Palestine Journal of Mathematics

Vol 14(2)(2025) , 714–725

SOME RESULTS ON ORTHOGONAL POLYNOMIALS GENERATED BY $A(t)F(xtA(t)-R(t))$

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Communicated by (NAME of the Editor who handled the paper)

MSC 2010 Classifications: 12E10, 33C47, 33C20.

Keywords and phrases: Generating functions, orthogonal polynomials, recurrence relations, Laguerre polynomials.

Dedicated to Professor Hacen Dib on the occasion of his retirement.

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Abstract In this paper, we consider polynomial sets, $\{P_n\}_{n\geq 0}$, with generating power series of the form $A(t)F(xtA(t)-R(t))$ and satisfying the 2-order recursion $xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x)$, where $n \geq 0$, $P_0(x) = 1$, $P_{-1}(x) = 0$, and $\{\beta_n\}$, $\{\omega_n\}$ are complex sequences. We give the relations between $\{\beta_n\}$, $\{\omega_n\}$ and the coefficients of the formal power series $A(t)$, $F(t)$ and $R(t)$. This provides another method to obtain orthogonal polynomials with generating power series of Sheffer and ultraspherical types. We also show that the Laguerre polynomials are the only orthogonal polynomials with generating power series of the form $\frac{1}{1-t}F\left(\frac{xt}{1-t}+\alpha\ln(1-t)\right)$.

1 Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is a monic polynomial set (MPS) if and only if $P_n(x) = x^n +$ lower terms, for all non-negative integers n.

In this paper, we consider the following characterization problem,
(P): Find all (MPSs) $\{P_n\}_{n\geq 0}$ with a generating function (GF) of the form

$$A(t)F\left(xtA(t)-R(t)\right) = \sum_{n\geq 0} \alpha_n P_n(x)t^n, \ (\alpha_n \neq 0 \text{ for } n \geq 0), \tag{1.1}$$

and satisfying the 2-order recursion

$$\begin{cases} xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x), & n \geq 0, \\ P_{-1}(x) = 0, \quad P_0(x) = 1. \end{cases} \tag{1.2}$$

Where $F(t) = \sum_{n\geq 0}\alpha_n t^n$, $A(t)$ and $R(t)$ are formal power series with $(A(0), R(0)) = (1, 0)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\omega_n\}$ are complex sequences.

Remark that if a (MPS) $\{P_n(x)\}_{n\geq 0}$ is a solution of **(P)**, then for $a \neq 0$ and b complex parameters, the rescaled (MPS) $\{a^{-n}P_n(ax+b)\}_{n\geq 0}$ is also a solution of **(P)**. In fact, it is generated by $\tilde{A}(t)F\left(t\tilde{A}(t)x - \tilde{R}(t)\right)$ where $(\tilde{A}(z), \tilde{R}(z)) = (A(z/a), R(z/a) - bzA(z/a)/a)$ and satisfies (1.2) with $((\beta_n - b)/a, a^{-2}\omega_n)$ instead of (β_n, ω_n). So, in such characterization, we take into account that the two (MPSs) are equivalent. We also consider (MPSs) which are not necessarily orthogonal. Recall that an orthogonal (MPS) satisfies (1.2) with $\omega_n \neq 0$ for $n \geq 1$ [8]. We note also that the (GF) (1.1) is a subclass case of the Rainville type (GF) $V(t)F(xtU(t) - R(t))$ which Rainville introduced in [17] as an extension of the Boas-Buck type (GF) $V(t)F(xtU(t))$ [7], where $U(t)$ and $V(t)$ are formal power series. We mention here the use of these (GFs) and their generated polynomials to construct sequences of positive linear operators in order to approximate functions, see [1, 16, 20] and the references therein.

Problem **(P)** has been treated in [2, 4, 5, 6] by different methods for the case where $F(t)$ is an arbitrary formal power series, $(A(t), R(t)) = (1, \alpha t^2)$, $\alpha \neq 0$, and $\{P_n\}_{n\geq 0}$ are orthogonal polynomials. It was shown that the only polynomials are the rescaled ultraspherical, Hermite, and Chebyshev polynomials of the first kind. Surprisingly, these same polynomials (besides the monomial set) are the only (MPSs) solutions to **(P)** in the case where $F(t)$ and $R(t)$ are arbitrary formal power series with $A(t)=1$ and $\{P_n\}_{n\geq 0}$, satisfying (1.2), are not necessary orthogonal [13]. For an extension of this result to (MPSs) generated by $F(xt-R(t))$ and satisfying a higher order recursion see [6, 14, 15].

Two distinguished (GFs) which can be put in the form (1.1) are the (GFs) of Sheffer type [12, 19], $V(t)\exp(xtU(t))$, and of ultraspherical type [3, 9], $V(t)(1-xtU(t))^{-\lambda}$, for some λ. Namely, we have the representations:

$$V(t)\exp(xtU(t)) = U(t)\exp(xtU(t) - \ln(U(t)/V(t))) \tag{1.3}$$

and (except for $\lambda = 1$)

$$V(t)(1-xtU(t))^{-\lambda} = A(t)\left(1-(xtA(t)-R(t))\right)^{-\lambda} \tag{1.4}$$

where $A(t) = (U(t))^{-\lambda/(1-\lambda)}\,V(t)^{1/(1-\lambda)}$ and $R(t) = A(t)/U(t) - 1$.
They can also be written as

$$V(t)\exp(xtU(t)) = \exp(xtU(t) + \ln(V(t)),$$

$$V(t)(1-xtU(t))^{-\lambda} = \left(1-\left(xtU(t)V(t)^{-1/\lambda} + 1 - V(t)^{-1/\lambda}\right)\right)^{-\lambda}$$

and according to a result stated in [7, Theorem 5], they are characterized by being the only (GFs) in both Rainville and Boas-Buck classes.

