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LINEAR PRESERVER OF WEAK COLUMN HADAMARD MAJORIZATION

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Abstract In this manuscript we study a novel- notion called weak column Hadamard majorization. Also we derive the structure of linear preservers and strong linear preservers of the same.

1 Introduction and Preliminaries

The theory of majorization originates in the early twentieth century during investigations into diverse and seemingly unrelated topics such as wealth distribution and inequalities related to convex functions. In finite-dimensional spaces, majorization theory plays a significant role in mathematics, statistics [10], and quantum mechanics [4, 14].

For two vectors $x, y \in \mathbb{R}^n$, we say that x is majorized by y, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \text{ for all } 1 \leq k \leq n-1 \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Here $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ is the non-increasing rearrangement of the components of x. It is well known that for $x, y \in \mathbb{R}^n$, x is majorized by y if and only if $x = Dy$ for some doubly stochastic matrix $D \in M_{n,n}$ ([10]). A square matrix is said to be doubly stochastic if all of its entries are non-negative and each row sum equals 1, and so is each column sum. In recent years, majorization theory extends to the space of matrices by using various types of stochastic matrices and different types of matrix multiplication. Like, for $A, B \in M_{m,n}$, A is said to be multivariate majorized by B if $A = DB$ for some doubly stochastic matrix D of order m ([10]). For more details on majorization and multivariate majorization, the reader may refer to [2, 3, 5, 6, 10]. Moreover, V. Kaftal and G.Weiss extended the notion of majorization in infinite dimensional space ([18]).

Let $M_{m,n}$ be the vector space of all $m \times n$ real matrices and let R be a relation on $M_{m,n}$. A linear operator T on $M_{m,n}$ is said to be a preserver (resp. strong preserver) of R if $R\big(T(X), T(Y)\big)$ whenever $R(X, Y)$ (resp. $R\big(T(X), T(Y)\big)$ if and only if $R(X, Y)$), for X, Y in $M_{m,n}$. One of the interesting problems in matrix theory is the linear preserver problem, which analyses all linear operators that preserve (and/or strongly preserve) the relation R. For more details, we refer to [11, 12, 15, 16]. In [1], Ando formulated the structure of linear operators which preserve majorization. Over the years, multiple studies generalized the notion of majorization using different settings and obtained the structure of linear preservers and strong linear preservers; see [7, 9, 13, 17]. Recently Kosuru and Saha introduced the notion of Hadamard circulant majorization on $M_{n,n}$ and derived the structure of such linear preservers ([8]).

Definition 1.1. ([8]) If X and Y are two square matrices of order n, X is said to be Hadamard circulant majorized by Y if there exists a circulant doubly stochastic matrix D of order n such that X is the entry-wise product of D and Y.

In this article, we study a novel notion – weak column Hadamard majorization on $M_{m,n}$. This notion generalizes the notion of Hadamard circulant majorization. Further, in this article we derive the structure of linear preservers and strong linear preservers of weak column Hadamard majorization.

2 Weak Column Hadamard Majorization on $M_{m,n}$

In this section, we define and study a few basic properties of the Weak Column Hadamard Majorization on $M_{m,n}$. The Hadamard product of A and B is defined as the entry-wise product of A and B. This is denoted by $A \odot B$ and is given by $[a_{ij}b_{ij}]$. A matrix $A \in M_{m,n}$ is considered column stochastic if all its entries are non-negative and the sum of each column is 1. Let $\hat{C} = \{A \in M_{m,n} : A \text{ is column stochastic}\}$. A matrix $A \in M_{m,n}$ is said to be a $(0,1)$ matrix if all the entries of A belong to $\{0,1\}$. Suppose $\mathcal{C}$ is the set of all $(0,1)$ column stochastic matrices of order $m \times n$. It is evident that $\mathcal{C}$ becomes extreme point of the convex set $\hat{C}$. Denote $E_{i,j}$ is a matrix whose only $(i,j)^{th}$ entry is 1 and rest entries are 0. For A, B in $M_{m,n}$, A is dominated by B if $A \odot B = A$. It is also evident that, if X and Y are dominated by M_1 and M_2 respectively, where $M_1, M_2 \in \mathcal{C}$ and $M_1 \neq M_2$, then $M_1 + M_2$ can not be dominated by any matrix in $\mathcal{C}$.

