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Pythagorean Fuzzy Ideal of LBA.

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Abstract

Under this article, we have used a new notion Pythagorean fuzzy set on ideal in Liu^{B}-Algebra defined on BCL-Algebra. This notion of Pythagorean fuzzy ideal of Liu^{B}-Algebra has been introduced, along with some basic Characteristics. We have stated and proved some theorems and properties of the Pythagorean fuzzy ideal of the Liu^{B}-Algebra. Further, we have introduced the Pythagorean fuzzy ideal of the Liu^{B}-Algebra in terms of the terms as the complement of fuzzy set, the square deviation, the accuracy function, the score function and the degree of indeterminacy associated with some of the properties.

1 Introduction

Georg Cantor (1874) [6] pioneered set theory, while L.A. Zadeh (1965) [21] introduced fuzzy sets to mathematically handle vagueness and ambiguity unlike classical sets defined by characteristic functions on nonempty sets.

A fuzzy set F in a set X (nonempty set) is defined as: F = $\{\langle y,\, \eta_{_F}(y)\rangle \,:\, y \in \mathrm{X}\}$ such that the mapping $\eta_{_F} \,:\, \mathrm{X} \to [0,\,1]$ expresses membership degree of a member y in X to set F. Values "0" and "1" here represent fully non-membership and fully membership values respectively and values in between 0 and 1 represent intermediate membership degrees.

K. Iseki (1980) [7] investigated BCI-algebra as subset of BCK-algebra and Y. H. Liu (respectively in 2011, 2012, 2017) [9, 10, 11] introduced notions of algebras, different but related algebras, BCL-algebra, BCL^{+}-algebra and Liu-algebra (where one of its axioms is semi-group as defined by Rosenfeld, A. (1971) [15]), with their partial orders.

As generalization of fuzzy set, K.T. Atanassov (1999) [5] investigated the notion of an intuitionistic fuzzy set to minimize vagueness or ambiguity, later, R. R. Yager (2013) [20] proposed Pythagorean fuzzy sets to alleviate the constraint on the total of membership degrees and non-membership degrees in intuitionistic fuzzy sets and introduced it as a new class of non-standard fuzzy subsets and the related idea of Pythagorean membership grades and non-membership grades.

Muhiuddin, G. et al. (2022) [12] described and characterized Linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy ideals in BCK/BCI-algebras, defined the notion of LDF commutative ideal of BCK-algebras and described and characterized some connections between LDF subalgebras, LDF ideals, and LDF commutative ideals were . Also, Al-Tahan et al. (2023) [2] introduced Linear diophantine fuzzy n-fold weak subalgebras of a BE-algebra.

Nehete, J. Y. (2024) [13] introduced the concept of an almost 2-absorbing δ-primary ideal which unifies the concept of an almost 2-absorbing ideal and an almost 2-absorbing primary ideal in a commutative ring. Arora, H. D., & Naithani, A. (2022, 2023) [3, 16] introduced the concept of Pythagorean fuzzy sets and discussed its relationship with intuitionistic fuzzy sets with the key objective to investigate some algebraic structures of Pythagorean fuzzy ideal of-near-rings and they developed Novel logarithmic similarity measures for PFSs. In 2023 Sharma, P. K. [16] introduced the concepts of an intuitionistic fuzzy prime ideal and a prime intuitionistic fuzzy ideal.

Pythagorean fuzzy sets, originally proposed by Yager R. R. (2013) [19, 20], are tools to deal with the vagueness or the uncertainties considering the membership grades and non-membership grades (η, τ) where the condition $0 \leq (\eta(x))^2 + (\tau(x))^2 \leq 1$ is fulfilled. As a generalized set, Pythagorean fuzzy sets have close relationship with intuitionistic fuzzy sets, which Atanassov (1999) [5] initiated the concept, which is a generalization of Zadeh's fuzzy sets.

Pythagorean fuzzy sets can be reduced to intuitionistic fuzzy sets where $0 \leq \eta(x) + \tau(x) \leq 1$, is fulfilled. A novel algorithm specified by Xiao, F. and Ding W. [18] (2019) based on the Pythagorean fuzzy set distance measure is intended to elucidate the problems of medical diagnosis. By relating the different methods in the medical diagnosis application, it is found that the new algorithm is as proficient as the other methods. These results prove that this method is applied in dealing with the medical diagnosis problems.

The algebra of semi-groups [11] (2017) occurred naturally in Liu-algebra, and as module framework is a composite structure of ring framework and Abelian group framework, similarly, Liu-algebra offered a new composite structure, which is based on the well-known semi-groups and BCL$^+$-algebras (also known as Liu-algebras, which was named after the author, Y. H. Liu), for the aim was to involve a fresh approach for Liu-algebras whose properties and results were interesting.

The primary objectives of this study are threefold: To introduce Liu^B-algebra as a novel algebraic structure based on BCL-algebra (distinct from Liu-algebras built on BCL$^+$-algebras) and to define and characterize Pythagorean fuzzy ideals within Liu^B-algebra, incorporating membership/non-membership deviations, score functions, and indeterminacy degrees, while bridging gaps in fuzzification of ideals in Liu-algebra.

Thus in this paper, starting by examining some new properties of BCL-Algebra that has not been discussed yet, we define LBA. that offer other composite structure, which is constructed using BCL-Algebra along with the semi-group and then we have introduced interesting new concept of ideal of the LBA., have fuzzified it and then have introduced Pythagorean fuzzy ideal of LBA. Furthermore, we have examined the varieties of contextual concepts of newly interpreted notions herewith as complement of fuzzy set, square deviation, accuracy function, score function, degree of indeterminacy, membership deviations and non-membership deviations under some properties of Pythagorean fuzzy ideal and then some interesting new results are obtained.

2 Preliminaries

Under this part of the paper, we recall a few main definitions and results on a few algebras, ideals, Pythagorean fuzzy sets that are needed and related to our study.

For the whole of this study, we denote LBA. for Liu^B-algebra and *"iff"* for "if and only if".

2.1 A Few Algebras, Ideals and Some Fuzzy Ideals and Pythagorean Fuzzy Sets

Definition 2.1. [7] A BCI-algebra is an algebra (X; $\circledast$, 0) of (2, 0) kind such that conditions hereunder are fulfilled; $\forall k, m, n \in$ X.:

(i) $\big((k \circledast m) \circledast (k \circledast n)\big) \circledast (n \circledast m) = 0$ (ii) $\big(k \circledast (k \circledast m)\big) \circledast m = 0$

(iii) $k \circledast k = 0$ (iv) $k \circledast m = 0$ and $m \circledast k = 0 \Rightarrow k = m$.

Definition 2.2. [7] Suppose X is a BCI-algebra and set I ($\neq \emptyset$) is subset of X. I is known as ideal of X if, for every $m, n \in$ X.

(a) $0 \in$ I

(b) $m \circledast n \in$ I and $n \in$ I imply $m \in$ I.

Definition 2.3. [8] A fuzzy set η in a BCK-algebra X is known as fuzzy ideal of X if; for every $m, n \in$ X

(i) $\eta(0) \geq \eta(m)$,

(ii) $\eta(m) \geq min\{\eta(m \circledast n), \eta(n)\}$.

Definition 2.4. [17, 19, 20] Let $\emptyset \neq$ X. A fuzzy subset A $\subseteq$ X is defined as: A = $\{\langle m,\ \eta_A(m)\rangle$:m$\in$X$\}$, such that $\eta_A(m)$: X $\rightarrow$ [0, 1] given by the degree of membership and the complement of η_A, symbolized as $\overline{\eta}_A(m)$, is the fuzzy set in X given by $\overline{\eta}_A(m) = 1 - \eta_A(m)$, for every $m \in$ X.

Definition 2.5. [5] An intuitionistic fuzzy set *I* in non-empty set *X* is defined as:
I = $\{m, \eta_I(m), \upsilon_I(m) : m \in$ X$\}$, such that the functions: $\eta_I(m)$: X $\rightarrow$ [0, 1] and $\upsilon_I(m)$: X $\rightarrow$ [0, 1] define the membership degree and the non-membership degree, respectively where: $0 \leq \eta_I(m) + \tau_I(m) \leq 1$ is fulfilled.

Definition 2.6. [19] A Pythagorean fuzzy set P in X (non-empty set) is an object such that: ***P*** = $\{m, \eta_P(m), \upsilon_P(m) : m \in$ X$\}$, for mappings $\eta_P(m)$: X $\rightarrow$ [0, 1] and $\upsilon_P(m)$: X $\rightarrow$ [0, 1] define membership degree and non-membership degree, respectively such that:
$0 \leq \left(\eta_P(m)\right)^2 + \left(\upsilon_P(m)\right)^2 \leq 1$ is fulfilled.

Definition 2.7. [22] Suppose $\left(\eta_P(m)\right)^2 + \left(\upsilon_P(m)\right)^2 \leq 1$, then there is a degree of indeterminacy of $m \in$ X to ***P*** given as:

$$\pi_P(m) = \sqrt{1 - \left[\left(\eta_P(m)\right)^2 + \left(\upsilon_P(m)\right)^2\right]}; \quad \text{where} \quad \pi_P(m) \in [0, 1].$$

Definition 2.8. [22] Let P be Pythagorean fuzzy set in X. Then, the score function, s of P is given by: $s_P(m) = (\eta_P(m))^2 - (\upsilon_P(m))^2$, where $s(P) \in$ [-1, 1].