Meixner [12] and Sheffer [19] characterized all orthogonal (PSs) with a (GF) of Sheffer type. They have shown that these polynomials satisfy the recursion (1.2) with $\beta_n = an+b$ and $\omega_n = n(cn+d)$. The orthogonal (PSs) with a (GF) of ultraspherical type have been characterized by different methods in [3, 9, 11]. Our results here, provide another method to obtain orthogonal (PSs) with a (GF) of Sheffer and ultraspherical types (see section 4).

In section 2 we give the main results theorem 2.1 and theorem 2.2. Theorem 2.1 contains the relations between $\{\beta_n\}$, $\{\omega_n\}$ and the coefficients of the formal power series $A(t)$, $F(t)$ and $R(t)$. Theorem 2.2 concerns the monic Laguerre polynomials (MLPs) $\{L_n^{(\alpha)}(x)\}$ defined by a (GF) of Sheffer type [17]

$$(1-t)^{-\alpha-1}e^{\frac{-xt}{1-t}} = \sum_{n\geq 0}\frac{(-1)^n}{n!}L_n^{(\alpha)}(x)t^n \tag{1.5}$$

which can be written in the form (1.1) as follows

$$\frac{1}{1-t}\exp\left(\frac{-xt}{1-t} - \alpha\ln(1-t)\right) = \sum_{n\geq 0}\frac{(-1)^n}{n!}L_n^{(\alpha)}(x)t^n \tag{1.6}$$

where $A(t) = 1/(1-t)$, $R(t) = -\alpha\ln(1-t)$ and $F(x) = \exp(-x)$. By taking $F(x)$ an arbitrary power series, the result of theorem 2.2 is that the (MLPs) are the only (PSs), except the rescaled monomials, satisfying (1.2) and generated by $\frac{1}{1-t}F\left(\frac{xt}{1-t} + c\ln(1-t)\right)$ with c a complex parameter. The proofs of these two theorems are given in section 3.

2 Main results

Before we give the main results, we define the sequences $\{S_n\}$, $\{R_n\}$ and $\{A_n^k\}$ by:

$$\frac{A'(t)}{A(t)} = \sum_{n\geq 0} S_n t^n, \tag{2.1}$$

$$\frac{R'(t)}{A(t)} = \sum_{n\geq 0} R_n t^n \tag{2.2}$$

and

$$A_n^0 = -S_0\frac{\alpha_{n-1}}{\alpha_n} + R_0,\ n \geq 1 \tag{2.3}$$

$$A_n^1 = -S_1\frac{\alpha_{n-2}}{\alpha_n} - S_0\frac{\alpha_{n-1}}{\alpha_n}\beta_{n-1} + R_1\frac{\alpha_{n-1}}{\alpha_n},\ n \geq 2 \tag{2.4}$$

$$A_n^k = -S_k\frac{\alpha_{n-k-1}}{\alpha_n} - S_{k-1}\frac{\alpha_{n-k}}{\alpha_n}\beta_{n-k} - S_{k-2}\frac{\alpha_{n-k+1}}{\alpha_n}\omega_{n-k+1} + R_k\frac{\alpha_{n-k}}{\alpha_n},\ \ k \geq 2,\ n \geq k+1. \tag{2.5}$$

Theorem 2.1. *For a (MPS) $\{P_n\}$ satisfying (1.1) and (1.2) with $(n-k)A_n^k = 0$ for $n = k \geq 0$, we have*

i.)

$$\beta_n \quad = \quad (n+1)A_{n+1}^0 - nA_n^0,\ n \geq 0. \tag{2.6}$$

ii.)

$$\omega_n \quad = \quad \frac{n}{2}A_{n+1}^1 - \frac{n-1}{2}A_n^1 - \frac{n}{2(n+1)}\left(\beta_n - A_n^0\right)^2,\ n \geq 1. \tag{2.7}$$

iii.)

$$\begin{aligned} A_{n+1}^2 - \frac{n-2}{n-1}A_n^2 \quad &= \quad \omega_n\left(\frac{1}{n+1}\beta_n + \frac{2}{n}\beta_{n-1} - \frac{n+2}{n}A_{n-1}^0 + \frac{n}{n+1}A_n^0\right) \\ &\quad + A_n^1\left(-\frac{n+2}{n+1}\beta_n + \frac{n-1}{n}\beta_{n-1} + \frac{1}{n+1}A_n^0 + \frac{1}{n}A_{n-1}^0\right),\ n \geq 2. \end{aligned} \tag{2.8}$$

iv.)

$$\begin{aligned} \frac{2}{n}\omega_n\omega_{n-1} \quad &= \quad A_{n+1}^3 - \frac{n-3}{n-2}A_n^3 + \frac{n+2}{n}\omega_n A_{n-1}^1 - \frac{n-1}{n}\omega_{n-1}A_n^1 - \frac{1}{n}A_n^1A_{n-1}^1 \\ &\quad - \left(-\frac{n+2}{n+1}\beta_n + \frac{n-2}{n-1}\beta_{n-2} + \frac{1}{n+1}A_n^0 + \frac{1}{n-1}A_{n-2}^0\right)A_n^2,\ n \geq 3. \end{aligned} \tag{2.9}$$

v.)

$$\begin{aligned} A_{n+1}^k - \frac{n-k}{n-k+1}A_n^k + \left(\frac{n+2}{n+1}\beta_n - \frac{n-k+1}{n-k+2}\beta_{n-k+1}\right)A_n^{k-1} + \frac{n+2}{n}\,\omega_n\,A_{n-1}^{k-2} \\ -\frac{n-k+2}{n-k+3}\,\omega_{n-k+2}\,A_n^{k-2} = \sum_{l=0}^{k-1}\frac{A_n^{k-l-1}\,A_{n-k+l+1}^l}{n-k+l+2},\quad n \geq k \geq 4. \end{aligned} \tag{2.10}$$

Theorem 2.2. *Let c be a complex number and $F(t) = \sum_{n\geq 0}\alpha_n t^n$ a formal power series where $\{\alpha_n\}$ is a complex sequence with $\alpha_0 = 1$. The only (MPSs) defined by*

$$\frac{1}{1-t}F\left(\frac{xt}{1-t} + c\ln(1-t)\right) = \sum_{n\geq 0}\alpha_n P_n(x)t^n, \tag{2.11}$$

and satisfying the three-term recursion relation (1.2) are $\{(x+\frac{1}{\alpha_1})^n\}_{n\geq 0}$ and the rescaled (MLPs).

