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When every regular ideal is flat

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Abstract In this paper, we introduce a new class of rings called RF-rings in which every regular ideal is flat. A ring with $wgldim(R) \leq 1$ is naturally an RF-ring; and in the domain context, these two forms coincide to become a Prüfer domain. We study the transfer of this notion to various context of commutative ring extensions such as localization, direct product, trivial ring extensions and pullbacks. Using these results, we construct several classes of examples of RF-rings.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring, we denote by $Reg(R)$ and $Z(R)$ the set of all regular elements of R and the set of all zero-divisors of R respectively. By a “local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

In 1932, Prüfer introduced and studied integral domains in which every non-zero finitely generated ideal is invertible [39]. In 1936, Krull [30] named these rings after H. Prüfer and stated equivalent conditions for a ring to be a Prüfer domain. Since then, “Prüfer domains have assumed a central role in the development of multiplicative ideal theory through numeral equivalent forms. These touched on many areas of commutative algebra, e.g., valuation theory, arithmetic relations on the set of ideals, $*$-operations, and polynomial rings; in addition to several homological characterizations" (Gilmer [17]).

The extension of this concept to rings with zero-divisors gives rise to five classes of Prüfer-like rings featuring some homological apsects (Bazzoni-Glaz [5] and Glaz [21]). At this point, we have :

Semi-hereditary $\Rightarrow$ weak global dimension $\leq 1 \Rightarrow$ Arithmetical $\Rightarrow$ Gaussian $\Rightarrow$ Prüfer

In the domain context, all these forms coincide with the definition of a Prüfer domain. Glaz [21] provides examples which show that all these notions are distinct in the context of arbitrary rings. See for instance, [5, 6, 20, 21, 31, 32, 41].

In this paper, we introduce a RF-notion which is another characterization of a Prüfer domain. A ring is called RF-ring, if every regular ideal is flat. A ring with $wgdim(R) \leq 1$ is naturally an RF-ring, and in the domain context, these two forms coincide and which is a Prüfer domain.

Let A be a ring and E an A-module. Then $A \propto E$, the trivial ring extension of A by E, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [18, 25]. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R; ideals of R need not be of this form [28, Example 2.5]. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [25, Theorem 25.1(3)]. If (A, M) is a local ring with maximal ideal M and E an A-module with $ME = 0$, then $R := A \propto E$ is local total ring of fractions from [28, Proof of Theorem 2.6]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties and for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [2, 9, 10, 11, 12, 13, 18, 25, 26, 28, 30, 34, 36, 37].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M . Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T, such that $D = R/M$. The ring R is known as the pullback associated to the following pullback diagram:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|_R} & D = R/M \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where i and j are the natural injections.
A particular case of this pullback is the $D + M$-construction, when the ring T is of the form $K + M$, where K is a field and M is a maximal ideal of T, and R takes the form $D + M$. See for instance [4, 5, 18, 33].

In this paper, we investigate the possible transfer of RF-property to the direct product of rings, to various trivial extension constructions, and to a particular pullbacks. Using these results, we construct several classes of examples of RF-rings.

2 Main Results

A ring is called an RF-ring, if every regular ideal is flat. Now we give the following natural results.

Proposition 2.1. *Let R be a ring. Then :*

(i) If $wgdim(R) \leq 1$, then R is an RF-ring.

(ii) Assume that R is an integral domain. Then R is an RF-ring if and only if $wgdim(R) \leq 1$ if and only if R is a Prûfer domain.

(iii) A total ring is an RF-ring.

Proof. Straightforward. □

First, we construct an RF-ring such that $wgdim(R) \geq 2$.

Example 2.2. Let $R = \mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$, where $E := (\mathbb{Z}/3\mathbb{Z})^{\infty}$ is a $(\mathbb{Z}/3\mathbb{Z})$-vector space with infinite rank. Then:

(i) R is an RF-ring.

(ii) $wgdim(R) \geq 2$.

