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On Nil-Symmetric Modules

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Abstract *In this paper, we have introduced the notion of nil-symmetric modules as a generalisation of symmetric modules and reduced modules by working on the context of nilpotent elements of a module and have also investigated some of its properties. We have also extended various results on symmetric and other classes of modules to that of nil-symmetric modules and have also shown that there is a module which is nil-symmetric but not symmetric. We prove that localizations of nil-symmetric modules are nil-symmetric. It has also been shown that* $_RM$ *is nil-symmetric if and only if* $_{T_n(R)}T_n(M)$ *is nil-symmetric.*

1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules over the ring R. $T_n(R)$ *denotes the ring of all* $n \times n$ *upper triangular matrices over R. Let* $T(M) = \{m \in M : rm = 0$ *for some non-zero divisors* $r \in R\}$*. Torsion of* M *is defined as* $Tor(M) = \{m \in M : rm = 0$ *for some non-zero* $r \in R\}$*. Clearly,* $T(M) \subseteq Tor(M)$*. If* R *is an integral domain, they are same.* $C(R)$ *denotes the centre of a ring* R *and defined by* $C(R) = \{r \in R : ra = ar$ *for all* $a \in R\}$*. Here,* D *denotes a non-commutative domain.* $Nil_R(M)$ *is the set of all nilpotent elements of a left* R*-module* M*.*

Recall in [2], J. Lambek introduced the notion of symmetric ring. A ring R is symmetric if whenever $a, b, c \in R$ *satisfy* $abc = 0$*, we have* $bac = 0$*; it is easily seen that this is left-right symmetric concept. U.S. Chakraborty and K. Das introduced the concept of nil-symmetric rings as a generalisation of symmetric rings and a particular case of nil-semicommutative rings in [11]. A ring* R *is called right (left) nil-symmetric if whenever, for every* $a, b \in nil(R)$ *and for every* $c \in R$ *satisfy* $abc = 0 (cab = 0)$*, we have* $acb = 0$*. A ring* R *is nil-symmetric if it is both right and left nil-symmetric. Thus, every symmetric ring is nil-symmetric but the converse need not be true in general as in [[11], Example 3], if* R *is a reduced ring, then* $T_2(R)$ *is a nil-symmetric ring but not symmetric.*

In [2] and [9], a module $_RM$ *is symmetric if whenever* $a, b \in R, m \in M$ *satisfy* $abm = 0$*, we have* $bam = 0$*. M. B. Rege and A. M. Buhphang studied various properties of symmetric modules. The relationship of symmetric modules with reduced modules were also studied in [8]. Symmetric modules were generalised to* α*-symmetric modules by Agayev, Halicioglu and Harmanci in [6].*

A ring R *is reduced if it has no non-zero nilpotent elements. The reduced ring concept was extended to modules by Lee and Zhou in [10]. In [5], the relationship of reduced modules with ZI-modules was studied by Agayev and Harmanci. A left* R*-module* M *is reduced if it satisfies any of the following conditions:*

(i) whenever $a \in R, m \in M$ *satisfy* $a^2m = 0$*, we have* $aRm = 0$*.*

(ii) whenever $a \in R, m \in M$ *satisfy* $am = 0$*, we have* $aM \cap Rm = 0$*. In [4], M. Dutta and Singh introduced the idea of weak reduced and weak rigid module as a generalisation of reduced*

and rigid module. They stated that a left R-module M is weak reduced if whenever $a^2m = 0$ $\forall a \in R$ and $m \in M$ implies $aRm \subseteq Nil_R(M)$ and a left R-module M is weak rigid if whenever $a^2m = 0$ $\forall a \in R$ and $m \in M$ implies $am \in Nil_R(M)$.

In [1], Ssevviiri and Groenewald introduced the concept of nilpotent elements of a module. A non-zero element $m \in M$ is said to be a nilpotent element of M if there exist $0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0$ but $rm \neq 0$. We take the zero element of M as a nilpotent element. In this paper the term "nil" is used to generalize symmetric module by using the definition of nilpotent elements of a module.

Recall, a left R-module M is called semicommutative (a ZI-module) if whenever $am = 0$ implies $aRm = 0$ for all $a \in R$ and $m \in M$. In [7], Ansari and Singh introduced weakly semicommutative module as a generalisation of semicommutative module. A left R-module M is said to be weakly semicommutative if whenever $am = 0$ implies $aRm \subseteq Nil_R(M)$ for all $a \in R$ and $m \in M$.