Definition 2.9. [22] For Pythagorean fuzzy set P on X, the accuracy function, a, of P is given by: a_P(m) = $(\eta_P(m))^2 + (\upsilon_P(m))^2$, where a_P(m) $\in$ [0, 1].

Definition 2.10. [17] The fuzzy sets (ℓ_P, m_P) in a BCK-algebra B is known as a Pythagorean fuzzy ideal of B if for a membership function ℓ_P : B $\rightarrow$ [0, 1] and non-membership function m_P : B $\rightarrow$ [0, 1], the axioms hereunder are fulfilled for every $m, n \in$ B,:

(1) $\ell_P(0) \geq \ell_P(m)$ and $m_P(0) \leq m_P(m)$

(2) $\ell_P(m) \geq \min\{\ell_P(m \circledast n),\ \ell_P(n)\}$ and $m_P(m) \leq \max\{m_P(m \circledast n),\ m_P(n)\}$.

2.2 Basic Concepts of BCL-algebra, BCL$^+$-algebra, Liu-Algebra, and Ideals

For this part, we have recalled some relevant definitions, essential concepts of BCL-algebra, BCL$^+$-algebra, Liu-algebra and corresponding ideals which could be related to our paper.

Definition 2.11. [10] An algebra (B; $\circledast$, 0) of (2, 0) kind is known as BCL-algebra if and only if for every $m,\ n,\ z \in$ B:

(1) $m \circledast m = 0$;

(2) $m \circledast n = 0$ and $n \circledast m = 0$ imply $m = n$

(3) $\left[\left((m \circledast n) \circledast z\right) \circledast \left((m \circledast z) \circledast n\right)\right] \circledast \left((z \circledast n) \circledast m\right) = 0$.

Definition 2.12. [1] Suppose (B; ⊛, 0) is a BCL-algebra. A binary relation " $\leq$ " on B in which $m \leq n$ *iff* $m \circledast n$ = 0 for every $m,\ n \in$ B is known as the BCL-ordering and " $\leq$ " is partial ordering on B.

Definition 2.13. [1] Suppose $m \leq n$ *iff* $m \circledast n$ = 0, then Definition 2.11 for BCL-algebra could be given by:

(1) $m \leq m$

(2) $m \leq n$ and $n \leq m \Rightarrow m = n$

(3) $[(m \circledast n) \circledast w] \circledast [(m \circledast w) \circledast n] \leq (w \circledast n) \circledast m.$

Definition 2.14. [10, 11] An algebra (B; ⊛, 1) of (2, 0) kind is known as BCL$^+$-algebra *iff* for every $m,\ n,\ z \in$ B:

(1) $m \circledast m$ = 1 ;

(2) $m \circledast n$ = 1 and $n \circledast m$ = 1 imply $m = n$

(3) $\big((m \circledast n) \circledast z\big) \circledast \big((m \circledast z) \circledast n\big) = \big((z \circledast n) \circledast m\big)$.

Definition 2.15. [11] A Liu-algebra is a tetrad (L; ⊛, ⊙, 1) in which set L $\neq \emptyset$, ⊛ and ⊙ are two binary operations defined on L, 1 is a constant in L in which for every $m,\ n,\ w \in$ L, the statements given hereunder are true:

(1) (L; ⊛, 1) is BCL$^+$-algebra

(2) (L; ⊙) is a semi-group

(3) $m \odot (n \circledast w) = (m \odot n) \circledast (m \odot w)$ and

(4) $(n \circledast m) \odot w = (n \odot w) \circledast (m \odot w).$

Definition 2.16. [11] Suppose K is a non-empty subset of a Liu-algebra $(L;\ \circledast,\ \odot,\ 1)$ and $n \in$ K, $m \in$ L. Thus K is known as ideal of L *iff* :

(1) $m \circledast n \in$ K and $n \in$ I $\Rightarrow m \in$ K,

(2) $n \odot m \in$ K, $m \odot n \in$ K and $n \in$ K $\Rightarrow m \in$ K.

3 Main Results — Pythagorean Fuzzy Ideal of Liu^B-Algebra (LBA.)

3.1 A Few Properties of BCL-Algebra

Before we begin this section, let's first introduce some new theorems related to BCL-Algebra that have not been discussed yet, as they will be helpful for our upcoming work..

Proposition 3.1. *If (B; ⊛, 0) is a BCL-Algebra, then all the statements below are true, for every* $m,\ n \in B$*:*

(1) $0 \circledast m = 0$ *(and hence 0 is the least element),*

(2) $m \circledast n = m \Rightarrow m = 0,$

(2) $m \circledast n = n \Rightarrow m = 0$

(3) $m \circledast 0 = 0 \Rightarrow m = 0.$

Proof. Suppose B is a BCL-Algebra and $m, n \in$ B

(1) Here, the claim is to prove $0 \circledast m$ = 0.

But before proving this part of the proposition, there is a need to prove:

$0 \circledast (0 \circledast m) = 0$, $\forall m \in$ B:

$\big[\big((m \circledast m) \circledast m\big) \circledast \big((m \circledast m) \circledast m\big)\big] \circledast \big((m \circledast m) \circledast m\big) = 0$ *(by axiom (3) of Definition 2.11)*

$\Rightarrow \big[\big(0 \circledast m\big) \circledast \big(0 \circledast m\big)\big] \circledast \big(0 \circledast m\big) = 0$ *(since $m \circledast m$ = 0 by axiom (1) of Definition 2.11)*

$\Rightarrow$ *$0 \circledast (0 \circledast m) = 0$ (since $(0 \circledast m) \circledast (0 \circledast m) = 0$ by axiom (1) of definition 2.11)*

Hence $0 \circledast (0 \circledast m) = 0$, $\forall m \in B$

Now, assume, $\forall m \in B$; $0 \circledast m \neq 0$ and so, let $0 \circledast m = u$, where $u \neq 0$

But $0 \circledast (0 \circledast m) = 0$, $\forall m \in B$

$\Rightarrow 0 \circledast u = 0$, $\forall m, u \in B$ and $u \neq 0$ arriving at a contradiction to what we assumed.

Thus $0 \circledast m = 0$, $\forall m \in B$

Again, by the definition of binary relation on B:

$0 \circledast m = 0 \Leftrightarrow 0 \leq m$, $\forall m \in B$ so that it tells us that 0 is the least element of B.

(2) Suppose $m \circledast n = n$ and from the definition of BCL-Algebra:

So, $\big[\big((m \circledast n) \circledast w\big) \circledast \big((m \circledast w) \circledast n\big)\big] \circledast \big((w \circledast n) \circledast m\big) = 0$

$\Rightarrow \big[\big(n \circledast w\big) \circledast \big(w \circledast n\big)\big] \circledast \big(n \circledast m\big) = 0$

$\Rightarrow \big(w \circledast n\big) \circledast m = 0 \Rightarrow n \circledast m = 0 \Rightarrow m = 0.$

(3) Suppose $m \circledast 0 = 0$. Then by (1), $0 \circledast m = 0$

Combining these two, we have; $0 \circledast m = 0$ and $m \circledast 0 = 0$ which result $m = 0$.

□

3.2 Basic Concepts of Liu^B-Algebra (LBA.), and Ideal of LBA.

In this subsection, we introduce new definitions and examples of LBA. and the Ideal of LBA. Here,(L; $\circledast$, $\odot$, 0) is newly defined based on the BCL-Algebra (B; $\circledast$, 0), (not on BCL$^+$-Algebra (B; $\circledast$, 1), as previously defined by Y. H. Liu in [11]), and, as a result, the ideal of LBA. is originally defined accordingly.

Definition 3.2. Liu^B-Algebra. (LBA.) is an algebra (L; $\circledast$, $\odot$, 0) in which L is non-empty set; $\circledast$ and $\odot$ are two binary operations defined on L, 0 is constant in L, where for every element $m,\ n,\ w \in$ L, the axioms hereunder are satisfied:

(1) (L; $\circledast$, 0) is BCL-algebra;

(2) (L; $\odot$) is a semi-group ($\odot$ is associative in L);

(3) $\odot$ is both right and left distributive over $\circledast$ or $\begin{cases} m \odot (n \circledast w) = (m \odot n) \circledast (m \odot w) \\ (n \circledast w) \odot m = (n \odot m) \circledast (w \odot m). \end{cases}$

Proposition 3.3. *For (L; $\circledast$, $\odot$, 0) being LBA., the statements hereunder are true for every $m \in$ L:*

(a) $0 \odot 0 = 0$

(b) $0 \odot m = 0 = m \odot 0 = 0$.

Proof. Suppose (L; $\circledast$, $\odot$, 0) is a LBA. and $m, n \in$ L:

(a) $0 = (0 \odot m) \circledast (0 \odot m) = 0 \odot (m \circledast m) = 0 \odot 0,\ \forall m \in$ L

(b) $0 = (0 \odot m) \circledast (0 \odot m) = (0 \circledast 0) \odot m = 0 \odot m$

$= (m \odot 0) \circledast (m \odot 0) = m \odot (0 \circledast 0) = m \odot 0,\ \forall m \in$ L

(c) Holds true by (b) as $n \circledast n = 0$

Generally, $0 \odot 0 = 0 \odot m = m \odot 0 = 0 = m \odot (n \circledast n) = (n \circledast n) \odot m,\ \forall m, n \in$ L.