Proof. (i) Let $R = \mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$. It is clear that R is non-local since so is $\mathbb{Z}$. Now, we claim that $R - Z(R) = \{(n, e) \in R\, /\, n \notin 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}\}$.
Indeed, we have $(0, e)(0, e) = 0$ and $(3m, e)(0, e) = 0$ for every $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$ and $m \in \mathbb{Z}$. Hence, $R - Z(R) \subseteq \{(n, e) \in R\, /\, n \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}\}$. Conversely, let $(n, e) \in R$ such that $n \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$ and let $(m, f) \in R$ such that $(n, e)(m, f) = (0, 0)$.

Hence, $(0,0) = (n,e)(m,f) = (nm, nf + me)$ and so $nm = 0$ and $nf + me = 0$. Since $n \in \mathbb{Z} - 3\mathbb{Z}$ and $nm = 0$, then $m = 0$ and so $nf = 0$. On the other hand, since $n \in \mathbb{Z} - 3\mathbb{Z}$, two cases are then possible:

Case 1: $n = 3p + 1$ for some $p \in \mathbb{Z}$.
Hence, $0 = nf = (3p+1)f = 3pf + f = f$ and so $(m,f) = (0,0)$, as desired.

Case 2: $n = 3p + 2$ for some $p \in \mathbb{Z}$.
Hence, $0 = nf = (3p+2)f = 3pf + 2f = 2f$ and so $f = 4f = 2.2f = 2.0 = 0$. Hence, $(m,f) = (0,0)$, as desired.

In all cases, we have $(m,f) = (0,0)$, as desired.

Now, our aim is to show that R is an RF-ring. Let I be a regular ideal of R. Then, there exists $(p,e) \in I$ such that $p \in \mathbb{Z} - 3\mathbb{Z}$ and $e \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$. Hence, for every $f \in (\mathbb{Z}/3\mathbb{Z})^{\infty}$, we have $(p,e)(0,f) = (0,pf)$. Therefore, $0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty} \subseteq I$ (since $p \in \mathbb{Z} - 3\mathbb{Z}$) and so $I = J \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$, where J is a proper ideal of $\mathbb{Z}$, that is $J = n\mathbb{Z}$, where $n \in \mathbb{Z} - 3\mathbb{Z}$ since I is a regular ideal of R. Hence, $I = n\mathbb{Z} \propto (\mathbb{Z}/3\mathbb{Z})^{\infty} = R(n,0) \cong R$, as desired.

(ii) We claim that the ideal $J := 0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$ is not flat. Deny. Let $\{f_i\}_{i \in I}$ be a basis of the $(\mathbb{Z}/3\mathbb{Z})$-vector space $3\mathbb{Z})^{\infty}$ and consider the R-map $R^{(I)} \xrightarrow{u} J$ defined by $u((a_i, e_i)_{i \in I}) := \sum_{i \in I}(a_i, e_i)(0, f_i)(= (0, \sum_{i \in I} a_i f_i))$. Clearly, $Ker(u) = 0 \propto E^{(I)} = (0 \propto E)^{(I)}$, where $E := (\mathbb{Z}/3\mathbb{Z})^{\infty}$. Hence, by [40, Theorem 3.55], we obtain

$$(0 \propto E)^{(I)} = (0 \propto E^{(I)}) \cap (0 \propto E)R^{(I)} = (0 \propto E)^{(I)})(0 \propto E) = 0,$$

a desired contradiction.

Therefore, the ideal $0 \propto (\mathbb{Z}/3\mathbb{Z})^{\infty}$ is not flat and so $wgdim(R) \geq 2$, as desired.

□

Now, we construct an RF local ring such that $wgdim(R) \geq 2$.

Example 2.3. Let $A = K[[X_1, ..., X_n, ...]] = K + M$ be a power series local ring with infinite indeterminates $(X_i)_{i=1,...,n,...}$ over a field K with maximal ideal M and set $R := A/M^2$. Then:

(i) R is an RF-ring.

(ii) $wgdim(R) \geq 2$.