2 Nilpotent elements of modules

In [1], nilpotent elements of a module can be defined as:

Definition 2.1. An element $m \in M$ is said to be a nilpotent element if either $m = 0$ or there exist $0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0$ but $rm \neq 0$, i.e., $Nil_R(M) = \{m \in M | \exists 0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0, rm \neq 0\} \cup \{0\}$.

In [7], it is stated that if m is an element of a left R-module M, then the following conditions are equivalent:

(i) There exist $r \in R$ and $n \geq 2$ such that $r^n m = 0$ but $r^{n-1} m \neq 0$.

(ii) There exists $t \in R$ such that $t^2 m = 0$ but $tm \neq 0$.

In [7], we have, if $m \in M$ satisfies any of the above equivalent conditions, then m is a nilpotent element of the left R-module M.

Example 2.2. Some examples of nilpotent elements of modules are given below:

(i) Let $M = \mathbb{Z}_8$ and $R = \mathbb{Z}_8$.

Here, $2^3.\bar{1} = 0$ but $2.\bar{1} \neq 0$

$2^2.\bar{2} = 0$ but $2.\bar{2} \neq 0$

$2^3.\bar{3} = 0$ but $2.\bar{3} \neq 0$

$2^3.\bar{5} = 0$ but $2.\bar{5} \neq 0$

$2^2.\bar{6} = 0$ but $2.\bar{6} \neq 0$

$2^3.\bar{7} = 0$ but $2.\bar{7} \neq 0$

Clearly, $Nil_{\mathbb{Z}_8}(\mathbb{Z}_8) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{5}, \bar{6}, \bar{7}\}$.

(ii) If $a \in R$ is nilpotent (with degree $n \geq 3$) in the ring R, then we have $a^{n-1}.a = a^n = 0$ and $a.a = a^2 \neq 0$. Thus, a is nilpotent in the left R-module R.

3 Nil-symmetric modules

In this section, we introduced the class of nil-symmetric modules as a generalisation of symmetric modules and reduced modules. We also show that there are nil-symmetric modules which are not symmetric.

Definition 3.1. [2] A left R-module M is said to be symmetric if whenever $a, b \in R, m \in M$ satisfy $abm = 0$ implies $bam = 0$.

Definition 3.2. A left R-module M is said to be nil-symmetric if whenever $a, b \in R, m \in M$ satisfy $abm = 0$ implies $bam \in Nil_R(M)$.

Remark 3.3. From the definition, the following remarks can be obtained.

(1) All modules over commutative rings are nil-symmetric modules.

(2) Submodules of nil-symmetric modules are nil-symmetric.

Recall, in [3] the concept of generalized weakly symmetric rings were studied. A ring R is called generalized weakly symmetric if $abc = 0$ implies that bac is nilpotent for all $a, b, b \in R$.

Theorem 3.4. *If R is a generalized weakly symmetric ring with nilpotency index greater than 2, then the left R-module R is nil-symmetric.*

Proof: Let $a, b, m \in R$ with $abm = 0$. Since R is a nil-symmetric ring $\implies bam = 0 \in Nil(R) \implies (bam)^k = 0, k \in \mathbb{N} \implies (bam)^{k-1}(bam) = 0 \implies s^{k_0}bam = 0, sbam \neq 0$ where $s = bam, k_0 = k - 1 \implies bam \in Nil_R(R)$. Hence, ${}_RR$ is nil-symmetric.

Lemma 3.5. *[8] All reduced modules are symmetric modules.*

Theorem 3.6. *All symmetric modules are nil-symmetric modules.*

Proof: Let M be a symmetric module. Let $a, b \in R$ and $m \in M$ with $abm = 0$. Then, $bam = 0 \in Nil_R(M) \implies bam \in Nil_R(M)$.

Remark 3.7. The converse of Theorem 3.6 is not true in general which is shown in Example 3.30. The above Lemma 3.5 and Theorem 3.6 give Corollary 3.8.

Corollary 3.8. *All reduced modules are nil-symmetric modules.*

Lemma 3.9. *[8] Symmetric modules are semicommutative.*

Remark 3.10. Nil-symmetric modules are not semicommutative.