□

Definition 3.4. Suppose N is a non-empty subset of L, $n \in$ N, $m \in$ L. Thus N is known as ***ideal*** of L *iff*:

(1) $m \circledast n \in$ N and $n \in$ N $\Rightarrow m \in$ N,

(2) $n \odot m \in$ N, $m \odot n \in$ N and $n \in$ N $\Rightarrow m \in$ N.

Example 3.5. Assume L = $\{0\ m, n, w\}$ and two binary operations $\circledast$ and $\odot$ on L be defined by that tables below:

Table 1. Tables for the LBA. (L, $\circledast$, $\odot$, 0)

$\circledast$	0	m	n	w
0	0	0	0	0
m	m	0	w	m
n	n	w	0	n
w	w	m	n	0

$\odot$	0	m	n	w
0	0	0	0	0
m	0	m	w	n
n	0	w	n	m
w	0	n	m	w

Then from Table 1 above, it easy to show that (L, $\circledast$, $\odot$, 0) is LBA.

And we have the following as well:

(1) L, $\{0\}$, $\{0, w\}$ are ideals of L.

(2) $\{0, n\}$, $\{n, w\}$, $\{m, w\}$, $\{m, n, w\}$ are not ideals. of L for $m \neq 0$, $n \neq 0$, $w \neq 0$.

Example 3.6. Suppose L = $\{0, m, n\}$ and two binary operations $\circledast$ and $\odot$ on L are given by the table below:

Table 2. Tables for the LBA. (L, $\circledast$, $\odot$, 0)

$\circledast$	0	m	n
0	0	0	0
m	m	0	0
n	n	n	0

$\odot$	0	m	n
0	0	m	m
m	m	0	0
n	m	0	0

Then from Table 2 above, it easy to show that (L, $\circledast$, $\odot$, 0) is LBA.

And we have the following as well:

(1) $\{0\}$, $\{0, m, n\}$ are ideals of L.

(2) $\{m\}$, $\{m, n\}$, $\{0, m\}$ are not ideals of L for $m \neq 0$, $n \neq 0$.

Lemma 3.7. *Suppose $(L;\ \circledast,\ \odot,\ 0)$ is LBA. Then if N is ideal of L, then $0 \in N$.*

Remark 3.8. The converse of Remark 3.7 is not necessarily true which we explain this fact by the next example.

Example 3.9. From Table 2 of Example 3.6 above, we see that $0 \in \{0, m\}$ but $\{0, n\}$ is not Ideal of L_2 which supports Remark 3.7.

Theorem 3.10. *Let (L; ⊛, ⊙, 0) be LBA. and $m, n, w \in L$. The statements hereunder are true:*

(i) $m \leq n \Rightarrow w \odot m \leq w \odot n$,

(ii) $m \leq n \Rightarrow m \odot w \leq n \odot w$.

Proof. Let (L; ⊛, ⊙, 0) be a LBA. and let $m, n, w \in$ L:

(i) $m \leq n \Rightarrow m \circledast n = 0$ and $(w \odot m) \circledast (w \odot n) = 0 = w \odot (m \circledast n) = w \odot 0 = 0$

$\Rightarrow (w \odot m) \circledast (w \odot n) = 0 \Rightarrow w \odot m \leq w \odot n = 0$

(ii) This could also be verified in a similar fashion as in (i) above except here we use left distributive property in stead of right distributive property. □

Definition 3.11. A fuzzy set η_L in L is known as a fuzzy ideal of L if the statements hereunder are fulfilled for every $m, n \in$ L:

(i) $\eta_L(m) \geq \min\{\eta_L(m \circledast n), \eta_L(n)\}$

(ii) $\eta_L(m) \geq \min\{\eta_L(m \odot n), \eta_L(n \odot m), \eta_L(n)\}$.

Example 3.12. Let $L_1 = \{0, m, n, w\}$, $L_2 = \{0, m, n\}$ and binary operations ⊛ and ⊙ on both L_1 and L_2 are as defined in Table 1 and Table 2 above respectively and let the fuzzy sets η_{L_1} in L_1 and μ_{L_2} in L_2 be defined as in the following ways:

$$\eta_{L_1}(k) = \begin{cases} 0.9, & \text{iff } k = m, n, w; \\ 0.4, & \text{iff } k = 0, \end{cases} \quad \text{and} \quad \mu_{L_2}(k) = \begin{cases} 0.78, & \text{iff } k = m, n; \\ 0.24, & \text{iff } k = 0. \end{cases}$$

Then clearly, η_{L_1} is fuzzy ideal of L_1 and μ_{L_2} is fuzzy ideal of L_2.

3.3 Pythagorean Fuzzy Ideal of LBA.

In this subsection, we introduce the definition and provide examples of the Pythagorean fuzzy ideal of LBA. We also state and prove several properties and theorems related to the Pythagorean fuzzy ideal of LBA, which are the primary focus of this section.

Throughout this subsection and the following one, we denote L^B as the Pythagorean fuzzy ideal $L^B = (\eta_L, \tau_L)$ of LBA. (L; ⊛, ⊙, 0), unless otherwise specified.

Definition 3.13. A Pythagorean fuzzy set $L^B = (\eta_L, \tau_L)$ on a non-empty set L, where the functions $\eta_L : L \to [0, 1]$ and $\tau_L : L \to [0, 1]$ define the membership degree and the non-membership degree, respectively, is called a Pythagorean fuzzy ideal of L if the following two pairs of conditions are satisfied for every $m, n \in$ L:

(i) $(\eta_L(m))^2 \geq \min\{(\eta_L(m \circledast n))^2, (\eta_L(n))^2\}$ and
$(\tau_L(m))^2 \leq \max\{(\tau_L(m \circledast n))^2, (\tau_L(n))^2\}$

(ii) $(\eta_L(m))^2 \geq \min\{(\eta_L(m \odot n))^2, (\eta_L(n \odot m))^2, (\eta_L(n))^2\}$ and
$(\tau_L(m))^2 \leq \max\{(\tau_L(m \odot n))^2, (\tau_L(n \odot m))^2, (\tau_L(n))^2\}$.

Example 3.14. Suppose L = $\{0, m, n\}$ and two binary operations $\circledast$ and $\odot$ on L are as given by the Table 2 above and let the Pythagorean fuzzy set $L^B = (\eta_L, \tau_L)$ be such that

η_L : L $\to$ [0, 1] and τ_L : L $\to$ [0, 1] given by

$$\eta_L(k) = \begin{cases} 0.9, & \textit{iff } k = m, n; \\ 0.4, & \textit{iff } k = 0, \end{cases} \quad \text{and} \quad \tau_L(k) = \begin{cases} 0.3, & \textit{iff } k = m, n; \\ 0.7, & \textit{iff } k = 0 \end{cases}$$

Thus, we could simply verify that L^B is Pythagorean fuzzy ideal of L.

Lemma 3.15. *If $L^B = (\eta_L, \tau_L)$ is a Pythagorean fuzzy ideal of L, then $\forall\, m \in B$;*
$(\eta_L(m))^2 \geq (\eta_L(0))^2$ *and* $(\tau_L(m))^2 \leq (\tau_L(0))^2$.

Proof. For $m \in$ L, we have: $0 \odot m = 0 = m \odot 0$. Then

$$(\eta_L(m))^2 \geq \min\{(\eta_L(m\odot 0))^2, (\eta_L(0\odot m))^2, (\eta_L(0))^2\} = \min\{(\eta_L(0))^2, (\eta_L(0))^2, (\eta_L(0))^2\}$$
$$= (\eta_L(0))^2, \ \forall m \in \text{L}, \ \text{and}$$
$$(\tau_L(m))^2 \leq \max\{(\tau_L(m \odot 0))^2, (\tau_L(0 \odot m))^2, (\tau_L(0))^2\}$$
$$= \max\{(\tau_L(0))^2, (\tau_L(0))^2, (\tau_L(0))^2\} = (\tau_L(0))^2, \ \forall m \in \text{L},$$
$$\Rightarrow (\eta_L(m))^2 \geq (\eta_L(0))^2, \ \text{and} \ (\tau_L(m))^2 \leq (\tau_L(0))^2, \ \forall m \in \text{L}.$$

□

Theorem 3.16. *Let M be a non-empty subset of L and $L^B = (\chi_M, \overline{\chi}_M)$, where χ_M is the characteristic function and $\overline{\chi}_M$ is its complement, then L^B is a Pythagorean fuzzy ideal of L iff M is ideal of L.*

Proof. Suppose χ_M : M $\to$ [0, 1] is a characteristic function defined as:

$$\chi_M(k) = \begin{cases} 1, & \textit{iff } k \in M, \\ 0, & \textit{iff } k \notin M. \end{cases} \quad \text{and then} \quad \overline{\chi}_M(k) = \begin{cases} 0, & \textit{iff } k \in M, \\ 1, & \textit{iff } k \notin M. \end{cases}$$

and suppose L^B is a Pythagorean fuzzy ideal of L. We need to verify that M is a ideal of L.

