**Palestine Journal of Mathematics**

**Vol 14(2)(2025)** , 684–703 

# EXISTENCE OF POSITIVE SOLUTIONS FOR A DISCRETE EIGENVALUE PROBLEM INVOLVING THE $\phi$-LAPLACIAN

L. Bouchal, K. Mebarki, S. Zahar and A. Kheloufi

Communicated by Martin Bohner

MSC 2010 Classifications: Primary 39A70; Secondary 47H10.

Keywords and phrases: Fixed point theorem, $\phi$-Laplacian operator, difference equation, boundary value problem, existence.

*The authors acknowledge the support of "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria.*

**Corresponding Author: L. Bouchal**

**Abstract** This work aims to study a discrete $\phi$-Laplacian eigenvalue boundary value problem at four points. Our results are based on the fixed point theory for the sum of two operators defined on cones of Banach spaces. We first establish the existence of at least one nontrivial nonnegative solution under a general polynomial growth condition on the nonlinearity. Then, under additional assumptions, specifically the super-linear and sub-linear cases of the nonlinearity, we establish the existence of at least one positive solution and the existence of two positive solutions. To support our theoretical results, we provide numerical examples at the end of the paper.

## 1 Introduction

In this work, we investigate the following discrete four point boundary value problem

$$\begin{aligned} \Delta(\phi(\Delta u(k-1))) + \lambda g(k) f(k, u(k)) &= 0, \quad k \in \{1, 2, \dots, T\} \\ \Delta u(0) - \alpha u(l_1) &= 0, \\ \Delta u(T) + \beta u(l_2) &= 0, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $l_1, l_2 \in \{1, 2, \dots, T\}$ with $T > 1$ is an integer, $l_1 < l_2$, $\alpha, \beta > 0$ and $\alpha l_1 \leq 1$, $\beta(T + 1 - l_2) \leq 1$. Throughout this paper, $\mathbb{T} = \{0, 1, \dots, T + 1\}$ denotes the set of integers in the interval $[0, T+1]$ and by a positive solution of problem (1.1) we mean a sequence $\{u(0), u(1), ..., u(T + 1)\}$ which satisfies (1.1) with $u(k) > 0$ on $\mathbb{T}$. Here $\Delta$ is the forward difference operator defined by $\Delta u(k) = u(k + 1) - u(k)$ for $k \in \{1, 2, \dots, T\}$. The nonlinear operator of derivation $\phi$: $\mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism such that $\phi(0) = 0$ and satisfies these two conditions:

$(A_1)$ there exists an increasing homeomorphism $\psi$ of $(0, \infty)$ onto $(0, \infty)$ such that

$$\phi^{-1}(\alpha\,\beta) \leq \psi^{-1}(\alpha)\,\phi^{-1}(\beta), \quad \text{for all } \alpha, \beta \in (0, \infty), \tag{1.2}$$

and

$(A_2)$ there exists an increasing homeomorphism $\chi$ of $(0, \infty)$ onto $(0, \infty)$ such that

$$\phi^{-1}(\alpha\,\beta) \geq \chi^{-1}(\alpha)\,\phi^{-1}(\beta), \quad \text{for all } \alpha, \beta \in (0, \infty). \tag{1.3}$$

Note that the conditions $(A_1)$ and $(A_2)$ hold true when there exist increasing homeomorphisms $\psi$ and $\chi$ of $(0, \infty)$ onto $(0, \infty)$ such that

$$\psi(\alpha)\,\phi(\beta) \leq \phi(\alpha\,\beta) \text{ and } \phi(\alpha\,\beta) \leq \chi(\alpha)\,\phi(\beta), \quad \text{for all } \alpha, \beta \in (0, \infty),$$

respectively. Such $\psi$ and $\chi$ exist in the particular case where both are equal to $\phi$.

**Example 1.1.** Let $p$ be an integer such that $p > 0$ and odd, if

$$\phi(x) = \begin{cases} x^p, & x \geq 0 \\ -(-x)^p, & x < 0 \end{cases}$$

we can easily verify that $\phi$ is an odd increasing homeomorphism with

$$\phi^{-1}(x) = \begin{cases} x^{\frac{1}{p}}, & x \geq 0 \\ -(-x)^{\frac{1}{p}}, & x < 0 \end{cases}$$

and for $\chi(x) = (3x)^p$, $\psi(x) = (\frac{x}{3})^p$, we verify that $\chi, \psi$ are increasing homeomorphism on $(0, \infty)$ onto $(0, \infty)$, with $\chi^{-1}(x) = \frac{1}{3}\, x^{\frac{1}{p}}$ and $\psi^{-1}(x) = 3\, x^{\frac{1}{p}}$.

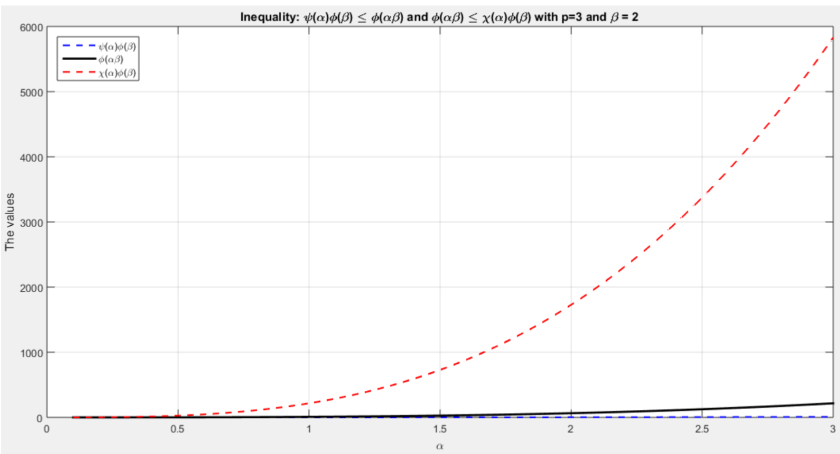


**Figure 1.** Graphical representation of the inequalities $\psi(\alpha)\,\phi(\beta) \leq \phi(\alpha\,\beta)$ and $\phi(\alpha\,\beta) \leq \chi(\alpha)\,\phi(\beta)$, for all $\alpha, \beta \in (0, \infty)$

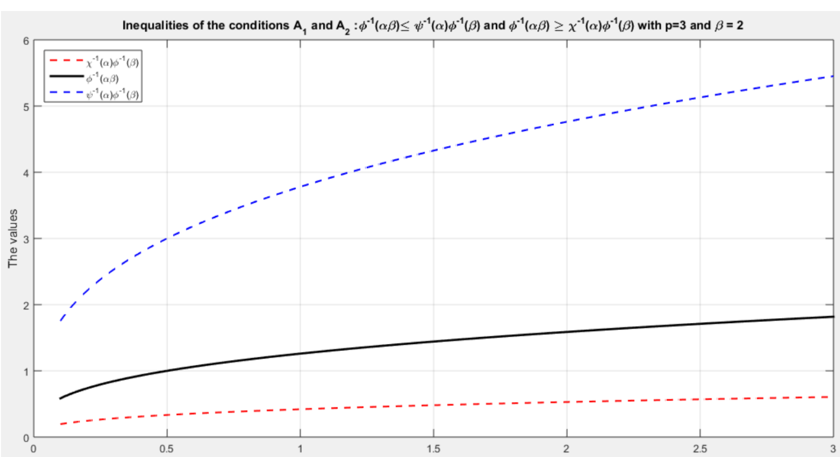


**Figure 2.** Graphical representation of conditions $(A_1)$ and $(A_2)$

Obviously, $\phi$ is an extension of the usual multiplicative $p$-Laplacian nonlinear operator $\phi_p(s) = |s|^{p-2}s$ for $p > 1$. In the case when $\phi(u) = u$, the problem (1.1) represents the classical second-order difference boundary value problem at four points.