Example 3.11. Let $M = \mathbb{Z}$. Then, M is nil-symmetric. Hence, ${}_{T_n(\mathbb{Z})}T_n(\mathbb{Z})$ is nil-symmetric by Theorem 3.28. Let $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $e_{11}e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But $e_{11}e_{12}e_{22} = e_{12}e_{22} = e_{12} \neq 0$. So, M is not semicommutative.

Theorem 3.12. *All nil-symmetric modules are weakly semicommutative.*

Proof: Let M be a nil-symmetric module. Let $a \in R, m \in M$ with $am = 0 \implies bam = 0$ for all $b \in R$. Since M is nil-symmetric $\implies abm \in Nil_R(M) \implies aRm \subseteq Nil_R(M)$. Hence, M is weakly semicommutative.

Next, we recall a torsion free module. A module having no non-zero torsion elements is called a torsion free-module,i.e., $0 \neq m$ is torsion free if $rm = 0, r \in R \implies r = 0$. We recall a result in [7].

Theorem 3.13. *If M is a torsion free left R-module, then $Nil_R(M) = \{0\}$.*

In [7], the converse of the above Theorem 3.13 need not be true in general,i.e., there exists a left R-module M such that $Nil_R(M) = 0$ but M is not torsion free by the following example.

Example 3.14. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p$, where p is a prime number. Then, $\bar{1} \in Tor({}_{\mathbb{Z}}\mathbb{Z}_p)$ as $p.\bar{1} = \bar{0}$. Thus, $Tor({}_{\mathbb{Z}}\mathbb{Z}_p) \neq 0$. Let $\bar{0} \neq \bar{a} \in Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. Then, by definition there exist $r \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $r^k\bar{a} = \bar{0}$ and $r\bar{a} \neq \bar{0}$ implies $p|r^ka$ which again implies $p|r^k$ or $p|a$. If $p|a$, then $r\bar{a} = 0$ and thus $\bar{a} \notin Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. Suppose $p|r^k$ which implies $p|r.r^{k-1}$. Again, $p|r$ or $p|r^{k-1}$. If $p|r$, then $r\bar{a} = \bar{0}$ and hence $\bar{a} \notin Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. On the other suppose $p|r^{k-1}$, then by continuing we get $p|r$ and hence $r\bar{a} = 0$. Thus, $Nil_{\mathbb{Z}}(\mathbb{Z}_p) = 0$.

Here, we have found some conditions for which symmetric and nil-symmetric modules are equivalent which is given below.

Theorem 3.15. *Let M be a torsion free left R-module. Then, M is symmetric if and only if M is nil-symmetric.*

Proof: Let M be nil-symmetric. Also, let $a, b \in R, m \in M$ with $abm = 0$. Since M is nil-symmetric, $bam \in Nil_R(M)$. Now, since M is torsion free, $Nil_R(M) = 0$. Therefore, $bam \in \{0\} \implies bam = 0$. Hence, M is symmetric.

The converse part follows from Theorem 3.6.

Theorem 3.16. *Let M be a nil-symmetric module over a domain D. Then, $T(M)$ is a submodule of M.*

Proof: Let $m_1, m_2 \in T(M)$. Then, there exist $0 \neq r_1, 0 \neq r_2 \in R$ such that $r_1 m_1 = 0, r_2 m_2 = 0 \implies r_2 r_1 m_1 = 0, r_1 r_2 m_2 = 0 \implies r_1 r_2 m_1 \in Nil_D(M), r_2 r_1 m_2 \in Nil_D(M)$. Then, there exist $0 \neq t \in D$ and $n \in \mathbb{N}$ such that $t^n r_1 r_2 m_1 = 0, t r_1 r_2 m_1 \neq 0$. Now, $t^n r_1 r_2 (m_1 - m_2) = t^n r_1 r_2 m_1 - t^n r_1 r_2 m_2 = 0$ which implies $m_1 - m_2 \in T(M)$. Also, let $m \in T(M) \implies rm = 0$ for some $0 \neq r \in D \implies arm = 0\ \forall a \in D$. Since M is a nil-symmetric module and $arm = 0 \implies ram \in Nil_D(M)$. Then, there exist $0 \neq t \in D$ and $n \in \mathbb{N}$ such that $t^n ram = 0, tram \neq 0$. Since D is domain, we have $t^n r \neq 0$. Thus, $am \in T(M)$. Hence proved.