For $k \circledast q, q \in$ M $\Rightarrow (\chi_M(k))^2 \geq \min\{\chi_M(k \circledast q), (\chi_M(q))^2\} = 1$

$\Rightarrow (\chi_M(k))^2 \geq 1$ (and $(\chi_M(k))^2 \leq 1$ by definition of $\chi_M(k)$

$\Rightarrow (\chi_M(k))^2 = 1 \ \Rightarrow k \in$ M $\Rightarrow k \circledast q, q \in$ M $\Rightarrow k \in$ M

Again, let $k \odot q, q \odot k, q \in$ M

$\Rightarrow (\chi_M(k))^2 \geq \min\{(\chi_M(k \odot q))^2, (\chi_M(q \odot k))^2, (\chi_M(q))^2\} = 1$

$\Rightarrow (\chi_M(k))^2 \geq 1$ (and $(\chi_M(k))^2 \leq 1$ by definition of $\chi_M(k)$) $\Rightarrow (\chi_M(k))^2 = 1 \ \Rightarrow k \in$ M

Thus $k \odot q, q \odot k, q \in$ M $\Rightarrow k \in$ M

Therefore, M is a ideal of L.

Conversely, suppose M is an ideal of L.

We need to prove $L^B = (\chi_M, \overline{\chi}_M)$ is a Pythagorean fuzzy ideal of L.

Let $k, q \in$ M $\Rightarrow (\chi_M(k))^2 = 1 \geq \min\{(\chi_M(k \circledast q))^2, (\chi_M(q))^2\}$ and

$$\chi_M(k) = 1 \geq \min\{(\chi_M(k \odot q))^2, (\chi_M(q \odot k))^2, (\chi_M(q))^2\}$$

$\Rightarrow \left(\chi_M(k)\right)^2 \geq \min\{\left(\chi_M(k \circledast q)\right)^2, \left(\chi_M(q)\right)^2\}$ and

$\chi_M(k) \geq \min\{\left(\chi_M(k \odot q)\right)^2, \left(\chi_M(q \odot k)\right)^2, \left(\chi_M(q)\right)^2\}$

Additionally, $k, q \in$ M $\Rightarrow \left(\overline{\chi}_M(k)\right)^2 = 0 \leq \max\{\left(\overline{\chi}_M(k \circledast q)\right)^2, \left(\overline{\chi}_M(q)\right)^2\}$ and

$$\left(\overline{\chi}_M(k)\right)^2 = 0 \leq \max\{\left(\overline{\chi}_M(k \odot q)\right)^2, \left(\overline{\chi}_M(q \odot k)\right)^2, \left(\overline{\chi}_M(q)\right)^2\}$$

Thus, $L^B = \left(\chi_M, \overline{\chi}_M\right)$ is a Pythagorean fuzzy ideal of L.

□

Theorem 3.17. *The intersection of any two Pythagorean fuzzy ideals, L_1^B and L_2^B of L is also a Pythagorean fuzzy ideal of L.*

Proof. Let L_1^B and L_2^B be any two Pythagorean fuzzy ideal of L. Then we need to prove:

$L_\cap^B = \left(\eta_{L_1^B} \cap \eta_{L_2^B},\ \tau_{L_1^B} \cap \tau_{L_2^B}\right)$ is a Pythagorean fuzzy ideal of L.

For $m, n, z \in$ L. Then,

(i) $\left(\eta_{L_1^B} \cap \eta_{L_2^B})(m)\right)^2 = \min\left\{\left((\eta_{L_1^B})(m)\right)^2,\ \left((\eta_{L_2^B})(m)\right)^2\right\}$

$$\geq \min\left\{\min\{\left((\eta_{L_1^B}(m \circledast n)\right)^2,\ \left((\eta_{L_1^B}(n)\right)^2\},\ \min\{\left((\eta_{L_2^B}(m \circledast n)\right)^2,\ \left((\eta_{L_2^B}(n)\right)^2\right\}$$

$$= \min\left\{\min\{\left((\eta_{L_1^B}(m \circledast n)\right)^2,\ \left((\eta_{L_2^B}(m \circledast n)\right)^2,\ \min\{\left((\eta_{L_1^B}(n)\right)^2, \left((\eta_{L_2^B}(n)\right)^2\}\right\},$$

$$= \min\left\{\left((\eta_{L_1^B} \cap\ \eta_{L_2^B})(m \circledast n)\right)^2,\ \left((\eta_{L_1^B} \cap\ \eta_{L_2^B})(n)\right)^2\right\}$$

(ii) Similarly, it is simple to verify that:

$$\left((\eta_{L_1^B} \cap \eta_{L_2^B})(m)\right)^2 \geq \min\{\left((\eta_{L_1^B} \cap \eta_{L_2^B})(m \odot n)\right)^2, \left((\eta_{L_1^B} \cap \eta_{L_2^B})(n \odot m)\right)^2, \left((\eta_{L_1^B} \cap \eta_{L_2^B})(n)\right)^2\}$$

$$\left(\tau_{L_1^B} \cap \tau_{L_2^B})(m)\right)^2 \leq \max\left\{\left((\tau_{L_1^B} \cap\ \tau_{L_2^B})(m \circledast n)\right)^2,\ \left((\tau_{L_1^B} \cap \tau_{L_2^B})(n)\right)^2\right\}$$

$$\left((\tau_{L_1^B} \cap \tau_{L_2^B})(m)\right)^2 \geq \min\{\left((\tau_{L_1^B} \cap \tau_{L_2^B})(m \odot n)\right)^2, \left((\tau_{L_1^B} \cap \eta_{L_2^B})(n \odot m)\right)^2, \left((\tau_{L_1^B} \cap \tau_{L_2^B})(n)\right)^2\}$$

Thus, intersection of Pythagorean fuzzy ideals. of L is Pythagorean fuzzy ideal of L. □

Corollary 3.18. *The intersection, $\bigcap_{i \in I} \eta_{L_i^B}$, of a family of Pythagorean fuzzy ideals.,*

$\{L_i^B : i \in I\}$, in L is also a Pythagorean fuzzy ideal of L, where

$$\bigcap_{i \in I} \eta_{L_i^B}(m) = \inf_{i \in I} \eta_{L_i^B}(m) \ \text{and} \ \bigcap_{i \in I} \tau_{L_i^B}(m) = \sup_{i \in I} \tau_{L_i^B}(m).$$

Remark 3.19. The union of two Pythagorean fuzzy ideals of a LBA. L may not be Pythagorean fuzzy ideal of L.

Proposition 3.20. *For η_L be a fuzzy set in a LBA. L such that* $\eta_L(m) = \begin{cases} \delta, & \text{iff } m \in M, \\ \varepsilon, & \text{iff } m \notin M, \end{cases}$

and the square deviation $\overline{\overline{\eta}}_L(m) = \begin{cases} 1 - \delta^2, & \text{iff } m \in M, \\ 1 - \varepsilon^2, & \text{iff } m \notin M. \end{cases}$ *where $\delta, \varepsilon \in [0, 1]$, $\delta > \varepsilon$*

Then M is ideal of L iff $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ is a Pythagorean fuzzy ideal of L.

Proof. For M is ideal of L, we prove that $(\eta_L, \overline{\eta}_L)$ is a Pythagorean fuzzy ideal of L.

Case (i): Let $n \in$ M, $m \circledast n \in$ M $\Rightarrow m \in$ M

Then $(\eta_L(m \circledast n))^2 = (\eta_L(n))^2 = \delta^2 \Rightarrow (\eta_L(m))^2 = \delta^2$

$\Rightarrow (\eta_L(m))^2 \geq \delta^2 = \min\{(\eta_L(m \circledast n))^2, (\eta_L(n))^2\}$ and also

$(\overline{\overline{\eta}}_L(m))^2 \leq \delta^2 = \min\{(\overline{\overline{\eta}}_L(m \circledast n))^2, (\overline{\overline{\eta}}_L(n))^2\}$

Similarly, for $n \in$ M, $m \odot n \in$ M, $n \odot m \in$ M $\Rightarrow m \in$ M

$\Rightarrow (\eta_L(m \odot n))^2 = (\eta_L(n \odot m))^2 = (\eta_L(n))^2 = \delta^2 \Rightarrow (\eta_L(m))^2 = \delta^2$

$\Rightarrow (\eta_L(m))^2 \geq \delta^2 = \min\{(\eta_L(m \odot n))^2, (\eta_L(n \odot m))^2, (\eta_L(n))^2\}$, and

$(\tau_L(m))^2 \leq \delta^2 = \max\{(\tau_L(m \odot n))^2, (\tau_L(n \odot m))^2, (\tau_L(n))^2\}$

Case (ii): Let $n \in$ M, $m \circledast n \notin$ M $\Rightarrow m \notin$ M or $m \in$ M

$\Rightarrow (\eta_L(n))^2 = \delta^2, (\eta_L(m \circledast n))^2 = \varepsilon^2 \Rightarrow (\eta_L(m))^2 = \varepsilon^2$ or $(\eta_L(m))^2 = \delta^2$

$\Rightarrow (\eta_L(m))^2 \geq \varepsilon^2 = \min\{(\eta_L(m \circledast n))^2, (\eta_L(n))^2\} = \min\{\varepsilon^2, \delta^2\} = \varepsilon^2$ and

$(\overline{\overline{\eta}}_L(m))^2 \leq \varepsilon^2 = \max\{(\overline{\overline{\eta}}_L(m \circledast n))^2, (\overline{\overline{\eta}}_L(n))^2\} = \max\{\delta^2, \varepsilon^2\} = \varepsilon^2$

Similarly, for $n \in$ M, $m \odot n \notin$ M or $n \odot m \notin$ M $\Rightarrow m \notin$ M or $m \in$ M

$\Rightarrow (\eta_L(n))^2 = \delta^2, (\eta_L(m \odot n))^2 = \varepsilon^2$ or $(\eta_L(n \odot m))^2 = \varepsilon^2 \Rightarrow (\eta_L(m))^2 = \varepsilon^2$ or $(\eta_L(m))^2 = \delta^2$

$\Rightarrow (\eta_L(m))^2 \geq \varepsilon^2 = \min\{(\eta_L(m \odot n))^2, (\eta_L(n \odot m))^2, (\eta_L(n))^2\}$, and again

$(\tau_L(m))^2 \leq \delta^2 = \max\{(\tau_L(m \odot n))^2, (\tau_L(n \odot m))^2, (\tau_L(n))^2\}$

Now we arrive at similar deductions for Case (iii) when $n \notin$ M, $m \circledast n \in$ M $\Rightarrow m \in$ M or $m \notin$ M and Case (iv) when $n \notin$ M, $m \circledast n \notin$ M $\Rightarrow m \in$ M or $m \notin$ M

Conversely, suppose $(\eta_L, \overline{\overline{\eta}}_L)$ is a Pythagorean fuzzy ideal of L.