Note that the special case of the $\phi$-Laplacian boundary value problem (1.1) has been studied in [25] for $\phi(s) = \phi_p(s)$ and the nonlinearity $f$ is autonomous, where the authors discussed the existence of positive solutions by imposing restrictive conditions on the parameters in the boundary conditions of the problem (1.1) to ensure the nonnegativity of solutions even when both $f$

and $g$ are nonnegative; they also discussed cases where solutions fail to exist. In [5], the authors provide sufficient conditions to guarantee that there is at least one homoclinic solution for the following nonlinear second-order difference equation with p-Laplacian by using variational methods.

$$\begin{aligned} \Delta(a(k)\phi_p(\Delta(k-1))) + b(k)\phi_p(u(k)) =& f(u(k)), \quad k \in \mathbb{Z}, \\ u(k) \to 0, \ \ \text{as } |k| \to \infty \end{aligned} \tag{1.4}$$

with $a, b : \mathbb{Z} \to (0, +\infty)$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
In [3], the author used the lower and upper solutions method combined with fixed point index theory to study the existence, nonexistence, and multiplicity of positive solutions for the following discrete Dirichlet $\phi$-Laplacian eigenvalue problem

$$\begin{aligned} \Delta(\phi(\Delta(k-1))) + \lambda g(k) f(u(k)) =& 0, \quad k \in \{1, 2, \ldots, T\}, \\ u(0) = u(T+1) =& 0, \end{aligned} \tag{1.5}$$

where $\lambda$ is a positive parameter, $g : [1, T]_{\mathbb{Z}} \to (0, \infty)$ and $f : \mathbb{R}^+ \to (0, \infty)$ is continuous, with $\phi : \mathbb{R} \to \mathbb{R}$ is an odd and strictly increasing homeomorphism and $\lim\limits_{u\to\infty} \frac{f(u)}{\phi(u)} = \infty$.

In [4], Bai and Xu established the existence of three positive solutions for the problem (1.5), under some suitable assumptions imposed on the nonlinearity $f$ and the positive parameter $\lambda$ belonging to an explicit open interval. The multiplicity result is based on the Brouwer degree theory and the method of lower and upper solutions.
In [26], Zhou and Ling obtained some sufficient conditions on the existence of infinitely many positive solutions for the following second order $\phi_c$-Laplacian difference equation; the approach used is critical point theory.

$$\begin{aligned} -\Delta(\phi_c(\Delta(k-1))) =& \lambda f(k, u(k)), \quad k \in \{1, 2, \ldots, T\}, \\ u(0) = u(T+1) =& 0, \end{aligned} \tag{1.6}$$

where $\phi_c$ is a special $\phi$-Laplacian operator defined by $\phi_c = \frac{s}{\sqrt{1+s^2}}$.
In [8], the authors established the existence of two positive solutions for the following nonlinear Robin problem involving the discrete $p$-Laplacian using variational methods and truncation techniques.

$$-\Delta(\phi_p(\Delta u(k-1))) + g(k)\phi_p(u(k)) = \lambda f_k(u(k)), \quad k \in \{1, \ldots, T\} \tag{1.7}$$

$$u(0) = \Delta u(T) = 0, \tag{1.8}$$

where $\lambda$ is a positive parameter, $g : [1, T] \to [0, +\infty)$ and $f_k : \mathbb{R} \to \mathbb{R}$ are continuous mappings.

Difference equations have been extensively applied across various fields of applied sciences [13, 18]. In recent years, increasing attention has been given to the existence and multiplicity of positive solutions for boundary value problems involving the $\phi$-Laplacian and its special cases. To the best of our knowledge, no existing work has addressed on discrete four-point boundary value problems involving the generalized $\phi$-Laplacian operator. For recent developments in discrete eigenvalue problems, relevant studies can be found in the context of partial difference equations [12, 19, 21, 22, 23, 24], where the principal method employed is critical point theory. In the case of fractional difference equations [14, 15, 20], fixed point theorems and variational methods have been the main used methods. The fixed point approach was chosen for its adaptability to the problem's nonlinear structure.

In this work, we assume that:

$(\mathcal{H}_1)$ $g : \mathbb{T} \to (0, \infty)$ is a function such that $\sum\limits_{k=1}^{T} g(k) < \infty$ and $f : \mathbb{T} \times [0, \infty) \to \mathbb{R}$ is a continuous function such that $f(\bar{k}, 0) \neq 0$ for some $\bar{k} \in \mathbb{T}$, satisfying the growth condition

$$0 \leq f(k, u(k)) \leq a(k) + b(k)|u(k)|^p,$$

where $a, b \in \mathcal{C}(\mathbb{T}, \mathbb{R})$ are positive functions, $p$ nonnegative constant and $a(k), b(k) \leq M$ for some positive constant $M$.

$(\mathcal{H}_2)$ There exists a constant $R > 0$ such that

$$\mathcal{L}\,\psi^{-1}(\lambda)\phi^{-1}\left((M + MR^p)\sum_{i=1}^{T} g(i)\right) < R,$$

where $\mathcal{L} = \max\left(\frac{1}{\alpha} + T + 1, \frac{1}{\beta} + T + 1\right)$.

To ensure that $(\mathcal{H}_2)$ is satisfied, it suffices to show that

$$\sup_{z>0} \frac{z}{\mathcal{L}\,\psi^{-1}(\lambda)\phi^{-1}\left((M + Mz^p)\sum\limits_{i=1}^{T} g(i)\right)} > 1.$$

This work is devoted to discussing the existence and the multiplicity of positive solutions to the problem (1.1). We will make use of a recent fixed point theory for the sum of two operators on a suitable cone in some Banach space. In addition, two examples are given to illustrate some existence results. The paper is organized as follows: In the next section, we give some preliminary results we need in this paper. In section 3, we provide some auxiliary results and necessary lemmas. In section 4, we present our main existence results. In the last section, we give numerical examples to support our theoretical results.

## 2 Preliminaries

**Definition 2.1.** Let $(X, d)$ be a metric space and $D$ a subset of $X$. A mapping $T : D \to X$ is said to be $h$-expansive if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h\, d(x, y), \quad \forall\, x, y \in D.$$

Let $E$ be a real Banach space.

**Definition 2.2.** The mapping $K : E \to K$ is said to be completely continuous if it is continuous and it maps any bounded set into a relatively compact set.

**Lemma 2.3.** *(Discrete Ascoli-Arzelà Theorem, see [1, Theorem 17.1]). Let $C$ be a closed subset of the class of continuous maps $u : \mathbb{T} \to E$. If $C$ is uniformly bounded and the set $\{u(k) : u \in C\}$ is relatively compact for each $k \in \mathbb{T}$, then $C$ is compact.*

**Remark 2.4.** Recall that a map $f : \mathbb{T} \times (0, \infty) \to (0, \infty)$ is continuous if it is continuous as a map of the topological space $\mathbb{T} \times (0, \infty)$ into the topological space $(0, \infty)$. Throughout this paper the topology on $\mathbb{T}$ will be the discrete topology.

**Definition 2.5.** A closed, convex set $\mathcal{P}$ of $E$ is said to be cone if

(i) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,

(ii) $x, -x \in \mathcal{P}$ implies $x = 0$.

For two numbers $0 < r < R$, we set

$$\begin{aligned}\overline{\mathcal{P}}_R &= \{x \in \mathcal{P} : \|x\| \leq R\},\\ \mathcal{P}_{r,R} &= \{x \in \mathcal{P} : r < \|x\| < R\}.\end{aligned}$$

The following theorems will be utilized to demonstrate the existence of at least one nonnegative solution, one nonzero nonnegative solution as well as multiple nonnegative solutions to Problem (1.1). The proofs are based on the fixed point index theory on retracts of Banach spaces for the sum of two operators. This theory has proven to be a powerful tool in investigating the existence, positivity, multiplicity and localization of various boundary value problems, including those arising in differential equations, difference equations, dynamic equations on time scales; for some works in this direction, see [6, 7, 9, 10, 17]. For more details on this theory, readers are referred to [11] and [16] and references therein.

**Theorem 2.6.** *[16, Proposition 2.2.55] Let $U$ be a bounded open subset of a cone $\mathcal{P}$ and $\Omega$ a subset of $\mathcal{P}$. Assume that $T:\Omega\to E$ is an $h$-expansive mapping, $S:\overline{U}\to E$ is a completely continuous mapping and $tS(\overline{U})\subset(I-T)(\Omega)$ for all $t\in[0,1]$. If $(I-T)^{-1}0\in U$, and*

$$(I-T)x\neq\mu Sx \text{ for all } x\in\partial U\cap\Omega \text{ and } 0\leq\mu\leq 1, \tag{2.1}$$

*then $i_*(T+S,U\cap\Omega,\mathcal{P})=1$. Therefore, there exists $x^*\in U\cap\Omega$ such that*

$$Tx^*+Sx^*=x^*.$$

**Theorem 2.7.** *[16, Theorem 3.1.5]. Let $\Omega$ be a subset of a cone $\mathcal{P}$ with $0\in\Omega$; $\alpha,\beta>0$, $\alpha\neq\beta$; $r=\min(\alpha,\beta)$ and $R=\max(\alpha,\beta)$ such that $\mathcal{P}_{r,R}\cap\Omega\neq\emptyset$. Assume that $T:\Omega\to E$ is an $h$-expansive mapping and $S:\overline{\mathcal{P}}_R\to E$ is a completely continuous mapping. Suppose that $\|T0\|<(h-1)\beta$,*

$$tS(\overline{\mathcal{P}}_R)\subset(I-T)(\Omega) \text{ for all } t\in[0,1], \tag{2.2}$$