Lemma 3.17. *[4] If $_RN$ is a submodule of $_RM$, then $Nil_R(N) \subseteq Nil_R(M)$.*

Theorem 3.18. *A left R-module M is nil-symmetric if and only if every cyclic submodule of M is nil-symmetric.*

Proof: Let M be nil-symmetric. Since submodules of nil-symmetric modules are nil-symmetric, every cyclic submodule of M is nil-symmetric.

Conversely, let $a, b \in R, m \in M$ satisfying $abm = 0$. Since $m \in M, m = 1.m \in Rm$ which is cyclic $\implies m \in Rm \subseteq M \implies abm = 0$. Since Rm is a nil-symmetric module $\implies bam \in Nil_R(Rm) \implies bam \in Nil_R(M)$. Hence M is nil-symmetric.

Theorem 3.19. *A left R-module M is nil-symmetric if and only if every finitely generated submodule of M is nil-symmetric.*

Proof: Let M be nil-symmetric. Since submodules of nil-symmetric modules are nil-symmetric, every finitely generated submodule of M is nil-symmetric.

The converse is clear by Theorem 3.18.

In the next theorem, we give a condition on a submodule N of a left R-module M which is sufficient for the nil-symmetricity of $\frac{M}{N}$ to imply nil-symmetricity of M.

Theorem 3.20. *Let M be a left-R module over a commutatative ring R and N be a submodule of M such that $N \subseteq Nil_R(M)$. If $\frac{M}{N}$ is nil-symmetric, then M is nil-symmetric.*

Proof: Let $a, b \in R$ and $m \in M$ with $abm = 0$. Then, we have, $ab\bar{m} = 0$. Since $\frac{M}{N}$ is nil-symmetric, $ba\bar{m} \in Nil_R(\frac{M}{N})$. Then, there exist $r \in R, k \in \mathbb{N}$ such that $r^k ba\bar{m} = \bar{0}, rba\bar{m} \neq \bar{0} \implies r^k ba(m+N) = \bar{0}, rba(m+N) \neq \bar{0} \implies r^k bam + N = 0 + N, rbam + N \neq 0 + N \implies r^k bam \in N$. Since $N \subseteq Nil_R(M)$, we have, $r^k bam \in Nil_R(M)$. Then, there exist $p \in R, s \in \mathbb{N}$ such that $p^s r^k bam = 0, pr^k bam \neq 0$. Since R is commutative, we have, $(pr)^{max(s,k)} bam = 0, prbam \neq 0$ as $pr^k bam \neq 0 \implies bam \in Nil_R(M)$. Hence, M is nil-symmetric.

Theorem 3.21. *Let M be a left R-module over an integral domain R. If M is nil-symmetric, then $\frac{M}{T(M)}$ is symmetric.*

Proof: The proof is obvious as R is commutative.

Corollary 3.22. *Let M be a left R-module over an integral domain R. If M is nil-symmetric, then $\frac{M}{T(M)}$ is nil-symmetric.*

Theorem 3.23. *Let $\theta : R \to R'$ be a ring homomorphism and let M be an R'-module. Then, M can be made as an R-module by defining $am = \theta(a)m$. If θ is onto, the following are equivalent:*

(1) M is a nil-symmetric R'-module.

(2) M is a nil-symmetric R-module.

Proof: $(1) \implies (2)$ Let $abm = 0\ \forall a, b \in R, m \in M \implies \theta(ab)m = 0 \implies \theta(a)\theta(b)m = 0$ in $_{R'}M$. Since $_{R'}M$ is nil-symmetric, $\theta(b)\theta(a)m \in Nil_{R'}(M) \implies \exists t \in R'$ and $k \in \mathbb{N}$ such that $t^k\theta(b)\theta(a)m = 0, t\theta(b)\theta(a)m \neq 0$. Since θ is onto, there exists $l \in R$ such that $\theta(l) = t$. Now, $l^k bam = \theta(l)^k\theta(b)\theta(a)m = t^k\theta(b)\theta(a)m = 0$ and $t\theta(b)\theta(a)m \neq 0$ implies $\theta(l)\theta(b)\theta(a)m \neq 0$, and so $lbam \neq 0$. Therefore, $bam \in Nil_R(M)$. Hence, M is a nil-symmetric R-module.