3 Proof of main results

To prove theorem 2.1 we need the following lemmas.

Lemma 3.1. *Let $\{P_n\}_{n\geq 0}$ be a (MPS) generated by (1.1). Then we have*

$$\alpha_n xP_n'(x) - n\alpha_n P_n(x) = -\sum_{k=0}^{n-1}S_k\alpha_{n-k-1}\left[xP_{n-k-1}'(x) + P_{n-k-1}(x)\right] + \sum_{k=0}^{n-1}R_k\alpha_{n-k}P_{n-k}'(x),\ \ n \geq 1. \tag{3.1}$$

Proof. By combining the two derivatives $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial t}$ of the generating function $W(x,t) = A(t)F\left(tA(t)x - R(t)\right)$, we obtain

$$x\frac{\partial W}{\partial x} - t\frac{\partial W}{\partial t} = -\frac{tA'(t)}{A(t)}\left[x\frac{\partial W}{\partial x} + W\right] + \frac{R'(t)}{A(t)}\frac{\partial W}{\partial x}. \tag{3.2}$$

The substitution of $W(x,t) = \sum_{n\geq 0}\alpha_n P_n(x)t^n$, $\frac{tA'(t)}{A(t)} = \sum_{n\geq 0} S_n t^{n+1}$ and $\frac{R'(t)}{A(t)} = \sum_{n\geq 0} R_n t^n$ in (3.2) gives

$$\begin{aligned}\sum_{n\geq 0}\alpha_n x P_n'(x)t^n - \sum_{n\geq 0} n\alpha_n P_n(x)t^n \quad = \quad & -\left(\sum_{n\geq 0} S_n t^{n+1}\right)\left[\sum_{n\geq 0}\alpha_n x P_n'(x)t^n + \sum_{n\geq 0}\alpha_n P_n(x)t^n\right] \\ & + \left(\sum_{n\geq 0} R_n t^n\right)\left(\sum_{n\geq 0}\alpha_n P_n'(x)t^n\right). \end{aligned} \tag{3.3}$$

After a resummation procedure in the right-hand side of the later equation, namely:

$$\left(\sum_{n\geq 0} S_n t^{n+1}\right)\left(\sum_{n\geq 0}\alpha_n x P_n'(x)t^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n-1} S_k\alpha_{n-k-1}P_{n-k-1}'(x)\right)t^n,$$

$$\left(\sum_{n\geq 0} S_n t^{n+1}\right)\left(\sum_{n\geq 0}\alpha_n P_n(x)t^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n-1} S_k\alpha_{n-k-1}P_{n-k-1}(x)\right)t^n,$$

$$\left(\sum_{n\geq 0} R_n t^n\right)\left(\sum_{n\geq 0}\alpha_n P_n'(x)t^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n-1} R_k\alpha_{n-k}P_{n-k}'(x)\right)t^n,$$

and a t^n coefficients comparison in (3.3), the result (3.1) of lemma 3.1 follows.

□

Lemma 3.2. *For the (MPS) generated by* (1.1) *and satisfying* (1.2) *we have:*

$$xP_n'(x) - nP_n(x) = \sum_{k=0}^{n-1} A_n^k P_{n-k}'(x), \quad n\geq 1, \tag{3.4}$$

Proof. By differentiating (1.2) we get

$$xP_n'(x) + P_n(x) = P_{n+1}'(x) + \beta_n P_n'(x) + \omega_n P_{n-1}'(x). \tag{3.5}$$

Using this equation we can write (3.1) as

$$\begin{aligned}\alpha_n x P_n'(x) - n\alpha_n P_n(x) = \quad & -\sum_{k=0}^{n-1} S_k\alpha_{n-k-1}\left[P_{n-k}'(x) + \beta_{n-k-1}P_{n-k-1}'(x) + \omega_{n-k-1}P_{n-k-2}'(x)\right] \\ & + \sum_{k=0}^{n-1} R_k\alpha_{n-k}P_{n-k}'(x), \quad n\geq 1. \end{aligned} \tag{3.6}$$

Then

$$\begin{aligned}\alpha_n x P_n'(x) - n\alpha_n P_n(x) \quad = \quad & -\sum_{k=0}^{n-1} S_k\alpha_{n-k-1}P_{n-k}'(x) - \sum_{k=1}^{n-1} S_{k-1}\alpha_{n-k}\beta_{n-k}P_{n-k}'(x) \\ & -\sum_{k=2}^{n-1} S_{k-2}\alpha_{n-k+1}\omega_{n-k+1}P_{n-k}'(x) + \sum_{k=0}^{n-1} R_k\alpha_{n-k}P_{n-k}'(x), \\ = \quad & (-S_0\alpha_{n-1} + R_0\alpha_n)P_n'(x) + (-S_1\alpha_{n-2} - S_0\alpha_{n-1}\beta_{n-1} + R_1\alpha_{n-1})P_{n-1}'(x) \\ & + \sum_{k=2}^{n-1}\left[-S_k\alpha_{n-k-1} - S_{k-1}\alpha_{n-k}\beta_{n-k} - S_{k-2}\alpha_{n-k+1}\omega_{n-k+1}\right. \\ & \left. + R_k\alpha_{n-k}\right]P_{n-k}'(x)\end{aligned}$$

where we recognize the sequence $\{A_n^k\}$. □

Now, we are able to give the proof of theorem 2.1.