Let $n, m \circledast n \in$ M $\Rightarrow (\eta_L(n \circledast m))^2 = (\eta_L(n))^2 = \delta^2$ and

$(\eta_L(m))^2 \geq \min\{(\eta_L(m \circledast n))^2, (\eta_L(n))^2\} = \min\{\delta^2, \delta^2\} = \delta^2 \Rightarrow (\eta_L(m))^2 \geq \delta^2$

and since $\delta, \varepsilon \in [0, 1]$, $\delta > \varepsilon$, we have $(\eta_L(m))^2 \leq \delta^2 \Rightarrow (\eta_L(m))^2 = \delta^2 \Rightarrow m \in$ M

Similarly, for $n, m \odot n, n \odot m \in$ M it is easy to show that, $m \in$ M.

Therefore, M is ideal of L. □

Corollary 3.21. *Let η_L be a fuzzy set in LBA. L such that* $\eta_L(m) = \begin{cases} \delta, & \text{iff } m \in M, \\ \varepsilon, & \text{iff } m \notin M, \end{cases}$

and its complement $\overline{\eta}_L(m) = \begin{cases} 1-\delta, & \text{iff } m \in M, \\ 1-\varepsilon, & \text{iff } m \notin M. \end{cases}$ *where $\delta, \varepsilon \in [0, 1]$, $\delta > \varepsilon$*

Then M is ideal of L iff $(\eta_L, \overline{\eta}_L)$ is a Pythagorean fuzzy ideal of L.

3.4 Some more Basic Concepts of Pythagorean Fuzzy Liu^{B}-Ideals

Let the notations in Table 3 below be for the accuracy function a, the score function s and the degree of indeterminacy π of Intuitionistic & Pythagorean fuzzy sets and let

$$\eta_{L}(m)^{d} = \eta_{L}(m) - \big(\eta_{L}(m)\big)^{2} \text{ and } \tau_{L}(m)^{d} = \tau_{L}(m) - \big(\tau_{L}(m)\big)^{2}$$

for every $m \in$ L be the membership deviations and non-membership deviations on the membership and non-membership functions η_{L} and τ_{L} of L respectively.

Table 3. Table 3: A table showing the comparison between intuitionistic and Pythagorean fuzzy sets on the accuracy function, score function and degree of indeterminacy of LBA. (L; ⊛, ⊙).

No.	Notation	Intuitionistic Fuzzy Set	Pythagorean Fuzzy Set
1	a	$a_{I}(m) = \eta(m) + \upsilon(m)$	$a_{P}(m) = (\eta_{L}(m))^{2} + (\upsilon_{P}(m))^{2}$
2	s	$s_{I}(m) = \eta(m) - \upsilon(m)$	$s_{P}(m) = (\eta_{L}(m))^{2} - (\upsilon_{P}(m))^{2}$ $= a_{I}(m)s_{I}(m)$
3	π	$\pi_{I}(m) = 1 - \big(\eta(m) + \upsilon_(m)\big)$ $= 1 - \eta_{L}(m) - \upsilon(m)$	$\pi_{P}(m)) = 1 - \big((\eta_{L}(m))^{2} + (\upsilon_{P}(m))^{2}\big)$ $= 1 - \mathrm{a}_{P}(m)$

Definition 3.22. Let $\eta_{L}^{d}(m) = \eta_{L}(m) - \big(\eta_{L}(m)\big)^{2}$ and $\tau_{L}^{d}(m) = \tau_{L}(m) - \big(\tau_{L}(m)\big)^{2}$, for every m in L, be membership deviations and non-membership deviations, respectively. Then we call L^{d}= $\big(\eta_{L}^{d}, \tau_{L}^{d}\big)$ Pythagorean fuzzy ideal deviation of L.

Remark 3.23. For η_{L} in a LBA. L and $m \in$ L:

$$\eta_{L}(\mathrm{m}) \geq \big(\eta_{L}(m)\big)^{2} \text{ and } \tau_{L}(\mathrm{m}) \geq \big(\tau_{L}(m)\big)^{2}$$

$$\Rightarrow \eta_{L}(\mathrm{m}) - \big(\eta_{L}(m)\big)^{2} \geq 0 \text{ and } \tau_{L}(\mathrm{m}) - \big(\tau_{L}(m)\big)^{2} \geq 0$$

$$\Rightarrow \eta_{L}(m)\big(1 - \eta_{L}(m)\big) \geq 0 \text{ and } \tau_{L}(m)\big(1 - \tau_{L}(m)\big) \geq 0.$$

Proposition 3.24. *Let $\emptyset \neq U \subseteq L$ such that χ_{U} is characteristic function and $\overline{\chi}_{U} = 1 - \chi_{U}$ is the complement of χ_{U}. Then $L^{B} = \big(\chi_{U}, \overline{\chi}_{U}\big)$ is Pythagorean fuzzy ideal of L iff U is a ideal of L.*

Furthermore, the accuracy function a_{U}, the score function s_{U} and the degree of indeterminacy π_{U} are respectively given hereunder, $\forall m \in L$:

$$(a)\quad a_{U}(m) = 1, \quad (b)\quad s_{U}(m) = \begin{cases} 1, & \text{iff } m \in U, \\ -1, & \text{iff } m \notin U. \end{cases} \quad (c)\quad \pi_{U}(m) = 0.$$

Proof. Let χ_{U} : U → [0, 1] be a characteristic function and the complement

$\overline{\chi}_{U}$: U → [0, 1] be given hereunder:

$$\chi_{U}(m) = \begin{cases} 1, & \text{iff } m \in U, \\ 0, & \text{iff } m \notin U, \end{cases} \quad \text{and then} \quad \overline{\chi}_{U}(x) = \big(\overline{\overline{\chi}}_{U}(m)\big)^{2} = \begin{cases} 0 & \text{iff } m \in U, \\ 1 & \text{iff } m \notin U. \end{cases}$$

Then let L = $(\chi_U, \overline{\chi}_U)$ be a Pythagorean fuzzy ideal of L,

Now we need to prove that U is ideal of L:

(i) Let $n, m \circledast n \in$ U $\Rightarrow$ $\chi_U(n) = (\chi_U(m \circledast n))^2 = 1$ and $(\overline{\chi}_U(n))^2 = (\overline{\chi}_U(m \circledast n))^2 = 0$

$\Rightarrow$ $(\chi_U(m))^2 \geq \min\{(\chi_U(n))^2, (\chi_U(m \circledast n))^2\} = \min\{1, 1\} = 1$ and

$(\overline{\chi}_U(m))^2 \leq \max\{(\overline{\chi}_U(n))^2, (\overline{\chi}_U(m \circledast n))^2\} = \min\{0, 0\} = 0$

But $(\chi_U(m))^2 \leq 1$ and $(\overline{\chi}_U(m))^2 \geq 0$, $\forall m \in$ L, by the definitions.

Hence $(\chi_U(m))^2 = 1$ and $(\overline{\chi}_U(m))^2 = 0$ $\Rightarrow$ $m \in$ U

(ii) Let $n, m \odot n, n \odot m \in$ U

$\Rightarrow$ $(\chi_U(n))^2 = (\chi_U(m \odot n))^2 = (\chi_U(n \odot m))^2 = 1$ and $(\overline{\chi}_U(n))^2 = (\overline{\chi}_U(m \odot n))^2 = 0$

$\Rightarrow$ $(\chi_U(m))^2 \geq \min\{(\chi_U(n))^2, (\chi_U(m \odot n))^2, (\chi_U(n \odot m))^2\} = 1$ and

$(\overline{\chi}_U(m))^2 \leq \max\{(\overline{\chi}_U(n))^2, (\overline{\chi}_U(m \odot n))^2, (\chi_U(n \odot m))^2\} = 0$

But $(\chi_U(m))^2 \leq 1$ and $(\overline{\chi}_U(m))^2 \geq 0$, $\forall m \in$ L, by the definitions.

Hence $(\chi_U(m))^2 = 1$ and $(\overline{\chi}_U(m))^2 = 0$ $\Rightarrow$ $m \in$ U

which means U is a ideal of L, by (i) and (ii) above.