*and that there exists $u_0\in\mathcal{P}\backslash\{0\}$ such that the following conditions are satisfied:*

$$\begin{aligned} Sx &\neq (I-T)(x-\mu u_0) \quad \text{for all } \mu\geq 0,\ x\in\partial\mathcal{P}_\alpha\cap(\Omega+\mu u_0),\\ Sx &\neq \lambda(x-Tx) \quad \text{for all } \lambda\geq 1,\ x\in\partial\mathcal{P}_\beta. \end{aligned}$$

*Then $T+S$ has a fixed point $x\in\mathcal{P}_{r,R}\cap\Omega$.*

## 3 Auxiliary results

We consider the Banach space $E=\{y:\ y:\mathbb{T}\to\mathbb{R}\}$, endowed with the norm $\|y\|=\max_{k\in\mathbb{T}}|y(k)|$. Define

$$\begin{aligned} \mathcal{K}&=\{u\in E:\ u(k)\geq 0,\ \ k\in\mathbb{T}\},\\ \mathcal{P}&=\left\{u\in\mathcal{K}:\ \Delta^2u(k-1)\leq 0,\ \ k\in\{1,\dots,T\}\right\}. \end{aligned}$$

**Remark 3.1.** Let $\ell\in\{1,\dots,[\frac{T+1}{2}]\}$, where $[x]$ is the greatest integer less than $x$. For $u\in\mathcal{P}$, from [25, Lemma 2.4], we have

$$u(k)\geq\min\left(\frac{k}{T+1},1-\frac{k}{T+1}\right)\|u\|, \qquad \text{for } k\in\mathbb{T}.$$

In particular, we get

$$\min_{k\in\{\ell,T+1-\ell\}}u(k)\geq\frac{\ell}{T+1}\|u\|.$$

**Lemma 3.2.** *For each fixed $y\in\mathcal{K}$, the linear boundary value problem*

$$\Delta(\phi(\Delta u(k-1)))+\lambda g(k)f(k,y(k))=0, \quad k\in\{1,2,\dots,T\} \tag{3.1}$$

*subject to the boundary conditions*

$$\Delta u(0)-\alpha u(l_1)=0,\ \Delta u(T)+\beta u(l_2)\ \ =0, \tag{3.2}$$

*has a unique solution $u\in\mathcal{K}$. Furthermore, the solution $u$ can be expressed as:*

$$u(k)=u(0)+\sum_{s=1}^{k}\phi^{-1}\left(\phi(\mathcal{A}_y)-\sum_{i=1}^{s-1}\lambda g(i)f(i,y(i))\right),$$

*or*

$$u(k)=u(T)-\sum_{s=k+1}^{T}\phi^{-1}\left(\phi(\mathcal{B}_y)+\sum_{i=s}^{T}\lambda g(i)f(i,y(i))\right).$$

*Here, $\mathcal{A}_y$ is the unique constant determined by $\mathcal{A}_y = \phi^{-1}\left(\sum_{i=1}^{k_0} \lambda g(i) f(i, y(i))\right)$, for some $k_0 \in \mathbb{T}$, which satisfies*

$$\beta \mathcal{A}_y + \alpha\phi^{-1}\left(\phi(\mathcal{A}_y) - \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) + \alpha\beta \sum_{s=l_1+1}^{l_2} \phi^{-1}\left(\phi(\mathcal{A}_y) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right) = 0.$$

*Similarly, $\mathcal{B}_y$ is the unique constant determined by $\mathcal{B}_y = -\phi^{-1}\left(\sum_{i=k_1+1}^{T} \lambda g(i) f(i, y(i))\right)$, for some $k_1 \in \mathbb{T}$, which satisfies*

$$\alpha \mathcal{B}_y + \beta\phi^{-1}\left(\phi(\mathcal{B}_y) - \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) + \alpha\beta \sum_{s=l_1+1}^{l_2} \phi^{-1}\left(\phi(\mathcal{B}_y) + \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right) = 0.$$

*Moreover, the integers $k_0$ and $k_1$ are equal.*

***Proof.*** On one side, we sum (3.1) from 1 to $s-1$, one gets

$$\Delta u(s) = \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right).$$

Again summing (3) from 1 to $k$, it follows that

$$u(k) = u(0) + \sum_{s=1}^{k} \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right). \tag{3.3}$$

Considering $\Delta u(0) - \alpha u(l_1) = 0$, we obtain

$$u(0) = \frac{1}{\alpha}\Delta u(0) - \sum_{s=1}^{l_1} \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right).$$

Let $\Delta u(0) = \mathcal{A}_y$, where $\mathcal{A}_y$ satisfies $\Delta u(T) + \beta u(l_2) = 0$, i.e.,

$$\begin{aligned} & \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) \\ + \; & \beta\Bigg[\frac{1}{\alpha}\Delta u(0) - \sum_{s=1}^{l_1} \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right) \\ + \; & \sum_{i=1}^{l_2} \phi^{-1}\left(\phi(\Delta u(0)) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right)\Bigg] = 0. \end{aligned}$$

Next, define

$$G(c) = \beta c + \alpha\phi^{-1}\left(\phi(c) - \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) + \alpha\beta \sum_{s=l_1+1}^{l_2} \phi^{-1}\left(\phi(c) - \sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right),$$

then $G : \mathbb{R} \to \mathbb{R}$ is a continuous and increasing function satisfying

$$G(0) < 0 \quad \text{and} \quad G\left(\phi^{-1}\left(\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right)\right) > 0.$$

By the Intermediate Value Theorem (IVT), it follows that there exists a unique

$$\mathcal{A}_y \in \left(0, \phi^{-1}\left(\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right)\right)$$

such that $G(\mathcal{A}_y)=0$. This implies by applying the IVT again to the continuous and increasing function $k \mapsto \phi^{-1}\left(\sum_{i=1}^{k} \lambda g(i) f(i, y(i))\right)$, that there exists a unique $k_0=k_{0_y} \in\{0, \ldots, T\}$ such that

$$\mathcal{A}_y=\phi^{-1}\left(\sum_{i=1}^{k_0} \lambda g(i) f(i, y(i))\right).$$

On the other side, we sum (3.1) from $s$ to $T$, one gets

$$\Delta u(s-1)=\phi^{-1}\left(\phi(\Delta u(T))-\sum_{i=s}^{T} \lambda g(i) f(i, y(i))\right). \tag{3.4}$$

Again summing (3.4) from $k+1$ to $T$, it follows that

$$u(k)=u(T)-\sum_{s=k+1}^{T} \phi^{-1}\left(\phi(\Delta u(T))+\sum_{i=s}^{T} \lambda g(i) f(i, y(i))\right). \tag{3.5}$$

Considering $\Delta u(T)+\beta u(l_2)=0$, we obtain

$$u(T)=-\frac{1}{\beta} \Delta u(T)+\sum_{s=l_2}^{T} \phi^{-1}\left(\phi(\Delta u(T))+\sum_{i=s}^{T} \lambda g(i) f(i, y(i))\right).$$

Let $\Delta u(T)=\mathcal{B}_y$, where $\mathcal{B}_y$ satisfies $\Delta u(0)+\alpha u(l_1)=0$, i.e.,

$$\begin{aligned}
& \phi^{-1}\left(\phi(\Delta u(T))+\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) \\
- & \alpha\Big[-\tfrac{1}{\beta} \Delta u(T)+\sum_{s=l_2+1}^{T} \phi^{-1}\left(\phi(\Delta u(T))+\sum_{i=s}^{T} \lambda g(i) f(i, y(i))\right) \\
- & \sum_{i=l_1+1}^{T} \phi^{-1}\left(\phi(\Delta u(T))+\sum_{i=s}^{T} \lambda g(i) f(i, y(i))\right)\Big]=0
\end{aligned}$$

Now, define

$$H(c)=\alpha c+\beta \phi^{-1}\left(\phi(c)-\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right)+\alpha \beta \sum_{s=l_1+1}^{l_2} \phi^{-1}\left(\phi(c)+\sum_{i=1}^{s-1} \lambda g(i) f(i, y(i))\right)$$

then, $H: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and increasing function satisfying

$$H(0)>0 \quad \text{and} \quad H\left(-\phi^{-1}\left(\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right)\right)<0.$$

By the IVT, it follows that there exists a unique

$$\mathcal{B}_y \in\left(-\phi^{-1}\left(\sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right), 0\right)$$

such that $H(\mathcal{B}_y)=0$. This implies by applying the IVT again to the continuous and increasing function $k \mapsto-\phi^{-1}\left(\sum_{i=k+1}^{T} \lambda g(i) f(i, y(i))\right)$, that there exists a unique $k_1=k_{1_y} \in\{0, \ldots, T\}$ such that

$$\mathcal{B}_y=-\phi^{-1}\left(\sum_{i=k_1+1}^{T} \lambda g(i) f(i, y(i))\right).$$