Proof. By making the combinations $nEq(3.5)+Eq(3.4)$ and $Eq(3.5)-Eq(3.4)$ we obtain, respectively,

$$(n+1)xP_n'(x) = n\left(P_{n+1}'(x)+\beta_nP_n'(x)+\omega_nP_{n-1}'(x)\right)+\sum_{k=0}^{n-1}A_n^kP_{n-k}'(x) \tag{3.7}$$

and

$$(n+1)P_n(x) = P_{n+1}'(x)+\beta_nP_n'(x)+\omega_nP_{n-1}'(x)-\sum_{k=0}^{n-1}A_n^kP_{n-k}'(x). \tag{3.8}$$

Multiplying (3.8) by x and using (1.2) in the l.h.s gives

$$(n+1)\left(P_{n+1}(x)+\beta_nP_n(x)+\omega_nP_{n-1}(x)\right) = xP_{n+1}'(x)+\beta_nxP_n'(x)+\omega_nxP_{n-1}'(x)-\sum_{k=0}^{n-1}A_n^kxP_{n-k}'(x). \tag{3.9}$$

For the l.h.s (resp. the r.h.s) of (3.9) we use (3.8) (resp. (3.7)) to get

$$\begin{aligned}
&\frac{n+1}{n+2}\left(P_{n+2}'(x)+\beta_{n+1}P_{n+1}'(x)+\omega_{n+1}P_n'(x)\right)-\frac{n+1}{n+2}\sum_{k=0}^{n}A_{n+1}^kP_{n-k+1}'(x)\\
&+\beta_n\left(P_{n+1}'(x)+\beta_nP_n'(x)+\omega_nP_{n-1}'(x)\right)-\beta_n\sum_{k=0}^{n-1}A_n^kP_{n-k}'(x)\\
&+\frac{n+1}{n}\omega_n\left(P_n'(x)+\beta_{n-1}P_{n-1}'(x)+\omega_{n-1}P_{n-2}'(x)\right)-\frac{n+1}{n}\omega_n\sum_{k=0}^{n-2}A_{n-1}^kP_{n-k-1}'(x)=\\
&=\frac{n+1}{n+2}\left(P_{n+2}'(x)+\beta_{n+1}P_{n+1}'(x)+\omega_{n+1}P_n'(x)\right)+\frac{1}{n+2}\sum_{k=0}^{n}A_{n+1}^kP_{n-k+1}'(x)\\
&+\frac{n}{n+1}\beta_n\left(P_{n+1}'(x)+\beta_nP_n'(x)+\omega_nP_{n-1}'(x)\right)+\frac{1}{n+1}\beta_n\sum_{k=0}^{n-1}A_n^kP_{n-k}'(x)\\
&+\frac{n-1}{n}\omega_n\left(P_n'(x)+\beta_{n-1}P_{n-1}'(x)+\omega_{n-1}P_{n-2}'(x)\right)+\frac{1}{n}\omega_n\sum_{k=0}^{n-2}A_{n-1}^kP_{n-k-1}'(x)\\
&-\sum_{k=0}^{n-1}A_n^k\frac{n-k}{n-k+1}\left(P_{n-k+1}'(x)+\beta_{n-k}P_{n-k}'(x)+\omega_{n-k}P_{n-k-1}'(x)\right)\\
&-\sum_{k=0}^{n-1}\frac{A_n^k}{n-k+1}\sum_{l=1}^{n-k-1}A_{n-k}^lP_{n-k-l}'(x),
\end{aligned} \tag{3.10}$$

which can be simplified to

$$\begin{aligned}
&-\sum_{k=0}^{n}A_{n+1}^kP_{n-k+1}'(x)+\frac{1}{n+1}\beta_n\left(P_{n+1}'(x)+\beta_nP_n'(x)+\omega_nP_{n-1}'(x)\right)\\
&-\frac{n+2}{n+1}\beta_n\sum_{k=0}^{n-1}A_n^kP_{n-k}'(x)+\frac{2}{n}\omega_n\left(P_n'(x)+\beta_{n-1}P_{n-1}'(x)+\omega_{n-1}P_{n-2}'(x)\right)\\
&-\frac{n+2}{n}\omega_n\sum_{k=0}^{n-2}A_{n-1}^kP_{n-k-1}'(x)+\sum_{k=0}^{n-1}A_n^k\frac{n-k}{n-k+1}(P_{n-k+1}'(x)+\beta_{n-k}P_{n-k}'(x)\\
&+\omega_{n-k}P_{n-k-1}'(x))+\sum_{k=0}^{n-1}\frac{A_n^k}{n-k+1}\sum_{l=0}^{n-k-1}A_{n-k}^lP_{n-k-l}'(x)=0.
\end{aligned} \tag{3.11}$$

The results follow after a computation of the coefficients of $P'_{n+1}(x)$, $P'_n(x)$, $P'_{n-1}(x)$, $P'_{n-2}(x)$ and $\{P'_{n+1-k}(x)\}_{n\geq k\geq 4}$ in (3.11).

The relations obtained in theorem 2.1 are used in the proof of theorem 2.2 below.

