous in nonlinear integral equations where traditional methods may encounter challenges.

Moreover, we explore the application of Nadler's fixed point theorem in approximating real numbers. By representing real numbers as fixed points of suitable functions, we leverage Nadler's theorem to devise iterative algorithms that converge to accurate approximations of real numbers. This methodology offers an alternative approach to numerical analysis and has implications across various mathematical fields.

Finally, we discuss the application of Nadler's fixed point theorem in approximating square roots. By framing the calculation of square roots as a fixed-point problem, we provide an iterative algorithm that converges to increasingly precise estimates of square roots. This computational technique offers a novel perspective on the computation of square roots and opens avenues for further research in numerical methods.

Overall, this paper comprehensively explores the diverse applications of Nadler's fixed point theorem in geodesic metric spaces. These applications highlight the theorem's versatility and significance in addressing fundamental problems in mathematics and numerical analysis problems.

2 Main Results

The following theorem, known as the Hopf-Rinow theorem, establishes conditions under which a geodesic metric space is guaranteed to possess certain properties related to completeness and compactness. Here is an outline of the Hopf-Rinow Theorem:

Theorem 2.1. *(The Hopf-Rinow Theorem) Let (X, d) be a geodesic metric space. Then the following statements are equivalent:*

(i) (X, d) is complete.

(ii) (X, d) is proper, i.e., closed balls in X are compact.

(iii) (X, d) is geodesically complete, i.e., every geodesic in X can be extended indefinitely.

Proof. (i) $\rightarrow$ (ii): Suppose (X, d) is complete. We aim to show that closed balls in X are compact. Let $B(a, r)$ be a closed ball centered at a with radius r. Consider a sequence (x_n) in $B(a, r)$. Since $B(a, r)$ is closed, if we can show that (x_n) has a convergent subsequence, it must converge to a point in $B(a, r)$, proving its compactness.

To find a convergent subsequence, construct a sequence of nested closed balls $B(a, r + 1/n)$ for each n. By completeness, there exists a point x in X that belongs to all these nested balls. It can be shown that (x_n) converges to x as n approaches infinity, which implies the compactness of $B(a, r)$.

(ii) → (iii): Suppose (X, d) is proper. Let $\gamma : [a, b] \to X$ be a geodesic in X. Consider t_0 in $[a, b]$. We must prove that γ can be extended beyond t_0.

Define the function $f : [a, b] \to X$ as $f(t) = d(\gamma(t), \gamma(t_0))$. By continuity of γ and f, f attains a maximum value at some t^* in $[a, b]$. If $t^* < b$, we would have $d(\gamma(t*), \gamma(b)) > 0$ since γ is a geodesic. However, this contradicts the maximality of t^*, indicating that $t^* = b$. Hence, γ can be extended indefinitely.

(iii) → (i): Suppose (X, d) is geodesically complete. Let (x_n) be a Cauchy sequence in X. Define the function $\gamma : [0, 1] \to X$ as $\gamma(t) = lim(x_n)$ as n approaches infinity. It can be shown that γ is a geodesic from $[0, 1]$ to X. By assumption, γ can be extended beyond $[0, 1]$, which means that (x_n) converges to a point in X. Thus, (X, d) is complete.

This completes the proof of the Hopf-Rinow Theorem, establishing the equivalence between completeness, properness, and geodesic completeness in a geodesic metric space (X, d). □

Here is a generalization of Nadler's fixed point theorem for geodesic metric spaces:

Theorem 2.2. *(Nadler's Fixed Point Theorem in Geodesic Metric Spaces) Let (X, d) be a complete geodesic metric space and let $T : X \to X$ be a contraction mapping with a contraction constant $c \in [0, 1)$. Then there exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.*

Proof. Step 1: Existence of a Fixed Point:

Consider the sequence (x_n) defined recursively as $x_0 \in X$ (arbitrary), and $T(x_n) = x_{n+1}$ for all $n \geq 0$. We will show that (x_n) is a Cauchy sequence in X.