Finally, we show that $k_0=k_1$. We have,

$$\Delta u(0)=\mathcal{A}_y=\phi^{-1}\left(\sum_{i=1}^{k_0} \lambda g(i) f(i, y(i))\right).$$

From (3.5), we find

$$\Delta u(k) = \phi^{-1}\left(\phi(\Delta u(T)) + \sum_{i=k+1}^{T} \lambda g(i) f(i, y(i))\right).$$

$$\Delta u(k) = \phi^{-1}\left(\phi(\mathcal{B}_y) + \sum_{i=k+1}^{T} \lambda g(i) f(i, y(i))\right),$$

then,

$$\begin{aligned}\Delta u(0) =& \phi^{-1}\left(\phi(\mathcal{B}_y) + \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) \\ =& \phi^{-1}\left(-\sum_{i=k_1+1}^{T} \lambda g(i) f(i, y(i)) + \sum_{i=1}^{T} \lambda g(i) f(i, y(i))\right) \\ =& \phi^{-1}\left(\sum_{i=1}^{k_1} \lambda g(i) f(i, y(i))\right).\end{aligned}$$

Then, $k_0 = k_1$. □

For each $u \in \mathcal{K}$, define an operator $F : \mathcal{K} \to E$ by

$$Fu(k) = \begin{cases} \frac{1}{\alpha}\phi^{-1}\left(\lambda \sum\limits_{i=1}^{k_0} g(i) f(i, u(i))\right) - \sum\limits_{s=1}^{l_1} \phi^{-1}\left(\lambda \sum\limits_{i=s}^{k_0} g(i) f(i, u(i))\right) \\ + \sum\limits_{s=1}^{k} \phi^{-1}\left(\lambda \sum\limits_{i=s}^{k_0} g(i) f(i, u(i))\right), & k \le k_0, \\ \\ \frac{1}{\beta}\phi^{-1}\left(\lambda \sum\limits_{i=k_0+1}^{T} g(i) f(i, u(i))\right) - \sum\limits_{s=l_2+1}^{T+1} \phi^{-1}\left(\lambda \sum\limits_{i=k_0+1}^{s-1} g(i) f(i, u(i))\right) \\ + \sum\limits_{s=k+1}^{T+1} \phi^{-1}\left(\lambda \sum\limits_{i=k_0+1}^{s-1} g(i) f(i, u(i))\right), & k \ge k_0 + 1, \end{cases}$$

where $k_0 \in \mathbb{T}$ is an integer corresponding to $u$ satisfying

$$\Delta u(0) = \phi^{-1}\left(\sum_{i=1}^{k_0} \lambda g(i) f(i, u(i))\right)$$

and

$$\Delta u(T) = -\phi^{-1}\left(\sum_{i=k_0+1}^{T} \lambda g(i) f(i, u(i))\right).$$

Due to the uniqueness of $\mathcal{A}_y$ and $\mathcal{B}_y$, it follows that the operator $F$ is well-defined.

**Remark 3.3.** From Lemma 3.2, we can easily deduce that any fixed point of the operator $F$ on $\mathcal{K}$ is a positive solution to the boundary value problem (1.1), and conversely. Specifically, for each $u \in \mathcal{K}$, we have

$$\Delta Fu(k) = \begin{cases} \phi^{-1}\left(\lambda \sum\limits_{i=k+1}^{k_0} g(i) f(i, u(i))\right), & k \le k_0, \\ \\ -\phi^{-1}\left(\lambda \sum\limits_{i=k_0+1}^{k} g(i) f(i, u(i))\right), & k \ge k_0 + 1. \end{cases} \tag{3.6}$$

Hence, we obtain the boundary conditions:

$$\begin{aligned}\Delta Fu(0)-\alpha Fu(l_1)=&\phi^{-1}\left(\lambda\sum_{i=1}^{k_0}g(i)f(i,u(i))\right)-\alpha\bigg(\frac{1}{\alpha}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0}g(i)f(i,u(i))\right)\\&-\sum_{s=1}^{l_1}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0}g(i)f(i,u(i))\right)+\sum_{s=1}^{l_1}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0}g(i)f(i,u(i))\right)\bigg)\\=&0,\end{aligned}$$

and

$$\begin{aligned}\Delta Fu(T)+\beta Fu(l_2)=&-\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{T}g(i)f(i,u(i))\right)+\beta\bigg(\frac{1}{\beta}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{T}g(i)f(i,u(i))\right)\\&-\sum_{s=l_2+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1}g(i)f(i,u(i))\right)+\sum_{s=l_2+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1}g(i)f(i,u(i))\right)\bigg)\\=&0.\end{aligned}$$

### 3.1 Properties of the operator $F$

**Lemma 3.4.** *Assume that $(\mathcal{H}_1)$ holds. Then the operator $F$ maps $\mathcal{K}$ into $\mathcal{P}$.*

***Proof*.** Let $u\in\mathcal{K}$. (i) Firstly, we show that $Fu(k)\geq 0$ for $k\in\mathbb{T}$. We have

$$\Delta Fu(0)>0,\;\;\Delta Fu(T)<0,\;\;Fu(l_1)>0\;\;Fu(l_2)>0,$$

and

$$Fu(l_1)-Fu(0)=\sum_{i=0}^{l_1-1}\Delta Fu(i)\leq\Delta Fu(0)l_1=\alpha Fu(l_1)l_1\leq Fu(l_1).$$

So, $Fu(0)\geq 0$. Similarly,

$$Fu(T+1)-Fu(l_2)=\sum_{i=l_2}^{T}\Delta Fu(i)\geq-\beta(T+1-l_2)\,Fu(l_2)\geq -Fu(l_2).$$

So, $Fu(T+1)\geq 0$.

Now, let $k_0\in[l_1,l_2]$ be fixed; otherwise, the negative terms in the expression of $F$ are equal to zero.

- For $0<k\leq l_1$, we have

$$Fu(k)-Fu(0)=\sum_{s=1}^{k}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0}g(i)f(i,u(i))\right)\geq 0$$

  thus $Fu(k)\geq Fu(0)$ and since $Fu(0)\geq 0$, we get $Fu(k)\geq 0$.

- For $l_1<k\leq k_0$, we have

$$\sum_{i=1}^{k_0}g(i)f(i,u(i))\geq\sum_{i=s}^{k_0}g(i)f(i,u(i)),$$

  thus

$$-\sum_{s=1}^{l_1}\phi^{-1}\left(\sum_{i=1}^{k_0}g(i)f(i,u(i))\right)\leq-\sum_{s=1}^{l_1}\phi^{-1}\left(\sum_{i=s}^{k_0}g(i)f(i,u(i))\right).$$

Then,

$$\begin{aligned}
Fu(k) =&\frac{1}{\alpha}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right) - \sum_{s=1}^{l_1}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0} g(i)f(i,u(i))\right)\\
&+\sum_{s=1}^{k}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0} g(i)f(i,u(i))\right)\\
\geq&\frac{1}{\alpha}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right) - \sum_{s=1}^{l_1}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right)\\
&+\sum_{s=1}^{k}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0} g(i)f(i,u(i))\right)\\
=&\frac{1}{\alpha}\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right) - l_1\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right)\\
&+\sum_{s=1}^{k}\phi^{-1}\left(\lambda\sum_{i=s}^{k_0} g(i)f(i,u(i))\right)\\
\geq&(\frac{1}{\alpha}-l_1)\phi^{-1}\left(\lambda\sum_{i=1}^{k_0} g(i)f(i,u(i))\right)\\
\geq&0.
\end{aligned}$$

- For $k_0 \leq k \leq l_2$, we have

$$\begin{aligned}
Fu(l_2)-Fu(k) =& \sum_{s=l_2+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right) - \sum_{s=k+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right)\\
=&-\sum_{s=k+1}^{l_2}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right)\\
\leq&0,
\end{aligned}$$

thus $Fu(l_2) \leq Fu(k)$ and since $Fu(l_2) \geq 0$, we get $Fu(k) \geq 0$.