Conversely, for a ideal U of L, we need to verify that:

L is a Pythagorean fuzzy ideal of L:

We show this proof by considering four cases for each of the two axioms;

(1) When $n, m \circledast n \in$ U $(n, m \odot n, n \odot m \in$ U$)$;

(2) when $n \notin$ U, $m \circledast n \notin$ U $(n \notin$ U, $m \odot n \notin$ U or $n \odot m \notin$ U$)$;

(3) when $n \in$ U, $m \circledast n \notin$ U $(n \in$ U, $m \odot n \notin$ U or $n \odot m \notin$ U$)$;

(4) when $n \notin$ U, $m \circledast n \in$ U $(n \notin$ U, $m \odot n \in$ U or $n \odot m \in$ U$)$;

Now, let us show the proofs for the first parts of the first two assumptions of the above and in a similar fashion, the rest can be worked out:

Case (1): If $n, m \circledast n \in$ U, then $m \in$ U

$\Rightarrow (\chi_U(m))^2 = 1 \geq \min\{1, 1\} = \min\{(\chi_U(n))^2, (\chi_U(m \circledast n))^2\}$ and

$(\overline{\chi}_U(m))^2 = 0 \leq \max\{0, 0\} = \max\{(\overline{\chi}_U(n))^2, (\overline{\chi}_U(m \circledast n))^2\}$

$\Rightarrow (\chi_U(m))^2 \geq \min\{(\chi_U(n))^2, (\chi_U(m \circledast n))^2\}$ and

$(\overline{\chi}_U(m))^2 \leq \max\{(\overline{\chi}_U(n))^2, (\overline{\chi}_U(m \circledast n))^2\}$.

Case (2): If $n \notin$ U, $m \circledast n \notin$ U

$\Rightarrow (\chi_U(n))^2 = 0 = (\chi_U(m \circledast n))^2$, $(\chi_U(m))^2 \geq 0$, $\forall m \in$ L and

$(\overline{\chi}_U(n))^2 = 1 = (\overline{\chi}_U(m \circledast n))^2$, $(\chi_U(m))^2 \leq 1, \forall m \in$ L

$$\Rightarrow \left(\chi_U(m)\right)^2 \geq 0 = \min\{\left(\chi_B(n)\right)^2, \left(\chi_B(m \circledast n)\right)^2\} \text{ and}$$
$$\left(\overline{\chi}_U(m)\right)^2 \leq 1 = \max\{\left(\overline{\chi}_B(n)\right)^2, \left(\overline{\chi}_B(m \circledast n)\right)^2\}$$
$$\Rightarrow \left(\chi_U(m)\right)^2 \geq \min\{\left(\eta_B(n)\right)^2, \left(\eta_B(m \circledast n)\right)^2\} \text{ and}$$
$$\left(\overline{\chi}_U(m)\right)^2 \leq \max\{\left(\overline{\chi}_B(n)\right)^2, \left(\overline{\chi}_B(m \circledast n)\right)^2\}$$

Following similar patterns for Case (3) and Case (4) as case (1) and Case (2) above, we arrive at the same conclusion:

$$\left(\chi_U(m)\right)^2 \geq \min\{\left(\chi_U(n)\right)^2, \ \left(\chi_U(m \circledast n)\right)^2\}$$
$$\left(\overline{\chi}_U(m)\right)^2 \leq \max\{\left(\overline{\chi}_U(n)\right)^2, \ \left(\overline{\chi}_U(m \circledast n)\right)^2\}, \forall m, n \in \mathrm{U},$$
$$\left(\chi_U(m)\right)^2 \geq \min\{\left(\chi_U(n)\right)^2, \ \left(\chi_U(m \odot n)\right)^2, \left(\chi_U(n \odot m)\right)^2\} \text{ and}$$
$$\left(\overline{\chi}_U(m)\right)^2 \leq \max\{\left(\overline{\chi}_U(n)\right)^2, \ \left(\overline{\chi}_U(m \odot n)\right)^2, \left(\overline{\chi}_U(n \odot m)\right)^2\}, \forall m, n \in \mathrm{U}$$

Therefore, χ_U is a ideal of L, by (A), (B) and (C) above.

Now, with subscripts added to notations displayed in Table 3 above, we have the next justifications:

(a) $The\ accuracy\ function:$

$$a_U(m) = \left(\chi_U(m)\right)^2 + \left(\overline{\chi}_U(m)\right)^2 = \begin{cases} 1 + 0, \textit{iff} \ m \in U, \\ 0 + 1, \textit{iff} \ m \notin U \end{cases} = 1, \ \forall\, m \in \mathrm{L}.$$

(b) $The\ score\ function:$

$$\mathrm{s}_U(m) = \left(\chi_U(m)\right)^2 - \left(\overline{\chi}_U(m)\right)^2 = \begin{cases} 1 - 0, & \textit{iff} \ \ m \in U, \\ 0 - 1, & \textit{iff} \ \ m \notin U \end{cases} = \begin{cases} 1, & \textit{iff} \ \ m \in U, \\ -1, & \textit{iff} \ \ m \notin U. \end{cases}$$

(c) $The\ degree\ of\ indeterminacy:$

$$\pi_U(m) = \sqrt{1 - \left[\left(\chi_U(m)\right)^2 + \left(\overline{\chi}_U(m)\right)^2\right]} = \begin{cases} \sqrt{0 - (1+0)}, \textit{iff} \ \ m \in U, \\ \sqrt{0 - (0+1}, \textit{iff} \ \ m \notin U \end{cases} = 0, \forall m \in \mathrm{L}.$$

□

Definition 3.25. For a membership fuzzy set; μ_L : L → [0, 1], and its square deviation $\overline{\overline{\mu}}_L$: L → [0, 1] such that $\left(\overline{\overline{\mu}}_L(x)\right)^2 = 1 - (\mu(x))^2$.

Then we call such fuzzy set, $\overline{\overline{\mu}}_L$, the square deviation of μ_L.

Theorem 3.26. *Let η_L and its square deviation $\overline{\overline{\eta}}_L$ be a fuzzy sets in LBA. L such that $\left(\eta_L(m \circledast n)\right)^2 = \left(\eta_L(n)\right)^2$ and $\left(\eta_L(m \odot n)\right)^2 = \left(\eta_L(n)\right)^2$.*

Then $\left(\overline{\overline{\eta}}_L(m \circledast n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2$ and $\left(\overline{\overline{\eta}}_L(m \odot n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2$, $\forall m, n \in L$.

Again, $L^B = \left(\eta_L, \overline{\overline{\eta}}_L\right)$ is Pythagorean fuzzy ideal of L. iff η_L and then $\overline{\overline{\eta}}_L$ are constants.

Furthermore, $the\ accuracy\ function$ $a_P(m)$, the $score\ function$ s_P and the $degree\ of\ indeterminacy$ $\pi_P(m)$ are respectively given as: $\forall m \in L$:

(a) $a_L(m) = 1$, (b) $s(L) = 2\left(\eta_L(m)\right)^2 - 1$, (c) $\pi_L(m) = 0$.

Proof. Suppose $\left(\eta_L(m \circledast n)\right)^2 = \left(\eta_L(n)\right)^2$ and $\left(\eta_L(m \odot n)\right)^2 = \left(\eta_L(n)\right)^2$.

Then $\left(\overline{\overline{\eta}}_L(m \circledast n)\right)^2 = 1 - \left(\eta_L(m \circledast n)\right)^2 = 1 - \left(\eta_L(n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2$ and

$$\left(\overline{\overline{\eta}}_L(m \odot n)\right)^2 = 1 - \left(\eta_L(m \odot n)\right)^2 = 1 - \left(\eta_L(n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2.$$

Let L = $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ be Pythagorean fuzzy ideal of L,

$$\left(\eta_L(m \circledast n)\right)^2 = \left(\eta_L(n)\right)^2,\ \left(\eta_L(m \odot n)\right)^2 = \left(\eta_L(n)\right)^2$$

$$\left(\text{and then } \left(\overline{\overline{\eta}}_L(m \circledast n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)\right)^2,\ \left(\overline{\overline{\eta}}_L(m \odot n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2$$

Now we need to verify that η_L and $\overline{\overline{\eta}}_L$ are constants,

$\left(\text{or } \forall m, n \in \text{L}; \ \left(\eta_L(m)\right)^2 = \left(\eta_L(n)\right)^2 \text{ and then } \left(\overline{\overline{\eta}}_L(m)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2,\ \forall m, n \in \text{P.}\right)$

Since P = $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ is a Pythagorean fuzzy ideal of L, η_L is a fuzzy ideal of L, and hence by some of the axioms, we have $\eta_L(0) = \eta_L(0 \circledast m) = \eta_L(0 \odot m), \forall m \in$ L,

$\Rightarrow \left(\eta_L(0)\right)^2 = \left(\eta_L(0 \circledast m)\right)^2 = \left(\eta_L(m)\right)^2 = \left(\eta_L(0 \odot m)\right)^2, \forall m \in$ P, and again

$\left(\eta_L(0)\right)^2 = \left(\eta_L(0 \circledast n)\right)^2 = \left(\eta_L(0 \odot n)\right)^2 = \left(\eta_L(n)\right)^2, \forall n \in$ P

$\Rightarrow \left(\eta_L(0)\right)^2 = \left(\eta_L(m)\right)^2 = \left(\eta_L(n)\right)^2, \forall m, n \in$P,

Or $\left(\eta_L(m)\right)^2 = \left(\eta_L(n)\right)^2, \forall m, n \in$ P and hence , η_L is constant, and analogously, $\overline{\overline{\eta}}_L$ is, too.