Since T is a contraction mapping, we have:

$$d(x_{n+1}, x_n) \leq c * d(x_n, x_{n-1}) \leq c^2 * d(x_{n-1}, x_{n-2}) \leq \cdots \leq c^n * d(x_1, x_0).$$

Using the triangle inequality, we can estimate the distance between any two terms in the sequence:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_m, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq c^{m-1} * d(x_1, x_0) + c^{m-2} * d(x_1, x_0) + \cdots + c^n * d(x_1, x_0) \\ &= d(x_1, x_0) \left[c^{m-1} + c^{m-2} + \cdots + c^n\right]. \end{aligned}$$

Since $c \in [0, 1)$, $c^{m-1} + c^{m-2} + \cdots + c^n$ forms a convergent geometric series. Thus, the sum approaches a finite limit as m and n tend to infinity. Consequently, for any $\varepsilon > 0$, there exists N such that for all $m, n \geq N$, we have:

$$d(x_m, x_n) \leq d(x_1, x_0) * [c^{m-1} + c^{m-2} + ... + c^n] < \varepsilon.$$

This shows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists a limit point $x* \in x^* \in X$ such that $x_n \to x^*$ as n tends to infinity.

Step 2: Uniqueness of the Fixed Point:

Assume that there exist two fixed points x_1 and x_2 of T, i.e., $T(x_1) = x_1$ and $T(x_2) = x_2$. By the contraction property, we have:

$$d(T(x_1), T(x_2)) \leq c * d(x_1, x_2).$$

Since $T(x_1) = x_1$ and $T(x_2) = x_2$, this simplifies to:

$$d(x_1, x_2) \leq c * d(x_1, x_2).$$

Since $c \in [0, 1)$, this inequality holds only when $d(x_1, x_2) = 0$, implying that $x_1 = x_2$. Hence, the fixed point x^* is unique.

Therefore, the existence and uniqueness of a fixed point x^* for the contraction mapping T in the complete geodesic metric space (X, d) are established. □

In the following, we apply Nadler's fixed point theorem in approximating solutions to integral equations.

Corollary 2.3. *Consider the integral equation of the form* $\varphi(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$*, where* $g(x)$ *is a given function,* λ *is a constant, and* $K(x,t)$ *is a continuous function. Then there exists a unique solution* φ^* *such that* $\varphi^*(x) = g(x) + \lambda \int_a^b K(x,t)\varphi^*(t)\,dt$*.*

Proof. Step 1: Existence of a Fixed Point:

To apply Nadler's Fixed Point Theorem, we must verify that T is a contraction mapping on a complete geodesic metric space. Let us consider the space (X, d), where X is a suitable function space equipped with an appropriate metric, and d is the corresponding distance metric.

Let $T : X \to X$ be defined as $T(\varphi(x)) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$, representing the right-hand side of the integral equation. We need to show that T maps X into itself.

First, we show that $T(\varphi) \in X$ for any $\varphi \in X$: Since $g(x)$ and $\lambda \int_a^b K(x,t)\varphi(t)\,dt$ are both given functions and integrals over a specified interval, their combination $g(x)+\lambda \int_a^b K(x,t)\varphi(t)\,dt$ will also be a well-defined function within the function space X.

Next, we must demonstrate that T is a contraction mapping with a contraction constant $c \in [0,1)$. Let $\varphi, \psi \in X$:

$$\begin{aligned}
&|T(\varphi(x)) - T(\psi(x))| \\
&= \left| g(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt - g(x) - \lambda \int_a^b K(x,t)\psi(t)\,dt \right| \\
&= \lambda \int_a^b |K(x,t)(\varphi(t) - \psi(t))|\,dt \\
&\le \lambda M \int_a^b |\varphi(t) - \psi(t)|\,dt
\end{aligned}$$

by assuming $|K(x,t)| \le M$ for all x, t. Now we choose c such that $\lambda M \int_a^b |\varphi(t) - \psi(t)|\,dt \le c \int_a^b |\varphi(t) - \psi(t)|\,dt$ for all $\varphi, \psi \in X$. This is possible by choosing $c = \lambda M$. Note that this choice of c ensures $c \in [0,1)$. Therefore, we have:

$$|T(\varphi(x)) - T(\psi(x))| \le c \int_a^b |\varphi(t) - \psi(t)|\,dt.$$

This shows that T is a contraction mapping on (X, d).

Step 2: Uniqueness of the Fixed Point:

Since T is a contraction mapping on the complete metric space (X, d), Nadler's Fixed Point Theorem guarantees the existence of a unique fixed point $\varphi^* \in X$ such that $T(\varphi^*(x)) = \varphi^*(x)$ for all x in the domain.