- For $l_2 < k \leq T$, we have

$$\phi^{-1}\left(-\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right) \geq \phi^{-1}\left(-\lambda\sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right).$$

Hence

$$\begin{aligned}
Fu(k) =&\frac{1}{\beta}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right) - \sum_{s=l_2+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right)\\
&+\sum_{s=k+1}^{T+1}\phi^{-1}\left(\lambda\sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\beta}\phi^{-1}\left(\lambda \sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right) - (T+1-l_2)\phi^{-1}\left(\lambda \sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right) \\
&\quad + \sum_{s=k+1}^{T+1} \phi^{-1}\left(\lambda \sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right) \\
&=(\frac{1}{\beta} - T - 1 + l_2)\phi^{-1}\left(\lambda \sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right) + \sum_{s=k+1}^{T+1} \phi^{-1}\left(\lambda \sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right) \\
&\geq 0.
\end{aligned}
$$

(ii) Show that $\Delta^2 Fu(k-1) \leq 0$ for $k \in \{1, ..., T\}$. From (3.6), we have

$$
\Delta^2 Fu(k-1) = \begin{cases} \phi^{-1}\left(\lambda \sum_{i=k+1}^{k_0} g(i)f(i,u(i))\right) - \phi^{-1}\left(\lambda \sum_{i=k}^{k_0} g(i)f(i,u(i))\right), & k-1 \leq k_0, \\ -\phi^{-1}\left(\lambda \sum_{i=k_0+1}^{k} g(i)f(i,u(i))\right) + \phi^{-1}\left(\lambda \sum_{i=k_0+1}^{k-1} g(i)f(i,u(i))\right), & k-1 \geq k_0+1. \end{cases}
$$

The conclusion stems from the fact that $\phi^{-1}$ is an increasing function. □

**Lemma 3.5.** *Suppose that $(\mathcal{H}_1)$ holds and $B > 0$. If $u \in \mathcal{P}$ with $\|u\| \leq B$, then*

$$
\|Fu\| \leq \mathcal{L}\psi^{-1}(\lambda)\phi^{-1}\left((M + MB^p)\sum_{i=1}^{T} g(i)\right).
$$

***Proof*.**

$$
\begin{aligned}
\|Fu\| &= \max\{Fu(k_0), Fu(k_0+1)\} \\
&\leq \max\left\{\frac{1}{\alpha}\phi^{-1}\left(\lambda \sum_{i=1}^{k_0} g(i)f(i,u(i))\right) + \sum_{s=1}^{k_0} \phi^{-1}\left(\lambda \sum_{i=s}^{k_0} g(i)f(i,u(i))\right),\right. \\
&\qquad \left.\frac{1}{\beta}\phi^{-1}\left(\lambda \sum_{i=k_0+1}^{T} g(i)f(i,u(i))\right) + \sum_{s=k_0+2}^{T+1} \phi^{-1}\left(\lambda \sum_{i=k_0+1}^{s-1} g(i)f(i,u(i))\right)\right\} \\
&\leq \max\left\{\frac{1}{\alpha}\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right) + \sum_{s=1}^{T} \phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right),\right. \\
&\qquad \left.\frac{1}{\beta}\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right) + \sum_{s=1}^{T+1} \phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right)\right\} \\
&\leq \max\left\{\frac{1}{\alpha}\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right) + (T+1)\,\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right),\right. \\
&\qquad \left.\frac{1}{\beta}\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right) + (T+1)\,\phi^{-1}\left(\lambda(M+MB^p)\sum_{i=1}^{T} g(i)\right)\right\} \\
&\leq \max\left\{\left(\frac{1}{\alpha}+T+1\right), \left(\frac{1}{\beta}+T+1\right)\right\}\psi^{-1}(\lambda)\phi^{-1}\left((M+MB^p)\sum_{i=1}^{T} g(i)\right) \\
&\leq \mathcal{L}\,\psi^{-1}(\lambda)\phi^{-1}\left((M+MB^p)\sum_{i=1}^{T} g(i)\right).
\end{aligned}
$$

□

**Lemma 3.6.** *Assume that $(\mathcal{H}_1)$ holds. Then the operator $F$ is completely continuous.*

***Proof*.** (i) As demonstrated in [25, Lemma 2.4] and from remark 2.4, since $f$ and $g$ are continuous, we can show that $\mathcal{A}_y$ and $\mathcal{B}_y$ vary continuously with respect to $y$. Consequently, $F$ is a continuous operator.
(ii) According to the Ascoli-Arzelà compactness criterion (see Lemma 2.3), by invoking Lemmas 3.4 and 3.5, we conclude that $F : \mathcal{K} \to \mathcal{K}$ is a completely continuous operator. □

## 4 Main results

We are now ready to establish our first existence result by utilizing Theorem 2.6.

**Theorem 4.1.** *Suppose that* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold. Then the problem* (1.1) *has at least one nonnegative solution* $u^*$ *such that* $R > \|u^*\| \geq u^*(k) > 0$ *for all* $k \in \{1, ..., T\}$.

***Proof.*** Let $\varepsilon > 0$. For $u \in \mathcal{P}$, define the operators

$$Tu(k) = (1+\varepsilon)u(k),$$
$$Su(k) = -\varepsilon Fu(k), \quad k \in \mathbb{T}.$$

Note that any fixed point $u \in \mathcal{P}$ of the operator sum $T+S$ is a solution of the problem (1.1). Let

$$U = \{u \in \mathcal{P},\ \|u\| < R\},$$
$$\Omega = \{u \in \mathcal{P},\ \|u\| \leq 2R\}.$$

(i) For $u, v \in \Omega$, we have

$$\|Tu - Tv\| = (1+\varepsilon)\|u-v\|.$$

So, $T$ is $h$-expansive with constant $h = 1+\varepsilon > 1$.

(ii) From Lemma 3.6, the operator $S$ is completely continuous.

(iii) Let $t \in [0,1]$ and $u \in \overline{U}$ be arbitrary chosen. Then, by Lemmas 3.5 and 3.4, we get

$$z = tFu \in \mathcal{P}$$

and

$$\|z\| \leq \mathcal{L}\psi^{-1}(\lambda)\phi^{-1}\left((M+MR^p)\sum_{i=1}^{T} g(i)\right) \leq 2R.$$

Hence $z \in \Omega$. Next

$$\begin{aligned} tSu(k) &= -\, t\varepsilon Fu(k) \\ &= -\, \varepsilon z(k) \\ &= (I-T)z(k). \end{aligned}$$

Thus, $tS(\overline{U}) \subset (I-T)(\Omega)$.

(iv) We have that

$$(I-T)^{-1}0 = 0 \in U.$$

(v) Assume that there is $u \in \partial U \cap \Omega$ and $\mu \in [0,1]$ such that

$$(I-T)u = \mu Su.$$

- If $\mu = 0$, then $u = 0$, contradicting $u \in \partial U \cap \Omega$.
- If $\mu \in (0,1]$, then there would exist $\tilde{k} \in \mathbb{T}$ such that $u(\tilde{k}) = R$. We get

$$\begin{aligned} (I-T)u(\tilde{k}) &= -\, \epsilon u(\tilde{k}) \\ &= -\, \epsilon R \\ &= -\, \epsilon\mu Fx(\tilde{k}), \end{aligned}$$

which implies

$$R = \mu Fu(\tilde{k}) < \mu R \leq R,$$

which is a contradiction.

Consequently, by Theorem 2.6, the operator $T+S$ has a fixed point in $U$. Then there exists $u^* \in \mathcal{P}$ such that $Fu^* = u^*$ such that

$$0 \leq u^*(k) < R, \quad k \in \mathbb{T}.$$

Moreover, since $f(\bar{k}, 0) \neq 0$ for some $\bar{k} \in \mathbb{T}$, from Remark 3.1, we obtain

$$u^*(k) \geq \min\left(\frac{k}{T+1}, 1 - \frac{k}{T+1}\right) \|u^*\| > 0, \quad \text{for all } k \in \{1, ..., T\}.$$

□

**Remark 4.2.** The result of Theorem 4.1 remains correct even if the function $\phi$ does not satisfy the condition $(A_2)$.