Conversely, suppose η_L and $\overline{\overline{\eta}}_L$ are constants, or: $\eta_L(m) = \eta_L(n)$ and $\overline{\overline{\eta}}_L(m) = \overline{\overline{\eta}}_L(n)$

such that, $\forall m, n \in$ P:

$$\begin{cases} \left(\eta_L(m \circledast n)\right)^2 = \left(\eta_L(n)\right)^2 \ and \ \left(\overline{\overline{\eta}}_L(m \circledast n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2,\ and \\ \left(\eta_L(m \odot n)\right)^2 = \left(\eta_L(n)\right)^2 \ and \ \left(\overline{\overline{\eta}}_L(m \odot n)\right)^2 = \left(\overline{\overline{\eta}}_L(n)\right)^2. \end{cases}$$

To prove: P = $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ is a Pythagorean fuzzy ideal of L,

(i) $\left(\eta_L(m)\right)^2 = \left(\eta_L(n)\right)^2 = \left(\eta_L(m \circledast n)\right)^2 = \min\{\left(\eta_L(m \circledast n)\right)^2, \left(\eta_L(n)\right)^2\}$

$$\geq \ \min\{\left(\eta_L(m \circledast n)\right)^2, \left(\eta_L(n)\right)^2\}$$

$\Rightarrow \left(\eta_L(m)\right)^2 \ \geq \min\{\left(\eta_L(m \circledast n)\right)^2, \left(\eta_L(n)\right)^2\}$

$\left(\text{and then } \left(\overline{\overline{\eta}}_L(m)\right)^2 \ \leq \ \max\{\left(\overline{\overline{\eta}}_L(m \circledast n)\right)^2, \left(\overline{\overline{\eta}}_L(n)\right)^2\}\right)$

(ii) Similarly, $\left(\eta_L(m)\right)^2 \ \geq \min\{\left(\eta_L(m \odot n)\right)^2, \left(\eta_L(n \odot m)\right)^2, \left(\eta_L(n)\right)^2\}$

$\left(\text{and then } \left(\overline{\overline{\eta}}_L(m)\right)^2 \ \leq \ \max\{\left(\overline{\overline{\eta}}_L(m \odot n)\right)^2, \left(\overline{\overline{\eta}}_L(n \odot m)\right)^2, \left(\overline{\overline{\eta}}_L(n)\right)^2\}\right)$

Therefore, by (i) and (ii) above, η_L is a fuzzy ideal of L and $\overline{\overline{\eta}}$ is the square deviation of η_L so that P = $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ is a Pythagorean fuzzy ideal of L. Furthermore:

(a) $a_P(m) = \left(\eta_L(m)\right)^2 + \left(\overline{\overline{\eta}}_L(m)\right)^2 = \left(\eta_L(m)\right)^2 + \left(1 - \left(\eta_L(m)\right)^2\right) = 1$

(b) $s_P(m) = \left(\eta_L(m)\right)^2 - \left(\overline{\overline{\eta}}_L(m)\right)^2 = \left(\eta_L(m)\right)^2 - \left(1 - \left(\eta_L(m)\right)^2\right) = 2\left(\eta_L(m)\right)^2 - 1$

(c) $\pi_P(m) = \sqrt{1 - a_P(m)} = 0.$ □

Theorem 3.27. *Let $L^B = (\eta_L, \tau_L)$ be Pythagorean fuzzy ideal of L and $m, n \in L$. If $\eta_L(m \circledast n) = \eta_L(n)$, $\tau_L(m \circledast n) = \tau_L(n)$, $\eta_L(m \odot n) = \eta_L(n)$, $\tau_L(m \odot n) = \tau_L(n)$ then, $\forall m, n \in L$, the following hold:*

(1) The accuracy function: $\mathtt{a}(L) \le 2 - \left[(\eta_L(0))^2 + (\tau_L(0))^2\right]$,

(2) The score function: $\mathtt{s}(L) \le 1 - \left[\eta_L(0))^2 + \tau_L(0))^2)\right]$,

(3) The degree of indeterminacy: $\pi_L(m)$ *is such that*

$$\left(\pi_L(m)\right)^2 \ge (\eta_L(0))^2 + (\tau_L(0))^2 - 1.$$

Proof. Let $L^B = (\eta_L, \tau_L)$ be a Pythagorean fuzzy ideal of L;

$\left(\eta_L(m \circledast n)\right)^2 = \left(\eta_L(n)\right)^2$, $\left(\tau_L(m \circledast n)\right)^2 = \left(\tau_L(n)\right)^2$ and

$\left(\eta_L(m \odot n)\right)^2 = \left(\eta_L(n)\right)^2$ $\left(\tau_L(m \odot n)\right)^2 = \left(\tau_L(n)\right)^2$

Since, $\forall m \in L$, $0 = 0 \circledast m = 0 \odot m \Rightarrow \left(\eta_L(0)\right)^2 = \left(\eta_L(0 \circledast m)\right)^2 = \left(\eta_L(0 \odot m)\right)^2 = \left(\eta_L(m)\right)^2$

$\Rightarrow \left(\eta_L(0)\right)^2 = \left(\eta_L(m)\right)^2, \forall m \in L$

$\Rightarrow \left(\eta_L(0)\right)^2 = \left(\eta_L(m)\right)^2 = \left(\eta_L(n)\right)^2 = \left(\eta_L(z)\right)^2, \quad \forall m, n, z \in L$

Similarly, $\left(\tau_L(0)\right)^2 = \left(\tau_L(m)\right)^2 = \left(\tau_L(n)\right)^2 = \left(\tau_L(z)\right)^2, \forall m, n, z \in L$

$But \quad 0 \le (\eta_L(m))^2 + (\tau_L(m))^2 \le 1 \Rightarrow 0 \le (\eta_L(m))^2 + (\tau_L(0))^2 \le 1$

$\Rightarrow (\eta_L(m))^2 \le 1 - (\tau_L(0))^2.$ And,

$0 \le (\eta_L(m))^2 + (\tau_L(m))^2 \le 1 \Rightarrow 0 \le (\eta_L(0))^2 + (\tau_L(m))^2 \le 1$

$\Rightarrow (\tau_L(m))^2 \le 1 - (\eta_L(0))^2$

Then $\forall m \in L$, we have the following:

(1) $The\ accuracy\ function:$

$\mathtt{a}_P(m) = (\eta_L(m))^2 + (\tau_L(m))^2 \le (1 - (\eta_L(0))^2) + (1 - (\tau_L(0))^2) = 2 - (\eta_L(0))^2 - (\tau_L(0))^2$

(2) $The\ score\ function:$

$\mathtt{s}_P(m) = \left(\eta_L(m)\right)^2 - \left(\tau_L(m)\right)^2 \le \left(1 - \left(\eta_L(0)\right)^2\right) - \left(\tau_L(0)\right)^2 = 1 - \left(\eta_L(0)\right)^2 - \left(\tau_L(0)\right)^2$

(3) $The\ degree\ of\ indeterminacy:$

$$\left(\pi_P(m)\right)^2 = 1 - \left(\eta_L(m)\right)^2 - \left(\tau_L(m)\right)^2 = 1 - \mathtt{a}_P(m) \ge 1 - \left[2 - \left(\eta_L(0)\right)^2 - (\tau_L(0))^2)\right]$$

$$= (\eta_L(0))^2 + (\tau_L(0))^2 - 1.$$ □

Lemma 3.28. *In a Pythagorean fuzzy ideal deviation of L, $L^d = \left(\eta^d, \tau^d_L\right)$, we have:*

$\eta^d_L : L \to [0, 0.25]$ (or $\eta^d_L(x) \in [0, 0.25]$) and similarly $\tau^d_L : L \to [0, 0.25]$.

Proof. We prove this following five cases:

Case (i): For $\eta_L(m) = 0 \Rightarrow \eta^d_L(m) = \eta_L(m) - (\eta_L(m))^2 = 0 - 0 = 0 \in [0, 0.25]$.

$\Rightarrow 0 = \eta_L^d(m) \leq 0.25 \Rightarrow \eta_L^d(m) \in [0, 0.25]$

Case (ii): For $\eta_L(m) = 1 \Rightarrow \eta_L^d(m) = \eta_L(m) - (\eta_L(m))^2 = 1 - 1 = 0 \in [0, 0.25]$.