Proof. (i) R is a local total ring with maximal ideal M/M^2. In particular, R is an RF-ring.

(ii) We claim that $wgdim(R) \geq 2$. Deny. Then, R is a valuation domain since R is a local, a desired contradiction since R is a total ring with maximal ideal M/M^2. Therefore, $wgdim(R) \geq 2$, as desired.

□

Now, we study the transfer of RF notion to a direct product.

Proposition 2.4. *Let $R := \prod_{i=1}^{n} R_i$ the direct product of a rings R_i. Then R is an RF-ring if and only if so is R_i, for every $i = 1, ..., n$.*

Proof. By induction, it suffices to show the proof for $n = 2$. Assume that R_1 and R_2 are RF-rings and let J be a regular ideal of R. Then, it is easy to see that $J = I_1 \times I_2$, where I_i is a regular ideal of R_i for $i = 1, 2$. Hence, I_i is a flat ideal of R_i and so $J := I_1 \times I_2$ is a flat ideal of R, as desired.
Conversely, assume that R is an RF-ring and let I_1 be a regular ideal of R_1. Then, $I_1 \times R_2$ is a regular ideal of the RF-ring R, hence $I_1 \times R_2$ is a flat ideal of R. Therefore, I_1 is a flat ideal of R_1, as desired.
By the same argument, we show that R_2 is also an RF-ring which completes the proof. □

We know that a localization of a ring such that $wgdim(R) \leq 1$ has the same property. Now, we give an example showing that the localization of an RF-ring is not always an RF-ring.

Example 2.5. Let $A = K[[X_1, ..., X_n, ...]] = K + M$ be a local power series ring with infinite indeterminates $(X_i)_{i=1,...,n,...}$ over a field K, where M is its maximal ideal generated by $(X_i)_{i=1,...,n,...}$ over a field K. Set $E := (A/M)^{\infty}(= K^{\infty})$ be a K-vector space with infinite rank and set $R = A \propto E$ be the trivial ring extension of A by E. Let $S_0 := \{X_1^n/n \in \mathbb{N}\}$ be a multiplicatively set of A and set $S := S_0 \propto 0$ be a multiplicative set of R. Then:

(i) R is an RF-ring since R is a total ring.

(ii) $S^{-1}R \cong S_0^{-1}A$ is a non-Prûfer domain. In particular, $S^{-1}R$ is a non-RF-ring.

Proof. (i) Straightforward.

(ii) If we take $S_0 = \{X_1^n/n \in \mathbb{N}\}$ and $S = S_0 \propto 0$, we have $S^{-1}R \cong S_0^{-1}A = [S_0^{-1}(K[X_1])][X_2, ...X_n, ...]$ which is a non-Prûfer domain. Hence, $S^{-1}R$ is a non-RF-ring, as desired.

□

But a localization by a multiplicative set $S \subseteq Reg(R)$ of an RF-ring is an RF-ring.

Proposition 2.6. *Let S be a multiplicative set of a ring R such that $S \subseteq Reg(R)$. If R is an RF-ring, then so is $S^{-1}R$.*

Proof. Remark that if $S \subseteq Reg(R)$, then x/s is a regular element of $S^{-1}R$ if and only if x is a regular element of R, for every $x \in R$ and $s \in S$. The rest of the proof is straightforward. □

Now, we study the transfer of an RF-property in trivial ring extension.

Theorem 2.7. *Let A be a ring, E be an A-module and set $R := A \propto E$ be the trivial ring extension of A by E. Then:*

(i) Assume that $A \subseteq B$ be an extension of domains, $K := qf(A)$ and $E := B$. Then:
a) *$R := A \propto B$ is a an RF-ring if and only if A is a Prûfer domain and $K \subseteq B$.*
b) $wgdim(R) \geq 2$.

(ii) Assume that A be an integral domain, $K := qf(A)$ and E be a K-vector space. Then:
a) *$R := A \propto E$ is an RF-ring if and only if A is a Prûfer domain.*
b) $wgdim(R) \geq 2$.