Proof. By comparing the (GF) (2.11) to (1.1) we see that $A(t)=\frac{1}{1-t}$ and $R(t)=-c\ln(1-t)$. So, from (2.1) and (2.2) we have: $S_k=1$, for $k\geq 0$, $R_0=c$ and $R_k=0$, for $k\geq 1$. If we put $a_n=\frac{\alpha_n}{\alpha_{n+1}}$, then sequence $\{A_n^k\}$ becomes

$$\begin{aligned}
A_n^0 &= -a_{n-1}+c,\ n\geq 1, &(3.12)\\
A_n^1 &= -a_{n-1}a_{n-2}-a_{n-1}\beta_{n-1},\ n\geq 2, &(3.13)\\
A_n^2 &= -a_{n-3}a_{n-2}a_{n-1}-a_{n-2}a_{n-1}\beta_{n-2}-a_{n-1}\omega_{n-1},\ n\geq 3, &(3.14)\\
A_n^k &= -\prod_{i=n-k-1}^{n-1}a_i-\left(\prod_{i=n-k}^{n-1}a_i\right)\beta_{n-k}-\left(\prod_{i=n-k+1}^{n-1}a_i\right)\omega_{n-k+1},\ n\geq k+1, k\geq 2. &(3.15)
\end{aligned}$$

Note that the equation (3.15) can be written as

$$A_n^k=\left(\prod_{i=n-k+2}^{n-1}a_i\right)A_{n-k+2}^2,\quad k\geq 3,\ n\geq k+1, \tag{3.16}$$

and by using (3.13) and (3.14) we can write

$$A_n^1=\frac{A_{n+1}^2}{a_n}+\omega_n,\quad n\geq 2. \tag{3.17}$$

Now we use these expressions of A_n^k to get, for this case, the equations of theorem (2.1). By (3.12) and (3.13) the equations (2.6) and (2.7) become, respectively,

$$\beta_n=-(n+1)a_n+na_{n-1}+c,\ n\geq 0 \tag{3.18}$$

and

$$\omega_n=-\frac{c}{2}\left(na_n-(n-1)\,a_{n-1}\right)+\frac{n}{2}\left[(n+1)a_n-2na_{n-1}+(n-1)a_{n-2}\right]a_{n-1},\ n\geq 1. \tag{3.19}$$

According to (3.12), (3.13), (3.14), (3.18) and (3.19) the equation (2.8) takes, after multiplying by $\frac{2}{na_{n-1}a_{n-2}}$, the form

$$\begin{aligned}
c\left(\frac{3\,a_n}{a_{n-1}}+\frac{2\,a_n}{n\,a_{n-2}}-\frac{3\,a_{n-1}}{a_{n-2}}-\frac{2\,n-1}{n(n-1)}\right)-(n+1)\,a_n+3\,na_{n-1}\\
-3\,(n-1)\,a_{n-2}+(n-2)\,a_{n-3}=0,\ n\geq 2.
\end{aligned} \tag{3.20}$$

By substitution of (3.17) in (2.9) and by using (3.16), for $k=3$, we obtain

$$\begin{aligned}
\frac{\omega_{n-1}}{a_n}A_{n+1}^2 &= \left(a_n-\frac{1}{n}\frac{A_{n+1}^2}{a_na_{n-1}}+\frac{n+2}{n+1}\beta_n-\frac{n-2}{n-1}\beta_{n-2}+\frac{n+1}{n}\frac{\omega_n}{a_{n-1}}-\frac{A_n^0}{n+1}-\frac{A_{n-2}^0}{n-1}\right)A_n^2\\
&\quad -\frac{n-3}{n-2}a_{n-1}A_{n-1}^2,\quad n\geq 3.
\end{aligned} \tag{3.21}$$

For the last equation (2.10) we use (3.16) to get

$$\begin{aligned}
&a_nA_{n-k+3}^2-\frac{n-k}{n-k+1}A_{n-k+2}^2+\left(\frac{n+2}{n+1}\beta_n-\frac{n-k+1}{n-k+2}\beta_{n-k+1}\right)A_{n-k+3}^2+\frac{n+2}{n}\frac{\omega_n}{a_{n-1}}A_{n-k+3}^2\\
&-\frac{n-k+2}{n-k+3}\omega_{n-k+2}\,A_{n-k+4}^2=\frac{A_{n-k+3}^2\,A_{n-k+1}^0}{n-k+2}+\frac{A_{n-k+4}^2\,A_{n-k+2}^1}{n-k+3}+\frac{A_n^0\,A_{n-k+3}^2}{n+1}+\frac{A_n^1\,A_{n-k+3}^2}{na_{n-1}}\\
&\qquad+\sum_{l=2}^{k-3}\frac{A_{n-k+l+3}^2\,A_{n-k+3}^2}{(n-k+l+2)a_{n-k+l+1}a_{n-k+l+2}},\quad n\geq k\geq 4
\end{aligned} \tag{3.22}$$

In the sequel, we consider two cases :

Case 1. If $c = 0$, then the equation (3.20) leads, for $n = 2$ and for $n \geq 3$ respectively, to

$$a_2 - 2a_1 + a_0 = 0 \tag{3.23}$$

and

$$-(n+1)\,a_n + 3\,na_{n-1} - 3\,(n-1)\,a_{n-2} + (n-2)\,a_{n-3} = 0,\ n \geq 3. \tag{3.24}$$

By summing thrice the equation (3.24) and in virtue of (3.23) we obtain

$$a_n = \lambda\,n + \nu,\ n \geq 0$$

where $\lambda = a_1 - a_0$ and $\nu = a_0 = \alpha_0/\alpha_1 \neq 0$. So, the equations (3.18) and (3.19) give us

$$\beta_n = -2\,\lambda\,n - \nu, \tag{3.25}$$

and

$$\omega_n = \lambda\,n(\lambda\,n - \lambda + \nu). \tag{3.26}$$

The equations (3.21) and (3.22) are also satisfied, since in this case $A_n^2 = 0$, for $n \geq 3$. As $a_n = \alpha_n/\alpha_{n+1}$ we get

$$\alpha_n = \frac{\alpha_{n-1}}{\lambda\,(n-1) + \nu} = \frac{1}{\prod_{i=0}^{n-1}(\lambda\,i + \nu)},\ n \geq 1. \tag{3.27}$$