$(2) \implies (1)$ Let $a'b'm = 0\ \forall a', b' \in R', m \in M$. Since θ is onto, there exist $r \in R, l \in R$ such that $\theta(r) = a', \theta(l) = b'$. Now, $\theta(r)\theta(l)m = 0 \implies \theta(rl)m = 0 \implies rlm = 0 \implies lrm \in Nil_R(M)$. Then, there exist $t \in R$ and $n \in \mathbb{N}$ such that $t^n lrm = 0$ and $tlrm \neq 0$. Then, $b'a'm \in Nil_{R'}(M)$. Hence, M is a nil-symmetric R'-module.

Next, we study localisations. Recall that if R is a commutative ring and S is a multiplicatively closed subset of R consisting of $C(R)-\{0\}$ and without zero divisor, then $S^{-1}R$ has a ring structure with unity known as ring of fractions. If R is an integral domain and $S = R - \{0\}$, then the ring of fractions $S^{-1}R$ is called field of fractions. If M is a left R-module, then $S^{-1}M$ can be made as an $S^{-1}R$-module. By applying standard localisations techniques, we can prove Theorem 3.24 and Corollary 3.25.

Theorem 3.24. *Let R be a ring and S be a multiplicatively closed subset of $C(R)-\{0\}$. Then, M is a nil-symmetric R-module if and only if $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module.*

Proof: Consider M to be a nil-symmetric R-module. Let $\frac{a}{r}\frac{b}{s}\frac{m}{t} = 0$ in $S^{-1}M$ where $\frac{m}{t} \in S^{-1}M, \frac{a}{r}, \frac{b}{s} \in S^{-1}R \implies u_1abm = 0$ for some $u_1 \in R \implies abm = 0$. Since M is a nil-symmetric R-module, we have $bam \in Nil_R(M)$. Then, there exist $0 \neq t \in R$ and $n \in \mathbb{N}$ such that $t^n bam = 0$ and $tbam \neq 0$. Now, $t^n\frac{b}{s}\frac{a}{r}\frac{m}{t} = \frac{t^n bam}{srt} = 0$ and $t\frac{b}{s}\frac{a}{r}\frac{m}{t} = \frac{tbam}{srt} \neq 0$ as $tbam \neq 0$. Therefore, $\frac{b}{s}\frac{a}{r}\frac{m}{t} \in Nil_{S^{-1}R}(S^{-1}M)$. Hence, $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module.

Conversely, let $a, b \in R$ and $m \in M$ with $abm = 0 \implies \frac{a}{1}\frac{b}{1}\frac{m}{1} = 0$. Since $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module, we have $\frac{b}{1}\frac{a}{1}\frac{m}{1} \in Nil_{S^{-1}R}(S^{-1}M)$. Then, there exist $\frac{t}{s} \in S^{-1}R$ and $n \in \mathbb{N}$ such that $(\frac{t}{s})^n\frac{b}{1}\frac{a}{1}\frac{m}{1} = 0 \implies t^n bam = 0 \implies u_1(t^n bam - 0) = 0$ for some $u_1 \in S \implies u_1t^n bam = 0 \implies t^n bam = 0$ and $\frac{t}{s}bam \neq 0 \implies u(tbam - 0.s) \neq 0$ for all $u \in S \implies utbam \neq 0$ for all $u \in S \implies tbam \neq 0$ for $u = 1$. Therefore, $bam \in Nil_R(M)$. Hence, M is a nil-symmetric R-module.

Corollary 3.25. *For a left R-module M, ${}_{R[x]}M[x]$ is nil-symmetric if and only if ${}_{R[x,x^{-1}]}M[x, x^{-1}]$ is nil-symmetric.*

Proof: Let $S = \{1, x, x^2, ...\}$. Then, S is a multiplicatively closed subset of $R[x]$ consisting of central elements of $R[x]$. Since $S^{-1}M[x] = M[x, x^{-1}]$ and $S^{-1}R[x] = R[x, x^{-1}]$, the result is clear from Theorem 3.24.

Lemma 3.26. *[4] Let M be a left R-module. Then, $Nil_{M_n(R)}M_n(M) = M_n(M)$ for $n \geq 2$.*

Theorem 3.27. *For a left R-module M, ${}_{M_n(R)}M_n(M)$ is nil-symmetric for $n \geq 2$.*

Proof: Let $ABL = 0\ \forall A, B \in M_n(R)$ and $L \in M_n(M)$. Then, $BAL \in M_n(M) = Nil_{M_n(R)}M_n(M) \implies BAL \in Nil_{M_n(R)}M_n(M)$. Hence, ${}_{M_n(R)}M_n(M)$ is a nil-symmetric module.