Definition 2.1. Let X and Y be two matrices of order $m \times n$. We say that X is weak column Hadamard majorized by Y, denoted by $X \prec_{WCH} Y$, if there exists a column stochastic matrix C of order $m \times n$ such that $X = C \odot Y$.

It is clear from the definition that $X \prec_{WCH} Y$ if

$$X = \begin{bmatrix} \frac{1}{3} & 2 & 3 \\ 0 & \frac{3}{2} & \frac{6}{5} \\ \frac{14}{3} & 0 & \frac{9}{5} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Let X and Y be two matrices of order $M_{m,n}$. Then $X \prec_{WCH} X$ if and only if X is dominated by a $(0,1)$ column stochastic matrix . Also, if $X \prec_{WCH} Y$ and $Y \prec_{WCH} X$, $X = D \odot C \odot X$, for some column stochastic matrices $C = [c_{ij}]$ and $D = [d_{ij}]$. Hence $c_{ij} = 1 = d_{ij}$ whenever $x_{ij} \neq 0$. Thus we have the following:

Let $X, Y \in M_{m,n}$. Then $X \prec_{WCH} Y$ and $Y \prec_{WCH} X$ if and only if $Y = X$ and X is dominated by a $(0,1)$ column stochastic matrix.

Let $A = \begin{bmatrix} \frac{1}{9} & \frac{2}{25} \\ \frac{16}{9} & \frac{4}{5} \\ 0 & \frac{28}{25} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{8}{3} & 2 \\ 0 & \frac{14}{5} \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$. It is evident that $A \prec_{WCH} B$ and $B \prec_{WCH} C$ but A is not weak column Hadamard majorized by C. Thus the relation " $\prec_{WCH}$ " is non-transitive. Also for $A \in M_{m,m}$ and $B \in M_{n,n}$, it clear that the matrices $A(X \odot Y)B$ and $(AXB) \odot (AYB)$ are not necessarily the same, where $X, Y \in M_{m,n}$. Remarkably for permutation matrices these two are equal, thus we have the following:

Let P and Q be permutation matrices of order m and n respectively. Then $P(X \odot Y)Q = (PXQ) \odot (PYQ)$ for $X, Y \in M_{m,n}$.

3 Linear Preservers of Weak Column Hadamard Majorization

Let $X = [x_{ij}]$, $Y = [y_{ij}]$ be two matrices in $M_{m,n}$. The inner product of X and Y is given as $\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}y_{ij}$. Let $T : M_{m,n} \to M_{m,n}$ be a linear operator. We say that T preserves weak column Hadamard majorization if $T(X) \prec_{WCH} T(Y)$ whenever $X \prec_{WCH} Y$, for X, Y in $M_{m,n}$. Throughout this article we denote by $\mathbf{0}$ the zero matrix of order $m \times n$. The next remark gives some properties of weak column Hadamard majorization on $M_{m,n}$.

Remark 3.1. Let $X, Y \in M_{m,n}$. Then the following statements hold:

(i) Let P and Q be two permutation matrices of order m and n respectively, then the linear operator $X \mapsto T(X)$ preserves weak column Hadamard majorization if and only if the linear operator $X \mapsto PT(X)Q$ preserves weak column Hadamard majorization.

(ii) Let $A = [a_{ij}] \in M_{m,n}$ such that $a_{ij} \neq 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then the linear operator $X \mapsto T(X)$ is a preserver of $\prec_{WCH}$ if and only if $X \mapsto T(X) \odot A$ is a preserver of $\prec_{WCH}$.