If $\lambda = 0$ then $\beta_n = -\nu = -a_0$ for $n \geq 0$, and $\omega_n = 0$ for $n \geq 1$. The power series $F(t)$ takes the form $F(t) = \frac{1}{1-\frac{t}{\nu}}$ which generates the rescaled monomials $P_n(x) = (x+\nu)^n$. If $\lambda \neq 0$, then $F(t)$ becomes

$$F(t) = {}_1F_1\left(\begin{array}{c} 1 \\ \\ \gamma+1 \end{array};\frac{t}{\lambda}\right) \tag{3.28}$$

where $\gamma + 1 = \frac{\nu}{\lambda}$ is a non-negative integer since $a_n = \lambda\,n + \nu = \lambda\,(n + \gamma + 1) \neq 0$ for $n \geq 0$. Here

$${}_1F_1\left(\begin{array}{c} a \\ \\ b \end{array};t\right) = \sum_{k\geq 0}\frac{(a)_k}{(b)_k}\frac{t^k}{k!} \tag{3.29}$$

is the confluent hypergeometric series where a, b are complex numbers and the symbol $(a)_k$ stands for the shifted factorials, i.e., $(a)_0 = 1,\ (a)_k = a(a+1)\cdots(a+k-1),\ \ k \geq 1$.

The polynomials with the recursion (1.2) where $\beta_n = -\lambda\,(2n + 1 + \gamma)$ and $\omega_n = \lambda^2 n\,(n+\gamma) = \lambda n a_{n-1} \neq 0$ for $n \geq 1$, given in (3.25) and (3.26) respectively, are the rescaled orthogonal (MLPs) $(-\lambda)^n\,L_n^{(\gamma)}\,(-x/\lambda)$. Notice that the orthogonal (MLPs) satisfy the three-term recursion relation

$$\begin{cases} xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + (2n+1+\alpha)\,L_n^{(\alpha)}(x) + n\,(n+\alpha)\,L_{n-1}^{(\alpha)}(x)\,, & n \geq 0, \\ L_{-1}^{(\alpha)}(x) = 0,\ \text{ and } L_0^{(\alpha)}(x) = 1. \end{cases} \tag{3.30}$$

and have the following generating function [10, p.263]

$$\frac{1}{1-t}\,{}_1F_1\left(\begin{array}{c} 1 \\ \\ \gamma+1 \end{array};-\frac{xt}{1-t}\right) = \sum_{n\geq 0}(-1)^n\frac{L_n^{(\gamma)}(x)}{(\gamma+1)_n}t^n, \tag{3.31}$$

which is of the form (2.11) with $F(t) = {}_1F_1\left(\begin{array}{c} 1 \\ \\ \gamma+1 \end{array};-t\right)$ and $c = 0$.

Case 2. If $c \neq 0$.

Assume that $A_n^2=0$ for all $n\geq 3$. Then according to (3.18), (3.19) and (3.14), we have

$$c\left(\frac{a_{n-1}}{na_{n-2}}-\frac{1}{n-1}\right)-a_{n-1}+2\,a_{n-2}-a_{n-3}=0,\ n\geq 3. \tag{3.32}$$

Denoting by $E_1(n)$ the equation (3.32) and making the operation $(n-2)E_1(n)-(n+1)E_1(n+1)$ we obtain

$$c\left(-\frac{a_n}{a_{n-1}}+\frac{(n-2)a_{n-1}}{na_{n-2}}+\frac{2n-1}{n(n-1)}\right)+(n+1)a_n-3na_{n-1}+3(n-1)a_{n-2}-(n-2)a_{n-3}=0. \tag{3.33}$$

Again denoting the equations (3.33) and (3.20) by $E_2(n)$ and $E_3(n)$, respectively. Then, the operation $n\frac{a_{n-2}}{a_n}(E_2(n)+E_3(n))$ gives

$$-(n+1)\,\frac{a_{n-1}}{a_n}+n\frac{a_{n-2}}{a_{n-1}}+\,1=0,\ n\geq 3. \tag{3.34}$$

By summing the equation (3.34) we find, for $n\geq 2$,

$$\frac{a_{n-1}}{a_n}=\frac{n-1+b}{n+1},$$

which leads to the expression

$$a_n=\lambda\frac{(n+1)!}{(b)_n}, \tag{3.35}$$

where $\lambda=ba_1/2$ and $b=3a_1/a_2-1$. The substitution of (3.35) in both equations $E_2(n)$ and $E_3(n)$ gives, for $n\geq 2$,

$$\pm\,(b-1)\,[\lambda\,(b-2)\,((b-3)\,n-2\,b+3)\,(n-1+b)\,(n-2+b)\,n! \\ -c\left(n^2+(2b-3)\,n-b+2\right)(b)_n]=0.$$

Thus $b=1$ and for $n\geq 1$,

$$a_n=\lambda\,(n+1). \tag{3.36}$$

Using (3.32) again for $n=3$ and in virtue of (3.36) it follows that $a_0=\lambda$. Therefore,

$$\alpha_n=\frac{1}{\lambda^n\,n!},\quad \text{for } n\geq 1, \tag{3.37}$$

and then

$$F(t)=e^{t/\lambda}. \tag{3.38}$$

Moreover, $\beta_n=-\lambda(2n+1-c/\lambda)$ and $\omega_n=\lambda^2\,n(n-c/\lambda)$ which give again the rescaled (MLPs) $(-\lambda)^n\,L_n^{(\frac{-c}{\lambda})}(-x/\lambda)$.