(iii) Assume that (A, M) is a local ring and E is an (A/M)-vector space. Then R is an RF-ring.

Proof. (i) **a)** Assume that A is a Prûfer domain, $K \subseteq B$ and let J be a proper regular ideal of R. Then there exists $(a, e) \in J$ such that $a \neq 0$ (since $(0 \propto E)(0, e) = 0$). Since $(a, e)R = aA \propto E$ since $aE = aB = B = E$ (since $a \in A \subseteq K$ and $K \subseteq B$). Therefore, $J := I \bigotimes_A R = IR = I \propto E$ for some proper ideal I of A. Hence, I is a flat ideal of A (since A is a Prûfer domain) and so $J := I \bigotimes_A R = IR = I \propto E$ is a flat ideal of R since R is a flat A-module, as desired.

Conversely, assume that R is an RF-ring. Our aim is to show that $K \subseteq B$.
First, we wish to show that $K \subseteq B$ in the case when A is local. Let $x \neq 0 \in A$ and let $I := ((x, 0), (x, 1))R$, a finitely generated regular ideal of R. Then I is projective and hence principal (since R is local too). Write $I = (a, b)R$ for some $a \in A$ and $b \in B$. Clearly, $a = ux$ for some invertible element u in A, hence $I = (ux, b)R = (x, u^{-1}b)R$. Further $(x, 0) \in I$ yields $u^{-1}b = b'x$ for some $b' \in B$. It follows that $I = (x, b'x)R = (x, 0)(1, b')R = (x, 0)R$ since $(1, b')$ is invertible. But $(x, 1) \in I$ yields $1 = xb''$ for some $b'' \in B$. Therefore $K \subseteq B$.

Now, suppose that A is not necessarily local and let $q \in Spec(B)$ and $p := q \cap A$. Clearly, $S := (A \setminus p) \times 0$ is a multiplicatively closed subset of R with the feature that $\frac{r}{1}$ is regular in $S^{-1}R$ if and only if r is regular in R. So regular ideals of $S^{-1}R$ originate from regular ideals of R. Hence $A_p \propto B_p = S^{-1}R$ is an RF-ring. Whence $K = qf(A_p) \subseteq B_p \subseteq B_q$. It follows that $K \subseteq B = \bigcap B_q$, where q ranges over $Spec(B)$, as desired.

It remains to show that A is a Prûfer domain and let I be a proper ideal of A. Hence, $J := I \bigotimes_A R = IR = I \propto E$ is a regular ideal of an RF-ring R, so J is a flat ideal of R. Therefore, I be a flat ideal of A since R is a faithfully flat A-module and $J := I \bigotimes_A R = IR = I \propto E$, as desired.

b) We claim that $wgdim(R) \geq 2$. Indeed, set $I := R(0, e)$, where $e \in E - \{0\}$ and set $T := S \propto 0$ be a multiplicatively closed subset of R, where $S := A - \{0\}$. Remark that $T^{-1}R = K \propto S^{-1}E(= K \propto S^{-1}B)$ is a local ring and $T^{-1}I = 0 \propto Ke$ is a non-flat ideal of a local ring $T^{-1}R$ since $T^{-1}I(= 0 \propto Ke)$ is not principal generated by a regular element (since $(0, e)T^{-1}I = 0$). Therefore, I is a non-flat ideal of R since $fd_{T^{-1}R}(T^{-1}I) \leqslant fd_R(I)$, as desired.

(ii) Argue as 1) above.

(iii) Straightforward since R is a (local) total ring and this completes the proof of Theorem 2.7. □

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8. *Let A be a domain, $K := qf(A)$, and $R := A \propto K$. Then the following statements are equivalent:*

(i) R is an RF-ring.

(ii) A is a Prûfer domain.

Now, we construct a new examples of RF-rings R such that $wgdim(R) \geq 2$ by using Theorem 2.7.