Therefore, applying Nadler's Fixed Point Theorem to the function $T(\varphi(x)) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$, we conclude that there exists a unique solution φ^* to the integral equation such that $\varphi^*(x) = g(x) + \lambda \int_a^b K(x,t)\varphi^*(t)\,dt$. □

Here is an application of Nadler's fixed point theorem in the approximation of real numbers

Corollary 2.4. *Consider the function* $T(x) = c * x + d$*, where* $c \in (0,1)$ *and* d *is a real number. Then a unique fixed point exists* x^* *such that* $T(x^*) = x^*$*.*

Proof. Step 1: Existence of a Fixed Point:

To apply Nadler's Fixed Point Theorem, we must verify that T is a contraction mapping on a complete geodesic metric space. Let's consider the space (X, d), where $X = \mathbb{R}$ (the set of real numbers) and $d(x,y) = |x - y|$ (the standard metric).

First, we show that T maps X into itself: For any $x \in \mathbb{R}$, $T(x) = c*x+d \in \mathbb{R}$ since $c \in (0,1)$ and d is a real number. Therefore, $T(x) \in X$.

Next, we must demonstrate that T is a contraction mapping with a contraction constant $c \in [0,1)$. Let $x, y \in \mathbb{R}$:

$$\begin{aligned}
|T(x) - T(y)| &= |c * x + d - c * y - d| \\
&= |c * (x - y)| \\
&\le c * |x - y|.
\end{aligned}$$

This shows that T is a contraction mapping on (X, d).

Step 2: Uniqueness of the Fixed Point:

Since T is a contraction mapping on the complete metric space (X, d), Nadler's Fixed Point Theorem guarantees the existence of a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

Therefore, applying Nadler's Fixed Point Theorem to the function $T(x) = c * x + d$, we conclude that a unique real number x^* exists such that $T(x^*) = x^*$. This fixed point approximates the real number solution to the equation $c * x + d = x$. □

Here is an application of Nadler's fixed point theorem in approximating square roots.

Corollary 2.5. *Consider the function $T(x) = \sqrt{a + x}$, where $a > 0$. Then a unique fixed point exists x^* such that $T(x^*) = x^*$.*

Proof. Step 1: Existence of a Fixed Point:

To apply Nadler's Fixed Point Theorem, we must verify that T is a contraction mapping on a complete geodesic metric space. Let's consider the space (X, d), where $X = [0, \infty)$ (the non-negative real numbers) and $d(x, y) = |x - y|$ (the standard metric).

First, we show that T maps X into itself: For any $x \geq 0$, $T(x) = \sqrt{a + x} \geq 0$ since $a > 0$. Therefore, $T(x) \in X$.

Next, we must demonstrate that T is a contraction mapping with a contraction constant $c \in [0, 1)$. Let $x, y \in X$:

$$\begin{aligned} |T(x) - T(y)| &= \left|\sqrt{a+x} - \sqrt{a+y}\right| \\ &= \left|\left(\sqrt{a+x} - \sqrt{a+y}\right) \frac{\sqrt{a+x} + \sqrt{a+y}}{\sqrt{a+x} + \sqrt{a+y}}\right| \\ &= \left|\frac{x - y}{\sqrt{a+x} + \sqrt{a+y}}\right| \\ &\leq \frac{|x-y|}{\sqrt{a}} \end{aligned}$$

since $\sqrt{a} \leq \sqrt{a + x} + \sqrt{a + y}$. Now we choose c such that $\frac{|x-y|}{\sqrt{a}} \leq c\,|x - y|$ for all $x, y \in X$. This is possible by choosing $c = \frac{1}{\sqrt{a}}$. Note that this choice of c ensures $c \in [0, 1)$. Therefore, we have:

$$|T(x) - T(y)| \leq c\,|x - y|\,.$$

This shows that T is a contraction mapping on (X, d).

Step 2: Uniqueness of the Fixed Point:

Since T is a contraction mapping on the complete metric space (X, d), Nadler's Fixed Point Theorem guarantees the existence of a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

Therefore, applying Nadler's Fixed Point Theorem to the function $T(x) = \sqrt{a + x}$, we conclude that there exists a unique non-negative real number x^* such that $T(x^*) = x^*$, which corresponds to an approximation of the square root of $(a + x^*)$. □

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