Now, assume that $A_{n_0}^2\neq 0$ for some $n_0\geq 3$ and denote the equation (3.22) by $E(k,n)$. The operation $E(k+1,n+1)-E(k,n)$ will give, for $n\geq k\geq 4$,

$$\Bigg[a_{n+1}-a_n+\frac{n+3}{n+2}\beta_{n+1}-\frac{n+2}{n+1}\beta_n+\frac{n+3}{n+1}\,\frac{\omega_{n+1}}{a_n}-\frac{n+2}{n}\,\frac{\omega_n}{a_{n-1}} \\ -\frac{A_{n+1}^0}{n+2}+\frac{A_n^0}{n+1}-\frac{A_{n+1}^1}{(n+1)a_n}+\frac{A_n^1}{na_{n-1}}-\frac{A_{n+1}^2}{na_na_{n-1}}\Bigg]\,A_{n-k+3}^2=0,\quad n\geq k\geq 4. \tag{3.39}$$

Then for $k=n-n_0+3$ and $n\geq 4$ the equation (3.39) leads to

$$a_{n+1}-a_n+\frac{n+3}{n+2}\beta_{n+1}-\frac{n+2}{n+1}\beta_n+\frac{n+3}{n+1}\,\frac{\omega_{n+1}}{a_n}-\frac{n+2}{n}\,\frac{\omega_n}{a_{n-1}}-\frac{A_{n+1}^0}{n+2}+\frac{A_n^0}{n+1} \\ -\frac{A_{n+1}^1}{(n+1)a_n}+\frac{A_n^1}{na_{n-1}}-\frac{A_{n+1}^2}{na_na_{n-1}}=0. \tag{3.40}$$

According to (3.12), (3.13), (3.14), (3.18) and (3.19) the equation (3.40) can be written as

$$c\left(-\frac{(n+3)\,a_{n+1}}{(n+1)\,a_n}+\frac{a_n}{a_{n-1}}+\frac{2\,n+1}{n\,(n+1)}\right)+(n+2)\,a_{n+1}$$
$$-3\,(n+1)\,a_n+3\,na_{n-1}-(n-1)\,a_{n-2}=0,\;n\geq 4. \tag{3.41}$$

Denoting by $E_4(n)$ the equation (3.41) then the operation $E_3(n+1)+E_4(n)$ gives

$$\frac{n\,a_{n+1}}{(n+1)\,a_n}-\frac{a_n}{a_{n-1}}+\frac{a_{n+1}}{(n+1)\,a_{n-1}}=0,\;n\geq 4, \tag{3.42}$$

or equivalently, by multiplying by $(n+1)\frac{a_{n-1}}{a_{n+1}}$,

$$-(n+1)\,\frac{a_n}{a_{n+1}}+n\,\frac{a_{n-1}}{a_n}+1=0. \tag{3.43}$$

By summing the equation (3.43) we get

$$\frac{a_n}{a_{n+1}}=\frac{n+b}{n+1},$$

which leads to

$$a_n=\lambda'\frac{n!}{(b')_n}, \tag{3.44}$$

where λ' and b' are constants. The substitution of (3.44) in both equations (3.20) and (3.41) exhibits a contradiction.

□

Remark 3.3. If $A(t)=1/(1-t/a)$ and $R(t)=-c\ln(1-t/a)-bt/(a-t)$ then the set of polynomials is also the rescaled (MLPs) $\{a^{-n}L_n^{\alpha}(ax+b)\}_{n\geq 0}$ for some α. Indeed, we have

$$\frac{1}{1-\frac{t}{a}}F\left(\frac{xt}{1-\frac{t}{a}}+c\ln\left(1-\frac{t}{a}\right)+\frac{b\,t}{a-t}\right)=$$
$$\frac{1}{1-\frac{t}{a}}F\left(\frac{(ax+b)t}{a-t}+c\ln\left(1-\frac{t}{a}\right)\right)=\sum_{n\geq 0}\alpha_n a^{-n}L_n^{\alpha}(ax+b)t^n.$$

4 The Sheffer and Ultraspherical Classes

4.1 The Sheffer Class

The Sheffer class are polynomials generated by $V(t)\exp(xtU(t))$ which can be written in the form $A(t)F(xtA(t)-R(t))$ as follows:

$$V(t)\exp(xU(t))=U(t)\exp\left(xtU(t)-\ln\left(\frac{U(t)}{V(t)}\right)\right),$$

with $A(t)=U(t)$, $F(t)=\exp(t)$ and $R(t)=\ln\left(U(t)/V(t)\right)$. If they satisfy the 2-order recursion (1.2), then equations (2.3) and (2.4) become

$$A_n^0=-S_0\frac{\alpha_{n-1}}{\alpha_n}+R_0=-S_0 n+R_0,\;n\geq 1, \tag{4.1}$$

and

$$A_n^1=-S_1 n(n-1)-S_0 n\beta_{n-1}+R_1 n,\;n\geq 2. \tag{4.2}$$

By theorem (2.1) we find that

$$\beta_n\;=\;(n+1)A_{n+1}^0-nA_n^0=-2S_0 n-S_0+R_0=an+b,\;n\geq 0. \tag{4.3}$$

and

$$\omega_n=n(cn+d).$$

We recognize here the Jacobi-Szegö parameters of the Meixner polynomials [12, 18].