Example 2.9. Let $R := \mathbb{Z} \propto \mathbb{Q}$ be the trivial ring extension of $\mathbb{Z}$ by $\mathbb{Q}$. Then:

(i) R is an RF-ring.

(ii) $wgdim(R) \geq 2$.

Now, we study the transfer of RF-property in a particular case of pullbacks.

Theorem 2.10. *Let $T = K + M$ be a local ring, where K is a field and M is a maximal ideal of T such that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (take for instance $M^n = 0$ for some a positive integer n). Let $D \subseteq K$ be a subring of K and set $R = D + M$. Then R is an RF-ring if and only if D is a Prûfer domain.*

Proof. Assume that R is an RF-ring and let I be a proper ideal of D. Set $J = IR = I + M$ (since $aM = M$ for every $a \in K$) an ideal of R and we claim that J is a regular ideal of R. Indeed, let $d \in I - \{0\} \subseteq J$ and let $a + m \in R$ such that $d(a + m) = 0$, where $a \in D$ and $m \in M$. Then $0 = da + dm$ and so $da = 0$ in D and $dm = 0$ in M. Therefore, $a = 0$ since D is an integral domain and $d \in D - \{0\}$ and $m = 0$ since $0 = dm \in M$ and d is invertible in K, hence d is a regular element in J. Therefore, J is a flat ideal of R since R is an RF-ring and so I a flat ideal of D. Hence, D is a Prûfer domain.

Conversely, assume that D is a Prûfer domain and let J be a proper regular ideal of R. Then $J \subsetneq M$ since J is a regular ideal of R and so there exists $d + m \in J$, where $d \in D - \{0\}$ and $m \in M$. Hence, $J \supseteq (d+m)M = dM+mM = M$ (since $mM \subseteq M = dM$) and so $J = I+M$, where I is a proper ideal of D. Hence, I is a flat ideal of a Prûfer domain D. Therefore, J is a flat ideal of R and so R is an RF-ring which completes the proof of Theorem 2.10. □

Now, we construct a non-total RF-rings R such that $wgdim(R) \geq 2$ by using the above Theorem 2.10.

Example 2.11. Let $T = \frac{\mathbb{Q}[[\mathbb{X}]]}{<X^n>} = \mathbb{Q} + XT$, where X is an indeterminates over $\mathbb{Q}$, $\mathbb{Q}[[\mathbb{X}]]$ is the power series ring over $\mathbb{Q}$, and $< X^n >= X^n\mathbb{Q}[[X]]$ where n is a positive integers. Set $R = \mathbb{Z} + XT$. Then:

(i) R is an RF-ring.

(ii) $wgdim(R) \geq 2$.

Proof. (i) R is an RF-ring by Theorem 2.10.

(ii) The ring R is non-total since every $n \in \mathbb{Z} - \{0\}$ is regular in R. Also, we claim that the ideal XT is not flat. Deny. Let $S := \{2^n/n \in \mathbb{N}\}$ be a multiplicative subset of R. Hence, $S^{-1}(XT)$ is a flat ideal of a local ring $S^{-1}R$ (since $S^{-1}R = (\mathbb{Z}_{2\mathbb{Z}}) \propto XT$), a desired contradiction since $X^{n-1}(XT) = 0$.
Therefore, the ideal XT is not flat and so $wgdim(R) \geq 2$, as desired.

□

We know that R have weak global dimension ≤ 1 ($wgdim(R) \leq 1$) if every (resp., finitely generated) ideal of R is flat. Also, by using [5, Proposition 2.5] and since a flatness is a locally property, we have :

Proposition 2.12. *Let R be a ring and I be a finitely generated and regular ideal of R. Then the following statements are equivalent:*

(i) I is invertible.

(ii) I is projective.

(iii) I is locally principal.

(iv) I is flat.

We know that in an RF-ring, any finitely generated regular ideal is flat. We are led to make the following conjecture

Conjecture : Is a ring in which every finitely generated regular ideal is flat, is an RF-ring ?

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