Now, we will apply Theorem 2.7 to establish criteria for the existence of positive solutions satisfying both lower and upper bounds on the norm. In the sequel, let

$$\Gamma \quad \geq \quad \tfrac{1}{\lambda}\,\chi\left(\tfrac{T+1}{\ell}\right) \max\left(\frac{1}{g(\ell)}, \frac{1}{g(T+1-\ell)}\right).$$

**Theorem 4.3.** *Assume that $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold together with*

$$(\mathcal{H}_3) \quad \liminf_{u \to +\infty} \min_{k \in \{\ell, T+1-\ell\}} \frac{f(k,u)}{\phi(u)} > \Gamma\cdot$$

*Then the problem (1.1) has at least one positive solution.*

***Proof.*** Let $R$ be as defined by the assumption $(\mathcal{H}_2)$. By the inequality of $(\mathcal{H}_3)$, for $\varepsilon > 0$, there exists an $\eta > \frac{\ell}{T+1} R > 0$ such that

$$\min_{k \in \{\ell, T+1-\ell\}} f(k,u) \geq (\Gamma + \varepsilon)\,\phi(u), \quad \forall\, u \geq \eta.$$

Hence,

$$f(k,u) \geq (\Gamma + \varepsilon)\,\phi(u), \quad \text{for any } u \geq \eta \text{ and } k \in \{\ell, T+1-\ell\}. \tag{4.1}$$

Let $L = \max(2R, \frac{(T+1)\eta}{\ell})$ and define the following sets

$$\mathcal{P}_R = \{u \in \mathcal{P},\ \|u\| < R\},$$
$$\mathcal{P}_L = \{u \in \mathcal{P}\colon\ \|u\| < L\},$$
$$\Omega = \{u \in \mathcal{P},\ \|u\| \leq 2L\}.$$

Hereafter, we check the hypotheses of Theorem 2.7.
According to the above, it remains to show that there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that

$$Su \neq (I-T)(u - \mu u_0), \text{ for all } \mu \geq 0,\ u \in \partial\mathcal{P}_L \cap (\Omega + \mu u_0).$$

On the contrary, for any $u_0 \in \mathcal{P}\backslash\{0\}$ there would exist some $\mu \geq 0$ and $w \in \partial\mathcal{P}_L \cap (\Omega + \mu u_0)$ such that

$$Sw = (I-T)(w - \mu u_0),$$

or

$$Fw = w - \mu u_0.$$

From Remark 3.1, we have $\min\limits_{k \in \{\ell, T+1-\ell\}} w(k) \geq \frac{\ell}{T+1}\|w\| = \frac{\ell}{T+1} L \geq \eta$ so condition (4.1) implies that

$$f(k, w(k)) \geq (\Gamma + \varepsilon)\,\phi(w(k)), \quad \forall\, k \in \{\ell, T+1-\ell\}. \tag{4.2}$$

To reach a contradiction, we will use (4.2) and discuss three distinct cases, considering the definition of the operator $F$.
(1) For $k_0 \in \{\ell, T+1-\ell\}$, the following estimates are straightforward:

$$\begin{aligned}
L = \|w\| \geq w(k_0) =& Fw(k_0) + \mu u_0(k_0) \\
\geq& (\frac{1}{\alpha} - l_1)\phi^{-1}\left(\lambda \sum_{i=1}^{k_0} g(i) f(i, w(i))\right) \\
\geq& (\frac{1}{\alpha} - l_1)\phi^{-1}\left(\lambda \sum_{i=\ell}^{k_0} g(i) f(i, w(i))\right) \\
\geq& (\frac{1}{\alpha} - l_1)\phi^{-1}\left(\lambda\,(\Gamma+\varepsilon) \sum_{i=\ell}^{k_0} g(i)\phi(w(i))\right) \\
\geq& (\frac{1}{\alpha} - l_1)\,\chi^{-1}\left(\lambda\,(\Gamma+\varepsilon) \sum_{i=\ell}^{k_0} g(i)\right)\,\phi^{-1}\left(\phi(\min_{k\in\{\ell, T+1-\ell\}} w(k))\right) \\
>& (\frac{1}{\alpha} - l_1)\,\chi^{-1}\left(\lambda\Gamma \sum_{i=\ell}^{k_0} g(i)\right)\min_{k\in\{\ell, T+1-\ell\}} w(k) \\
\geq& (\frac{1}{\alpha} - l_1)\,\chi^{-1}\left(\lambda\Gamma \sum_{i=\ell}^{k_0} g(i)\right)\frac{l}{T+1}\|w\| \\
\geq& (\frac{1}{\alpha} - l_1)\,\chi^{-1}\left(\lambda\Gamma g(\ell)\right)\frac{l}{T+1}L \\
\geq& \chi^{-1}\left(\lambda\Gamma g(\ell)\right)\frac{l}{T+1}L \\
\geq& L.
\end{aligned}$$

(2) For $k_0 > T+1-\ell$, we get

$$\begin{aligned}
L \geq w(T+1-\ell) =& Fw(T+1-\ell) + \mu u_0(T+1-\ell) \\
\geq& \sum_{s=1}^{T+1-\ell} \phi^{-1}\left(\lambda \sum_{i=s}^{T+1-\ell} g(i) f(i, w(i))\right) \\
\geq& \phi^{-1}\left(\lambda \sum_{i=\ell}^{T+1-\ell} g(i) f(i, w(i))\right) \\
\geq& \phi^{-1}\left(\lambda\,(\Gamma+\varepsilon) \sum_{i=\ell}^{T+1-\ell} g(i)\phi(w(i))\right) \\
>& \chi^{-1}\left(\lambda\Gamma \sum_{i=\ell}^{T+1-\ell} g(i)\right)\min_{k\in\{\ell, T+1-\ell\}} w(k) \\
\geq& \chi^{-1}\left(\lambda\Gamma \sum_{i=\ell}^{T+1-\ell} g(i)\right)\frac{l}{T+1}L \\
\geq& \chi^{-1}\left(\lambda\Gamma g(T+1-\ell)\right)\frac{l}{T+1}L \\
\geq& L.
\end{aligned}$$

(3) Similarly, for $k_0 < \ell$, we get

$$\begin{aligned}
L \geq w(\ell) =& Fw(\ell) + \mu u_0(\ell) \\
\geq& \sum_{s=\ell+1}^{T+1} \phi^{-1}\left(\lambda \sum_{i=k_0+1}^{s-1} g(i) f(i, w(i))\right) \\
\geq& \sum_{s=\ell+1}^{T+2-\ell} \phi^{-1}\left(\lambda \sum_{i=\ell+1}^{s-1} g(i) f(i, w(i))\right) \\
\geq& \phi^{-1}\left(\lambda \sum_{i=\ell}^{T+1-\ell} g(i) f(i, w(i))\right) \\
\geq& \phi^{-1}\left(\lambda\left(\Gamma+\varepsilon\right) \sum_{i=\ell+1}^{T+1-\ell} g(i) \phi(w(i))\right) \\
>& \chi^{-1}\left(\lambda\Gamma \sum_{i=\ell+1}^{T+1-\ell} g(i)\right) \min_{k\in\{\ell, T+1-\ell\}} w(k) \\
\geq& \chi^{-1}\left(\lambda\Gamma \sum_{i=\ell+1}^{T+1-\ell} g(i)\right) \|w\| \\
\geq& \chi^{-1}\left(\lambda\Gamma \sum_{i=\ell+1}^{T+1-\ell} g(i)\right) \frac{l}{T+1} L \\
\geq& \chi^{-1}\Big(\lambda\Gamma g(T+1-\ell)\Big) \frac{l}{T+1} L \\
\geq& L.
\end{aligned}$$

Which is a contradiction, this implies that the statement holds true. By Theorem 2.7, we finally deduce that the mapping $T+S$ has at least one fixed point $v^*$, which belongs to $\mathcal{P}_{R,L}$ and hence is a solution to the problem (1.1) that satisfies $0 < R < \|v^*\| < L$. Moreover, we have

$$v^*(k) > \min\left(\frac{k}{T+1}, 1 - \frac{k}{T+1}\right) R, \qquad \text{for all } k \in \mathbb{T}.$$

Then $v^*$ is a positive solution of the problem (1.1).

□

The following result deals with the other polynomial growth case and can be proved in a similar manner.

**Theorem 4.4.** *Assume that $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold together with*

$$(\mathcal{H}_4) \quad \liminf_{u\to 0} \min_{k\in\{\ell, T+1-\ell\}} \frac{f(k,u)}{\phi(u)} > \Gamma.$$

*Then the problem (1.1) has at least one positive solution.*

***Proof.*** Let $R$ be as defined by the assumption $(\mathcal{H}_2)$. By the inequality of $(\mathcal{H}_4)$, for $\varepsilon > 0$, there exists an $r_0 > 0$ such that

$$f(k,u) \geq (\Gamma+\varepsilon)\,\phi(u), \qquad \text{for any } \ u \leq r_0 \ \text{ and } \ k \in \{\ell, T+1-\ell\}. \tag{4.3}$$

Let $r = \min(\frac{R}{2}, r_0)$ and consider the sets

$$\begin{aligned}
\mathcal{P}_R &= \{u \in \mathcal{P}, \ \|u\| < R\}, \\
\mathcal{P}_r &= \{u \in \mathcal{P}\colon \ \|u\| < r\}, \\
\Omega &= \{u \in \mathcal{P}, \ \|u\| \leq 2R\}.
\end{aligned}$$

Similarly, the conclusion follows from Theorem 2.7. Hence the problem $(1.1)$ has at least one solution $v^* \in \mathcal{P}$ such that $0 < r < \|v^*\| < R$. Moreover, we have

$$v^*(k) > \min\left(\frac{k}{T+1}, 1 - \frac{k}{T+1}\right) r, \qquad \text{for all } k \in \mathbb{T}.$$

Then $v^*$ is a positive solution of the problem (1.1).