The following lemmas will help us to formulate the structure of linear preservers of weak column Hadamard majorization on $M_{m,n}$.

Lemma 3.2. *Let $n \geq 2$ and $T : M_{m,n} \to M_{m,n}$ be a linear operator. If T preserves weak column Hadamard majoriation, then $T(E_{ij}) \odot T(E_{rs}) = \mathbf{0}$ for $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$ with $(i,j) \neq (r,s)$.*

Proof. The proof is similar with the proof of the Lemma 1 of [8]. □

Lemma 3.3. *Let $T : M_{m,n} \to M_{m,n}$ be a linear operator. If T preserves weak column Hadamard majorization, then $T(E_{ij})$ and $T(E_{rs})$ can not simultaneously have non-zero entries in the same column for $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$ with $j \neq s$.*

Proof. Let $A = T(E_{ij}) = [a_{lk}]$ and $B = T(E_{rs}) = [b_{lk}]$ for some fixed $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$ with $j \neq s$. If either $A = \mathbf{0}$ or $B = \mathbf{0}$, then there is nothing to prove. Suppose $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$. We show that A and B do not have simultaneously non-zero entry in the same column. Suppose without loss of generality $a_{11} \neq 0$. By Lemma 3.2, it is clear that $b_{11} = 0$. We need to show that $b_{j1} = 0$ for $2 \leq j \leq m$. Suppose there exist k such that $b_{k1} \neq 0$. Therefore $a_{k1} = 0$. Let $K = E_{ij} + E_{rs}$. As $j \neq s$, there exists a $(0,1)$ column stochastic matrix C such that $K = C \odot K$. Thus $K \prec_{WCH} K$. Hence by our assumption $T(K) = A + B \prec_{WCH} T(K) = A + B$. Therefore there exists a $(0,1)$ column stochastic matrix C' such that $A + B = C'(A + B)$, but this a contradiction. Hence first column of B is zero. □

Let $\mathbb{R}^n$ be the set of all $n \times 1$ column vectors. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, then $[T]$ denotes the matrix representation of T with respect to the standard orthonormal basis of $\mathbb{R}^n$. The following results characterize all linear operators on $\mathbb{R}^n$ which preserve weak column Hadamard majorization.

Lemma 3.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T preserves weak column Hadamard majorization if and only if $[T]$ is dominated by a permutation matrix of order n. Thus T preserves $\prec_{WCH}$ if and only if there exists a permutation matrix P of order n and a vector $\alpha \in \mathbb{R}^n$ such that $T(x) = (Px) \odot \alpha$.*

Proof. Let $[T] = A = [a_{ij}]$, then $T(x) = Ax$. First assume that A is dominated by a permutation matrix of order n. Let $x, y \in \mathbb{R}^n$ be such that $x \prec_{WCH} y$. Then there exists a column stochastic matrix $C = [c_{i1}]$ such that $x = C \odot y$. Now $T(x) = (a_{1\sigma(1)}x_{\sigma(1)}, a_{2\sigma(2)}x_{\sigma(2)}, \ldots, a_{n\sigma(n)}x_{\sigma(n)})^t$ for some permutation σ on $\{1, 2, 3, \ldots, n\}$. Clearly

$$
\begin{aligned}
T(x) &= (a_{1\sigma(1)}x_{\sigma(1)}, a_{2\sigma(2)}x_{\sigma(2)}, \ldots, a_{n\sigma(n)}x_{\sigma(n)})^t \\
&= (a_{1\sigma(1)}c_{\sigma(1)1}y_{\sigma(1)}, a_{2\sigma(2)}c_{\sigma(2)1}y_{\sigma(2)}, \ldots, a_{n\sigma(n)}c_{\sigma(n)1}y_{\sigma(n)})^t \\
&= C' \odot (a_{1\sigma(1)}y_{\sigma(1)}, a_{2\sigma(2)}y_{\sigma(2)}, \ldots, a_{n\sigma(n)}y_{\sigma(n)})^t \\
&= C' \odot T(y),
\end{aligned}
$$

where $C' = (c_{\sigma(1)1}, c_{\sigma(2)1}, \ldots c_{\sigma(n)1})^t$. Since σ is a permutation, C' is column stochastic matrix and hence $T(x) \prec_{WCH} T(y)$.