$\Rightarrow 0 = \eta_L^d(m) \in [0, 0.25]$

Case (iii) For $\eta_L(m) = 0.5 = 0.5 \Rightarrow \eta_L^d(m) = \eta_L(m) - (\eta_L(m))^2 = 0.5 - 0.25 = 0.25$

$\Rightarrow 0.25 = \eta_L^d(m) \Rightarrow \eta_L^d(m) \in [0, 0.25]$

Case (iv): For $\eta_L(m) = 0.5 + \varepsilon$, where $\varepsilon \in [0, 0.5]$

$\Rightarrow (\eta_L(m))^2 = 0.5 + \varepsilon)^2 = 0.25 + \varepsilon + \varepsilon^2$

$\Rightarrow \eta_L^d(m) = \eta_L(m) - (\eta_L(m))^2 = 0.5 + \varepsilon) - 0.25 + \varepsilon + \varepsilon^2) = 0.25 - \varepsilon^2 \leq 0.25 = 0.25$

$\Rightarrow 0 \leq \eta_L^d(m) \leq 0.25 \Rightarrow \eta_L^d \in [0, 0.25]$

Case (v): For $\eta_L(m) = 0.5 - \varepsilon$, where $\varepsilon \in [0, 0.5]$

$\Rightarrow (\eta_L(m))^2 = (0.5 - \varepsilon)^2 = 0.25 - \varepsilon + \varepsilon^2$

$\Rightarrow \eta_L^d(m) = \eta_L(m) - (\eta_L(m))^2 = \left(0.5 - \varepsilon\right) - (0.25 - \varepsilon + \varepsilon^2) = 0.25 - \varepsilon^2 \leq 0.25$

$\Rightarrow 0 \leq \eta_L^d(m) \leq 0.25 \Rightarrow \eta_L^d \in [0, 0.25]$

Now, suppose $\eta_L^d > 0.25 = 0.25 \Rightarrow \eta_L^d = \eta_L(m) - \left(\eta_L(m)\right)^2 > 0.25$

$\Rightarrow \; -\left(\eta_L(m)\right)^2 + \eta_L(m) - 0.25 > 0$ and this quadratic inequality has no real solution so that there is no $\eta_L(m)$ satisfying this inequality under the real number system and hence $\eta_L^d \in [0, 0.25]$

Therefore, by the above five cases, we exhaustively shown that $\eta_L^d \in [0, 0.25]$.

Following similar steps for τ_L^d as η_L^d above, we can easily show that $\tau_L^d \in [0, 0.25]$. □

Theorem 3.29. *If* $L = \left(\eta_B, \tau_B\right)$ *is a Pythagorean fuzzy ideal of L then the ordered pairs hereunder are each also Pythagorean fuzzy ideals*

(1) $\left(\eta_L, \overline{\eta}_L\right)$ *(2)* $\left(\overline{\tau}_L, \tau_L\right)$ *(3)* $\left(\overline{\tau}, \ \overline{\eta}_L\right)$

(4) $\left(\eta_L, \overline{\overline{\eta}}_L\right)$ *(5)* $\left(\overline{\overline{\tau}}_L, \tau_L\right)$ *(6)* $\left(\overline{\overline{\tau}}, \ \overline{\eta}_L\right)$

Proof.

(i) $\eta_B, \overline{\tau}_B$ are ideals and $\overline{\eta}_L, \tau_L$ their corresponding complements.

Recall that $(\eta_B, \overline{\eta}_B)$ is Pythagorean fuzzy ideal is previously proved above.

and then $(\overline{\tau}_B, \tau_B)$ is also Pythagorean fuzzy ideal

(ii) By the hypothesis, $L^B = \left(\eta_B, \tau_B\right)$ is Pythagorean fuzzy ideal

Then $\overline{\eta}_{B}, \overline{\tau}_{B}$ are corresponding complements so that $(\overline{\tau}_{B}, \overline{\eta}_{B})$ is Pythagorean fuzzy ideal. For sure, to confirm that, if say (η_{L}, τ_{L}) is Pythagorean fuzzy ideal of L η_{L} and τ_{L} are membership and non-membership functions respectively) where $(\overline{\eta}_{L}, \overline{\tau}_{L})$ is ordered pair of corresponding complements then $(\overline{\tau}_{L}, \overline{\eta}_{L})$ is Pythagorean fuzzy ideal of L.

Suppose (η_{L}, τ_{L}) is Pythagorean fuzzy ideal of L such that $(\overline{\eta}_{L}, \overline{\tau}_{L})$ is ordered pair of corresponding complements

$$\begin{cases} (i)\ \eta_{L}(m) \geq min\{\eta_{L}(m \circledast n),\ \eta_{L}(n)\}, \\ (ii)\ \eta_{L}(m) \geq min\{\eta_{L}(m \odot n),\ \eta_{L}(n \odot m),\ \eta_{L}(n)\} \end{cases}$$

$$\Rightarrow \begin{cases} (i)\ -\eta_{L}(m) \leq -\ min\{\eta_{L}(m \circledast n),\ \eta_{L}(n)\}, \\ (ii)\ -\eta_{L}(m) \leq -\ min\{\eta_{L}(m \odot n),\ \eta_{L}(n \odot m),\ \eta_{L}(n)\} \end{cases}$$

$$\Rightarrow \begin{cases} (i)\ 1-\eta_{L}(m) \leq 1-min\{\eta_{L}(m \circledast n),\ \eta_{L}(n)\}, \\ (ii)\ 1-\eta_{L}(m) \leq 1-min\{\eta_{L}(m \odot n),\ \eta_{L}(n \odot m),\ \eta_{L}(n)\} \end{cases}$$

$$\Rightarrow \begin{cases} (i)\ \overline{\eta}_{L}(m) \leq max\{\overline{\eta}_{L}(m \circledast n),\ \overline{\eta}_{L}(n)\}, \\ (ii)\ \overline{\eta}_{L}(m) \leq max\{\overline{\eta}_{L}(m \odot n),\ \overline{\eta}_{L}(n \odot m),\ \overline{\eta}_{L}(n)\}. \end{cases}$$

That means η_{L} is membership function which guides us to $\overline{\eta}_{L}$ as non-membership function and hence $(\eta_{L}, \overline{\eta}_{L})$ is Pythagorean fuzzy ideal of L.

Now, going back from the last to the first of the preceding steps for the non-membership function τ_{L} to get the membership function $\overline{\tau}_{L}$, we have got $(\overline{\tau}_{L}, \tau_{L})$ is Pythagorean fuzzy ideal of L.

Therefore, $\left(\eta_{L}, \tau_{L}\right)$ is Pythagorean fuzzy ideal of L implies $\left(\overline{\tau}_{L}, \overline{\eta}_{L}\right)$ is also Pythagorean fuzzy ideal of L and hence all the claims in this theorem hold true. □

4 Conclusion

In this article, new results as LBA. defined based on BCL−algebra (not based on BCL^{+}−algebra unlike Liu−algebra which has been defined based on BCL^{+}−algebra), ideal, fuzzy ideal and Pythagorean fuzzy ideal in depth, which have not been introduced so far, are introduced and following all these new introductions, some new results are obtained. We state and prove new properties and theorems (specially as widely as possible for Pythagorean fuzzy ideal of LBA.) which yield new fuzzified results which have not been addressed so far.

This study rigorously establishes Liu^{B}−algebra (LBA.) as a robust framework for handling uncertainty through Pythagorean fuzzy ideals. Key contributions include: Axiomatization of LBA. and its ideals, fuzzification of ideals with Pythagorean membership grades, novel metrics (score/accuracy functions, indeterminacy degrees) for decision−making under vagueness. And the future Work:Extend LBA. to neutrosophic fuzzy ideals for broader uncertainty modeling, develop algorithms for real−world applications (e.g., medical diagnosis in Table 3), investigate topological properties of Pythagorean fuzzy ideals in algebraic structures.

5 Author Contribution

All authors have contributed equally to the completion and success of this manuscript at each step. This article presents groundbreaking advancements in the realm of algebraic structures by introducing and rigorously analyzing Liu$^B-$algebra (LBA.), a novel framework rooted in BCL$-$algebra, distinct from the conventional BCL$^+-$algebra foundation of Liu$-$algebra. Diverging from prior studies, this work pioneers the conceptualization of ideals, fuzzy ideals, and Pythagorean fuzzy ideals within LBA.$-$concepts previously unexplored in this context$-$while establishing their theoretical and practical significance. By anchoring LBA. in BCL$-$algebra, the study circumvents the limitations of BCL$^+-$algebra$-$based systems, thereby enhancing structural flexibility and broadening applicability in uncertainty modeling.

The core contributions are threefold: First, the axiomatization of LBA. and its ideals provides a rigorous mathematical foundation, enabling precise formalization of algebraic properties. Second, the fuzzification of ideals is elevated through Pythagorean fuzzy sets, which extend classical fuzzy logic by incorporating dual membership grades (satisfaction and dissatisfaction degrees), thereby capturing nuanced uncertainties. This innovation is complemented by the introduction of novel metrics$-$including score/accuracy functions and indeterminacy degrees—that empower decision$-$making in ambiguous environments by quantifying and ranking uncertainty. Third, the study proves original theorems that unravel the behavior of Pythagorean fuzzy ideals in LBA., such as their closure properties under algebraic operations and their interaction with lattice structures, yielding fuzzified results previously unaddressed in literature.

Looking ahead, the research outlines transformative future directions: extending LBA. to neutrosophic fuzzy ideals for multi $-$ dimensional uncertainty representation, designing algorithms for real$-$world applications (e.g., medical diagnosis systems leveraging diagnostic criteria), and probing the topological properties of Pythagorean fuzzy ideals to uncover deeper algebraic$-$topological synergies. By bridging abstract algebra and computational intelligence, this work positions LBA. as a versatile tool for modeling vagueness in domains like artificial intelligence, automated decision systems, and data analytics, while inviting interdisciplinary exploration of its untapped potential.

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7 Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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