□

Our final result is concerned of the existence of at least two positive solutions.

**Theorem 4.5.** *Assume that $(\mathcal{H}_1) - (\mathcal{H}_4)$ hold. Then the problem (1.1) has at least two positive solutions.*

***Proof.*** Using the same arguments as in Theorems 4.3 and 4.4, we consider the following sets

$$\begin{aligned} \mathcal{P}_r &= \{u \in \mathcal{P}\colon\ \|u\| < r\}, \\ \mathcal{P}_R &= \{u \in \mathcal{P},\ \|u\| < R\}, \\ \mathcal{P}_L &= \{u \in \mathcal{P}\colon\ \|u\| < L\}, \\ \Omega &= \{u \in \mathcal{P},\ \|u\| \leq 2L\}. \end{aligned}$$

Here $r < \min(R, r_0)$ and $L > \max(R, \frac{(T+1)\eta}{\ell})$, where $r$ and $L$ are given in (4.3) and (4.1), respectively. By Theorem 2.7, the problem $(1.1)$ has at least two solutions $u^*$, $v^* \in \mathcal{P}$ such that

$$0 < r < \|u^*\| < R < \|v^*\| < L.$$

Moreover, we have $u^*(k) > \min\left(\frac{k}{T+1}, 1 - \frac{k}{T+1}\right) r$, for all $k \in \mathbb{T}$. Then $u^*$ and $v^*$ are positive solutions of the problem (1.1). □

As a consequence, we obtain

**Corollary 4.6.** *Assume that $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold together with*

$$(\mathcal{H}_5) \quad \liminf_{u\to 0} \min_{k\in\{\ell, T+1-\ell\}} \frac{f(k,u)}{\phi(u)} = \liminf_{u\to+\infty} \min_{k\in\{\ell, T+1-\ell\}} \frac{f(k,u)}{\phi(u)} = +\infty\cdot$$

*Then the problem (1.1) has at least two positive solutions.*

## 5 Examples

**Example 5.1.** Consider the following boundary value problem

$$\begin{aligned} \Delta(\phi(\Delta u(k-1))) + \lambda g(k) f(k, u(k)) &= 0, \quad k \in \{1, 2, \dots, T\} \\ \Delta u(0) - \alpha u(l_1) &= 0, \\ \Delta u(T) + \beta u(l_2) &= 0, \end{aligned} \tag{5.1}$$

with

$$T = 20, \quad l_1 = 4, \quad l_2 = 14, \quad \alpha = \beta = \frac{1}{8}, \quad \lambda = 2,$$

$$f(k,u) = C(1+k^2)(1+u^p) \text{ with } C = 6.10^{-12} > 0, \text{ and } g(k) = \frac{1}{\sqrt{1+k}},$$

$$\phi(x) = \psi(x) = x^q \text{ and } \chi(x) = x^p \text{ with } 0 < q = 3 < p = 6,$$

hence

$$\phi^{-1}(x) = \psi^{-1}(x) = x^{\frac{1}{q}} \text{ and } \chi^{-1}(x) = x^{\frac{1}{p}}.$$

We have

$$\psi(x)\phi(y) \leq \phi(xy) = \phi(x)\phi(y) \leq \chi(x)\phi(x), \quad \forall x, y \in (0, \infty),$$

so

$$\chi^{-1}(x)\phi^{-1}(y) \leq \phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y) \leq \psi^{-1}(x)\phi^{-1}(y), \quad \forall x, y \in (0, \infty).$$

Also, we have

$$\alpha\, l_1 = \frac{1}{2} \leq 1, \quad \beta(T+1-l_2) = \frac{1}{8}(20+1-14) = \frac{7}{8} = 0,875 \leq 1,$$

$$\mathcal{L} = \max\left(\frac{1}{\alpha} + T + 1, \frac{1}{\beta} + T + 1\right) = 8 + 21 = 29$$

and

$$\sum_{i=1}^{T} g(i) = \sum_{i=1}^{20} \left(\frac{1}{1+k}\right)^{\frac{1}{2}} = 6,8135 < 7.$$

Set $a(k) = b(k) = C(1+k^2),\ k \in \{0, 1, ..., 20\}$, we get $a(k), b(k) \leq M = C(1+20^2),\ k \in \{0, 1, ..., 20\}$.

Let

$$R = 10,$$

then

$$\begin{aligned} \mathcal{L}\,\psi^{-1}(\lambda)\phi^{-1}\left((M + MR^p)\sum_{i=1}^{T} g(i)\right) &\leq 29\lambda^{\frac{1}{q}}(7M(1+R^p))^{\frac{1}{q}} \\ &= 29.2^{\frac{1}{3}}(7C.(1+20^2)(1+10^6))^{\frac{1}{3}} \\ &\approx 9,3657 \\ &< 10. \end{aligned}$$

Then all conditions of Theorem 4.1 hold. Consequently, the problem (5.1) has at least one bounded solution $u^*$ such that

$$R = 10 > \|u^*\| \geq u^*(k) > 0 \text{ for all } k \in \{1, ..., 20\}.$$

Moreover,

$$\begin{aligned} \lim_{x\to+\infty} \frac{f(k,x)}{\phi(x)} &= \lim_{x\to+\infty} \frac{C\,(1+k^2)(1+x^p)}{x^q} \\ &\geq C \lim_{x\to+\infty} x^{p-q} = +\infty, \forall\, k \in \{0, ..., 20\} \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to 0} \frac{f(k,x)}{\phi(x)} &= \lim_{x\to 0} \frac{C\,(1+k^2)(1+x^p)}{x^q} \\ &\geq C \lim_{x\to 0} x^{-q} = +\infty, \forall\, k \in \{0, ..., 20\}. \end{aligned}$$

Then all conditions of Theorem 4.5 hold. Consequently, the problem (5.1) has at least two positive solutions.

**Example 5.2.** Consider the following boundary value problem

$$\begin{aligned} \Delta(\phi(\Delta u(k-1))) + \lambda g(k) f(k, u(k)) &= 0, \quad k \in \{1, 2, \ldots, T\} \\ \Delta u(0) - \alpha u(l_1) &= 0, \\ \Delta u(T) + \beta u(l_2) &= 0, \end{aligned} \tag{5.2}$$

with

$$T = 25, \quad l_1 = 2, \quad l_2 = 8, \quad \alpha = \frac{1}{12} \quad \beta = \frac{1}{20}, \quad \lambda = \frac{1}{2},$$

$$f(k,u) = C(1+u^6) \text{ with } C = 10^{-9} \text{ and } g(k) = \frac{1}{1+\sqrt{k}},$$

$$\phi(u) = |u|^{p^*} u, \quad \phi^{-1}(u) = |u|^{q^*} u \quad \text{ with } \frac{1}{p^*} + \frac{1}{q^*} = 1,$$

here we set $p^* = 4,\ q^* = \frac{4}{3}$. Let

$$\psi(x) = x^{\frac{1}{4}} \text{ and } \chi(x) = x^8$$

hence

$$\psi^{-1}(x) = x^4 \text{ and } \chi^{-1}(x) = x^{\frac{1}{8}}.$$

We have

$$\psi(x)\phi(y) = x^{\frac{1}{4}}\,|y|^4 y \le \phi(xy) = \phi(x)\phi(y) = |x|^4 x\,|y|^4 y, \quad \forall x,y \in (0,\infty),$$

and

$$\phi(xy) = \phi(x)\phi(y) = |x|^4 x\,|y|^4 y \le \chi(x)\phi(y) = x^8\,|y|^4 y, \quad \forall x,y \in (0,\infty),$$

so

$$\chi^{-1}(x)\phi^{-1}(y) = x^{\frac{1}{8}}\,|y|^{\frac{4}{3}}\,y\ \le \phi^{-1}(xy) = |x|^{\frac{4}{3}} x\,|y|^{\frac{4}{3}} y, \quad \forall x,y \in (0,\infty),$$

and

$$\phi^{-1}(xy) = |x|^{\frac{4}{3}} x\,|y|^{\frac{4}{3}} y \le \psi^{-1}(x)\phi^{-1}(y) = x^4\,|y|^{\frac{4}{3}}\,y, \quad \forall x,y \in (0,\infty).$$

Also, we have

$$\alpha\, l_1 = \frac{2}{12} = \frac{1}{6} \le 1, \quad \beta(T+1-l_2) = \frac{1}{20}(25+1-14) = \frac{12}{20} = 0{,}6 \le 1,$$

$$\mathcal{L} = \max\left(\frac{1}{\alpha} + T + 1, \frac{1}{\beta} + T + 1\right) = \frac{1}{\beta} + T + 1 = 46$$

and

$$\sum_{i=1}^{T} g(i) = \sum_{i=1}^{25} \frac{1}{1+\sqrt{k}} = 6,1460 \le 6,5.$$

Set $a(k) = Ck,\ b(k) = C,\ k \in \{0,1,...,25\}$, we get $a(k), b(k) \le Ck \le M = 25C,\ k \in \{0,1,...,25\}$.

