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On the Abian-Makowski-Wisniewski fixed point Theorem

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Abstract In this brief note, we enhance a fixed point theorem by Abian, Makowski, and Wisniewski by introducing a new statement and presenting a simple proof of the result.

1 Introduction

In 1968, Abian [1] proved that a mapping f from a finite set X into itself has a fixed point if and only if X is not the union of three mutually disjoint sets X_1, X_2, X_3 such that the intersection of each set with its image under f is empty. In 1969, Makowski and Wisniewski [2] demonstrated that the requirement of finiteness in Abian's original theorem is unnecessary. More precisely, they derived the following result, which we label as the Abian-Makowski-Wisniewski Theorem.

Abian-Makowski-Wisniewski Theorem. Let X be a nonempty set and let $f : X \to X$ be an operator. Then the following statements are equivalent:

(i) $F_f := \{x \in X | f(x) = x\} = \emptyset$.

(ii) There exists three mutually disjoint subsets such that:

  a. $X = X_1 \cup X_2 \cup X_3$.

  b. $X_i \cap f(X_i) = \emptyset$, for each $i \in \{1, 2, 3\}$.

The reader can also consult [3]. Our main objective here is to refine this theorem by adding a third statement and presenting a simple proof, which is completely different from the approach used by Makowski and Wisniewski.

2 Main results

As stated in the introduction, we aim to establish the following result.

Theorem 2.1. *Let X be a nonempty set and let $f : X \to X$ be an operator. Then the following statements are equivalent:*

(i) $F_f = \{x \in X | f(x) = x\} = \emptyset$.

(ii) There exists three mutually disjoint subsets such that:

  a. $X = X_1 \cup X_2 \cup X_3$.

  b. $X_i \cap f(X_i) = \emptyset$, *for each* $i \in \{1, 2, 3\}$.

(iii) There exists a function $g : X \to \{1, 2, 3\}$ *such that* $g(f(x)) \neq g(x)$ *for any* x *in* X.

Proof. (i)⇒(ii) Assume (i) and let us introduce the following set:

$$\Gamma := \{(g, A) | g : A \to \{1, 2, 3\}, f(A) \subseteq A,\ g(f(x)) \neq g(x) \forall x \in A\}.$$

It is evident that Γ is nonempty, since the empty function is an element of Γ. We define the following order relation on Γ:

$$\forall (g, A), (h, B) \in \Gamma,\ ((g, A) \leq (h, B) \Leftrightarrow A \subseteq B, g(x) = h(x)\ \forall x \in A).$$

Let $(g_i, A_i)_{i\in I}$ be a chain of elements in Γ. Set $A = \bigcup_{i\in I} A_i$ and $g : A \to \{1, 2, 3\}$ such that $g(x) = g_i(x)$ when $x \in A_i$. Observe that $f(A) = f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i) \subseteq \bigcup_{i\in I} A_i = A$. Furthermore, if $x \in A$, then there exists an $i \in I$ such that $x \in A_i$. Therefore, $g(f(x)) = g_i(f(x)) \neq g_i(x) = g(x)$. This reveals that $(g, A) \in \Gamma$ and, clearly, (g, A) is an upper bound for the chain $(g_i, A_i)_{i\in I}$. According to Zorn's Lemma, it follows that Γ has a maximal element (g_0, A_0). To conclude the proof, we just need to demonstrate that $A_0 = X$. Assume the contrary and let us take an element $a \in X \setminus A_0$. Inductively, we can define a function $h : A_0 \cup \{a, f(a), f^2(a), \cdots\} \to \{1, 2, 3\}$ that extends g_0 and continues to satisfy the conditions outlined earlier. For each nonnegative integer n, we assign $h(f^n(x))$ a value from $\{1, 2, 3\}$, ensuring that $h(f^n(x)) \neq h(f^{n-1}(x))$ and $h(f^n(x)) \neq h(f^{n+1}(x))$ if those values have already been assigned. The maximality of (g_0, A_0) implies that $h = g_0$ and $A_0 = A_0 \cup \{a, f(a), f^2(a), \cdots\}$. Therefore, $a \in A_0$, leading to the desired contradiction.
(iii)⇒(ii) Define X_i as the preimage of $\{i\}$ under the function g for $i \in \{1, 2, 3\}$. It follows that $X = X_1 \cup X_2 \cup X_3$ with $X_i \cap X_j = \emptyset$ for $i \neq j$. Let us suppose that $X_i \cap f(X_i) \neq \emptyset$ for some i. This means there exists an element $x \in X_i$ such that $f(x) \in X_i$. Thus, $g(x) = i = g(f(x))$ which contradicts our assumption.
(ii)⇒(i) Assume, for the sake of contradiction, that f has a fixed point x. Then $x \in X_i$ for some i, and since $x = f(x)$, it follows that $x \in X_i \cap f(X_i)$, providing the desired contradiction. □

References

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