4.2 The Ultraspherical Class

For the ultraspherical class generated by

$$A(t)(1 - (xtA(t) - R(t)))^{-\lambda}, \tag{4.4}$$

we have

$$\alpha_n = \frac{(\lambda)_n}{n!} \text{ and } a_n = \frac{\alpha_n}{\alpha_{n+1}} = \frac{n+1}{\lambda + n}.$$

If they satisfy the 2-order recursion (1.2), then from theorem (2.1) we get

$$\beta_n = R_0 - S_0 + \frac{S_0(\lambda - 1)^2}{(\lambda + n)(\lambda + n - 1)}. \tag{4.5}$$

So, by making the translation $x \to x + R_0 - S_0$, we can take $R_0 = S_0$ to obtain

$$\beta_n = \frac{S_0(\lambda - 1)^2}{(\lambda + n)(\lambda + n - 1)} \tag{4.6}$$

and

$$\begin{aligned} \omega_n &= \frac{R_1 - S_1}{2} - \frac{S_0^2(\lambda - 1)^4}{(\lambda + n - 1)^2} + \frac{(\lambda - 1)\left(2(S_0^2 + S_1)(\lambda - 1)^2 + \lambda(S_1 - R_1)\right)}{2(\lambda + n - 1)} \\ &- \frac{\lambda(\lambda - 1)(S_0^2(\lambda - 1)^2 + \lambda S_1 - R_1)}{2(\lambda + n)} + \frac{(\lambda - 2)(\lambda - 1)^2(S_0^2(\lambda - 1) - S_1)}{2(\lambda + n - 2)}. \end{aligned} \tag{4.7}$$

Remark that for $\lambda = 1$ we have $\beta_n = 0$ and $\omega_n = (R_1 - S_1)/2$. We meet here the Chebyshev polynomials of the second kind generated by $\left(1 - 2xt + t^2\right)^{-1}$.

Now assume that $\lambda \neq 1$ and consider two cases:

- **The symmetric case:** $S_0 = 0$

  From (2.9) we must have $2(R_3 - S_3) = S_1(R_1 - S_1)$. If $R_3 = S_3$ then $S_1 = 0$ or $S_1 = R_1$. For $S_1 = R_1$ we find from (2.9) that $R_3 = R_1 = 0$ and then $\omega_n = 0$. For $S_1 = 0$ we get the rescaled ultraspherical polynomials $(2R_1)^{n/2}\, \tilde{C}_n^{\lambda}\left(x/\sqrt{2R_1}\right)$ with

  $$\omega_n = \frac{R_1 n(n + 2\lambda - 1)}{2(\lambda + n)(\lambda + n - 1)}. \tag{4.8}$$

  We recall that for the monic ultraspherical polynomials $\tilde{C}_n^{\lambda}$ we have

  $$\omega_n = \frac{n(n + 2\lambda - 1)}{4(\lambda + n)(\lambda + n - 1)} \tag{4.9}$$

  and

  $$\sum_{n \geq 0} \frac{(\lambda)_n}{n!} \tilde{C}_n^{\lambda}(x) t^n = \left(1 - xt + \frac{t^2}{4}\right)^{-\lambda} \tag{4.10}$$

  with $A(t) = 1$ and $R(t) = t^2/4$ in (4.4). If $R_3 \neq S_3$ then from (2.9) we get $S_1 = R_1/\lambda$ which gives the rescaled ultraspherical (PS) $(2R_1)^{n/2}\, \tilde{C}_n^{\lambda-1}\left(x/\sqrt{2R_1}\right)$ with

  $$\omega_n = \frac{R_1(\lambda - 1)n(n + 2\lambda - 3)}{2\lambda(\lambda + n - 1)(\lambda + n - 2)}. \tag{4.11}$$

  Here we can proceed as in [9] to obtain the (GF)

  $$\sum_{n \geq 0} \frac{(\lambda)_n}{n!} \tilde{C}_n^{\lambda-1}(x) t^n = \left(1 - \frac{t^2}{4}\right)\left(1 - xt + \frac{t^2}{4}\right)^{-\lambda} \tag{4.12}$$

  which is (4.4) with $A(t) = \left(1 - t^2/4\right)^{1/(1-\lambda)}$ and $R(t) = \left(1 + t^2/4\right) A(t) - 1$.

- **The non symmetric case :** $S_0 \neq 0$

If $R_2 = S_2$ then (2.9) gives $S_1 = R_1 = 0$ and $\omega_n = 0$. If $R_2 \neq S_2$ then by (2.9) we get

$$\beta_n = \frac{S_0(\lambda - 1)^2}{(\lambda + n)(\lambda + n - 1)} \tag{4.13}$$

and

$$\omega_n = \frac{S_0^2(\lambda - 1)^2 n(n + 2\lambda - 2)}{(\lambda + n - 1)^2} \tag{4.14}$$

giving the rescaled Jacobi polynomials $\eta^n \tilde{P}_n^{(\lambda-1/2,\lambda-3/2)}(x/\eta) = (-\eta)^n \tilde{P}_n^{(\lambda-3/2,\lambda-1/2)}(-x/\eta)$ where $\eta = 2S_0(1 - \lambda)$. The Jacobi-Szegö parameters of the monic Jacobi polynomials $\tilde{P}_n^{(\alpha,\beta)}$ are given by [9]

$$\beta_n = \frac{\beta^2 - \alpha^2}{4(n + (\alpha + \beta)/2)(n + (\alpha + \beta)/2 + 1)} \tag{4.15}$$

and

$$\omega_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{4(n + (\alpha + \beta - 1)/2)(n + (\alpha + \beta + 1)/2)(n + (\alpha + \beta)/2)^2}. \tag{4.16}$$

The (GF) for $\tilde{P}_n^{(\lambda-1/2,\lambda-3/2)}$ has the form (see [9])

$$\sum_{n\geq 0} \frac{(\lambda)_n}{n!} \tilde{P}_n^{(\lambda-1/2,\lambda-3/2)}(x) t^n = \left(1 + \frac{t}{2}\right) \left(1 - xt + \frac{t^2}{4}\right)^{-\lambda} \tag{4.17}$$

which is (4.4) with $A(t) = (1 + t/2)^{1/(1-\lambda)}$ and $R(t) = \left(1 + t^2/4\right) A(t) - 1$.

□

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Received: 2024-01-29
Accepted: 2025-06-30