Let

$$R = 15,\ p = 6$$

then

$$\begin{aligned}
\mathcal{L}\,\psi^{-1}(\lambda)\phi^{-1}\left((M+MR^p)\sum_{i=1}^{T} g(i)\right) &\le 46.\lambda^4.(6,5.M(1+R^p))^{\frac{4}{3}}(6,5.M(1+R^p))\\
&= 46.\left(\tfrac{1}{2}\right)^4.(6,5.25.C.(1+15^6))^{\frac{7}{3}}\\
&\approx 12,0941\\
&< 15.
\end{aligned}$$

Then all conditions of Theorem 4.1 hold. Consequently, the problem (5.2) has at least one bounded solution $u^*$ such that

$$R = 15 > \|u^*\| \ge u^*(k) > 0 \text{ for all } k \in \{1,...,25\}.$$

Moreover,

$$\begin{aligned}
\lim_{x\to+\infty} \frac{f(k,x)}{\phi(x)} &= \lim_{x\to+\infty} \frac{C\,(1+x^6)}{x^5}\\
&\ge C \lim_{x\to+\infty} x = +\infty, \quad \forall\, k \in \{0,...,25\}
\end{aligned}$$

and

$$\lim_{x\to 0}\frac{f(k,x)}{\phi(x)}=\lim_{x\to 0}\frac{C\,(1+x^6)}{x^5}$$
$$\geq C\lim_{x\to 0}x^{-5}=+\infty,\quad \forall\, k\in\{0,...,25\}.$$

Then all conditions of Theorem 4.5 hold. Consequently, the problem (5.2) has at least two positive solutions.

## References


[1] R.P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Dordrecht, Kluwer Academic Publishers, (2001).

[2] F.M. Atici and J.M. Jonnalagadda, *An eigenvalue problem in fractional $h$-discrete calculus*, Fractional Calculus and Applied Analysis, **25(2)**, 630-647, (2022).

[3] D. Bai, *A global result for discrete $\phi$-Laplacian eigenvalue problems*, Advances in Difference Equations, **2013**, 1–10, (2013).

[4] D. Bai and X. Xu, *Existence and multiplicity of difference $\phi$-Laplacian boundary value problems*, Advances in Difference Equations, **2013**, 1–13, (2013).

[5] M. Bohner, G. Caristi, S. Heidarkhani and S. Moradi, *Existence of at least one homoclinic solution for a nonlinear second-order difference equation*, International Journal of Nonlinear Sciences and Numerical Simulation, **20**, 433–439, (2019).

[6] L. Bouchal and K. Mebarki, *New multiple fixed point theorems for sum of two operators and application to a singular generalized Sturm-Liouville multipoint BVP*, Matematički Vesnik, **76(3)**, 196–209, (2024).

[7] L.Bouchal, K. Mebarki and S.G. Georgiev, *Existence of solutions for boundary value problems for first order impulsive difference equations*, Boletim da Sociedade Paranaense de Matemática, **42**, 1-13, (2024).

[8] P. Candito, G. D'Aguì and E. Amoroso, *Two positive solutions for a nonlinear Robin problem involving the discrete $p$-Laplacian*, Dolomites Research Notes on Approximation, **15**, 1–7, (2022).

[9] S.G. Georgiev, *Existence of solutions for a class of wave dynamic equations on time scales*, Discrete and Continuous Dynamical Systems, **17(10)**, 2967-2981, (2024).

[10] S.G. Georgiev, A. Kheloufi and K. Mebarki, *Existence of Classical Solutions for a Class of Impulsive Hamilton-Jacobi Equations*, Palestine Journal of Mathematics, **13(4)**, 1084–1097,(2024).

[11] S.G. Georgiev and K. Mebarki, *On fixed point index theory for the sum of operators and applications to a class of ODEs and PDEs*, Applied General Topology, **22(2)**, 259–294, (2021).

[12] A. Ghobadi and S. Heidarkhanii, *Critical point approaches for doubly eigenvalue discrete boundary value problems driven by $\phi_c$-Laplacian operator*, Mathematical Communications, **30**, 39-59, (2025).

[13] O. Hammouti, *Existence of Three Solutions for a Discrete p-Laplacian Boundary Value Problem,* Palestine Journal of Mathematics, **13(2)**, 144–155, (2024).

[14] S. Heidarkhani, S. Moradi, D. Barilla, and G. Caristi, *Existence results for a discrete fractional boundary value problem driven by p-Laplacian operator*, Reasearch square, (2022).

[15] H. Li, L. Wang, and Y. Cui, *Positive solutions for a system of fractional q-difference equations with generalized $p$-Laplacian operators*, Electronic Research Archive, **32(2)**, 1044-1066, (2024).

[16] K. Mebarki, S.G. Georgiev, S. Djebali and K. Zennir, *Fixed Point Theorems with Applications*, CRC Press, (2023).

[17] A. Mouhous, K. Mebarki, *Existence of fixed points in conical shells of a Banach space for sum of two operators and application in ODEs*, Turkish Journal of Mathematics, **46(7)**, 2556-2571, (2022).

[18] I. Nyanquini and S. Ouaro, *On a Bi-nonlocal fourth-order difference problem involving the $p(k)$-Laplacian type operator,* Palestine Journal of Mathematics, **13(4)**, 483–503, (2024).

[19] T. Wang, M. Xu and Y. Li, *Positive solutions of discrete boundary value problem for a second-order nonlinear difference equation with singular $\phi$-Laplacian*, Journal of Applied Analysis & Computation, **15(2)**, 1020-1038, (2025).

[20] J. Wang, C. Yu, B. Zhang and S. Wang, *Positive solutions for eigenvalue problems of fractional $q$-difference equation with $\phi$-Laplacian*, Advances in Difference Equations, **2021**, 1-15, (2021).

[21] F. Xiong, Z.Zhou, Z, *Three positive solutions for a nonlinear partial discrete Dirichlet problem with $(p,q)$-Laplacian operator,* Boundary Value Problems, 2022:9, (2022).

[22] F. Xiong, *Infinitely many solutions for partial discrete Kirchhoff type problems involving p-Laplacian*, Mathematics **11(15)**, 3288, (2023).

[23] F. Xiong and W. Huang, *Positive solutions for dirichlet BVP of PDE involving $\phi_p$-Laplacian*, Fractal and Fractional, **8(3)**, p. 130 (2024).

[24] F. Xiong, *Infinitely many solutions for a perturbed partial discrete Dirichlet problem involving $\phi_c$-Laplacian*, Axioms, **12(10)**, p. 909, (2023).

[25] Y. Yang and F. Meng, *Eigenvalue problem for finite difference equations with p-Laplacian*, Journal of Applied Mathematics and Computing, **40(1)**, 319–340, (2012).

[26] Z. Zhou and J. Ling, *Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with $\phi_c$-Laplacian*, Applied Mathematics Letters, **91**, 28–34, (2019).

## Author information

L. Bouchal, Department of Mathematics, Faculty of Exact Sciences, University of Bouira, 10000 Bouira, Algeria,
Laboratory of Applied Mathematics, Faculty of Exact Sciences, University of Bejaia, 06000 Bejaia, Algeria.
E-mail: `l.bouchal@univ-bouira.dz, Bouchal_Lydia@outlook.com`

K. Mebarki, Université de Bejaia, Faculté des Sciences Exactes,
Laboratoire de Mathématiques Appliquées, 06000 Bejaia, Algérie.
E-mail: `karima.mebarki@univ-bejaia.dz, mebarqi_karima@hotmail.fr`

S. Zahar, Université de Bejaia, Faculté des Sciences de la Nature et de la Vie,
Laboratoire de Mathématiques Appliquées, 06000 Bejaia, Algérie.
E-mail: `samira.zahar@univ-bejaia.dz, zahar_samira@yahoo.fr`

A. Kheloufi, Université de Bejaia, Faculté des Sciences Exactes,
Laboratoire de Mathématiques Appliquées, 06000 Bejaia, Algérie.
E-mail: `arezki.kheloufi@univ-bejaia.dz, arezkinet2000@yahoo.fr`

Received: 2025-02-19
Accepted: 2025-04-27