Theorem 3.28. *A left R-module M is a nil-symmetric module if and only if for any $n \in \mathbb{N}$, ${}_{T_n(R)}T_n(M)$ is a nil-symmetric module.*

Proof: Consider M to be a nil-symmetric module. Let $A = (a_{ij}), B = (b_{ij}) \in T_n(R)$ and $L = (m_{ij}) \in T_n(M)$ with $ABL = 0$. Then, $a_{ii}b_{ii}m_{ii} = 0\ \forall\ 0 < i \leq n$. Since ${}_RM$ is nil-symmetric, we have $b_{ii}a_{ii}m_{ii} \in Nil_R(M)\ \forall\ 0 < i \leq n$.

$$\text{Now, } BAL = \begin{bmatrix} b_{11}a_{11}m_{11} & * & * & \cdots & * \\ 0 & b_{22}a_{22}m_{22} & * & \cdots & * \\ 0 & 0 & b_{33}a_{33}m_{33} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn}a_{nn}m_{nn} \end{bmatrix}.$$

Since $b_{nn}a_{nn}m_{nn} \in Nil_R(M)$, there exist $t_n \in R$ and $n \in \mathbb{N}$ such that $t_n^k b_{nn}a_{nn}m_{nn} = 0$ and $t_nb_{nn}a_{nn}m_{nn} \neq 0$. Choose $T = diag(0, 0, ..., t_n)$, we have $T^kBAL = 0$ and $TBAL \neq 0$.

The converse part is easily seen that submodules of nil-symmetric modules are nil-symmetric, then so is ${}_RM$.

Corollary 3.29. *Let ${}_RM$ be a symmetric module. Then, for any $n \in \mathbb{N}, {}_{T_n(R)}T_n(M)$ is a nil-symmetric module.*

Here, we have given an example of a module which is nil-symmetric but not symmetric.

Example 3.30. Let $M = \mathbb{Z}, R = \mathbb{Z}$. Then, ${}_{\mathbb{Z}}\mathbb{Z}$ is a nil-symmetric module by Remark 3.3(1). So, ${}_{T_2(\mathbb{Z})}T_2(\mathbb{Z})$ is a nil-symmetric module but it is not symmetric module as let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then, $ABC = 0$. But $BAC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. For a left R-module M and $B = (a_{ij}) \in M_n(R)$, let $MB = \{(a_{ij}m) : m \in M\}$. For unit matrices $\{E_{ij} : 1 \leq i, j \leq n\}$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ for $n \geq 2$. Let $V_n(R) = RI_n + RV + RV^2 + ... + RV^{n-1}$ and $V_n(M) = MI_n + MV + MV^2 + ... + MV^{n-1}$. Then, $V_n(R)$ forms a ring and $V_n(M)$ forms a left R-module over $V_n(R)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\theta : V_n(R) \to \frac{R[x]}{(x^n)}$ given by $\theta(r_o I_n + r_1 V + ... + r_{n-1}V^{n-1}) = r_o + r_1 x + ... + r_{n-1}V^{n-1} + (x^n)$ and an abelian group isomorphism $\phi : V_n(M) \to \frac{M[x]}{M[x](x^n)}$ defined by $\phi(m_o I_n + m_1 V + ... + m_{n-1}V^{n-1}) = m_o + m_1 x + ... + m_{n-1}V^{n-1} + M[x](x^n)$ such that $\phi(AW) = \theta(A)\phi(W)$ for all $A \in V_n(R)$ and $W \in V_n(M)$.

Theorem 3.31. *Let M be a left R-module. If M is nil-symmetric module, then for any $n \geq 2$, $\frac{M[x]}{M[x](x^n)}$ is a nil-symmetric module over $\frac{R[x]}{(x^n)}$.*

Proof: From the above remark we can easily prove that if ${}_RM$ is nil-symmetric, then ${}_{V_n(R)}V_n(M)$ is a nil-symmetric for $n \geq 2$. Thus, the proof follows from Theorem 3.28 given above.

4 Conclusion remarks

Remark 4.1. We conclude this note with the following questions.

(1) Is a direct product of nil-symmetric modules nil-symmetric?

(2) Is there any relation between $Nil_R(M)[x]$ and $Nil_{R[x]}M[x]$?

(3) Is a direct sum of nil-symmetric modules nil-symmetric?

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