Conversely assume that T preserves weak column Hadamard majorization. Since $e_i \prec_{WCH} e_i$ for all $1 \leq i \leq n$, we have $T(e_i) \prec_{WCH} T(e_i)$. Hence $T(e_i)$ is dominated by some $(0,1)$ column stochastic matrix. Also $T(e_i) \odot T(e_j) = 0$ for $i \neq j$ and 0 is the zero vector in $\mathbb{R}^n$. Therefore $[T]$ is dominated by some permutation matrix. □

Let $X \in M_{m,n}$, then X can be represented as $\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$, where $x_1, x_2, \ldots, x_n$ are column vectors of X, i.e. $x_i \in \mathbb{R}^m$ for $1 \leq i \leq n$. Suppose T is a linear operator on $M_{m,n}$, then $T(X) = T\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n T_{j1}(x_j) & \sum_{j=1}^n T_{j2}(x_j) & \cdots & \sum_{j=1}^n T_{jn}(x_j) \end{bmatrix}$, where $T_{ij} = \alpha^j T \alpha_i$ and $\alpha^j : M_{m,n} \to \mathbb{R}^m$, $\alpha_i : \mathbb{R}^m \to M_{m,n}$ are defined by:

$\alpha^j(X) = Xe_j$ and $\alpha_i(x) = xe_i^t$, for $1 \le i, j \le n$, $X \in M_{m,n}$ and $x \in \mathbb{R}^m$.
Now we are ready to derive the structure of linear operators on $M_{m,n}$ which preserve weak column Hadamard majorization.

Theorem 3.5. *Let T be a linear operator on $M_{m,n}$. T preserves weak column Hadamard majoriation if and only if there exists $A \in M_{m,n}$ and permutation matrices $P_1, P_2, \ldots, P_n$ of order m such that*

$$T(X) = \begin{bmatrix} P_1x_{k_1} & P_2x_{k_2} & \cdots & P_nx_{k_n} \end{bmatrix} \odot A \text{ for all } X \in M_{m,n} \tag{3.1}$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_n}$ are some columns of X (not necessarily distinct).

Proof. If $X, Y \in M_{1,n}$, $X \prec_{WCH} Y$ implies $X = Y$ and hence every linear operator T on $M_{1,n}$ preserves $\prec_{WCH}$.

Let T be a linear operator in $M_{m,n}$ and T preserves $\prec_{WCH}$. Clearly $T(X) = \begin{bmatrix} \sum_{j=1}^{n} T_{j1}x_j & \sum_{j=1}^{n} T_{j2}x_j & \cdots & \sum_{j=1}^{n} T_{jn}x_j \end{bmatrix}$, where T_{ij} defined earlier. Now we show that for each k $(1 \le k \le n)$ at most one T_{jk} $(1 \le j \le n)$ is non zero. Suppose T_{ik} and T_{jk} are non zero for some $i, j \in \{1, 2, 3, \ldots, n\}$ with $i \ne j$. Then there exist $a, b \in \mathbb{R}^m$ and two permutation matrices P, Q of order m such that $T_{ik}(x) = (Px) \odot a$ and $T_{jk}(x) = (Qx) \odot b$ for $x \in \mathbb{R}^m$. Since a and b non-zero there exist $r, s \in \{1, 2, 3, \ldots, m\}$ such that $a_r \ne 0$ and $b_s \ne 0$, where $a = (a_1, a_2, \ldots, a_m)^t$, $b = (b_1, b_2, \ldots, b_m)^t$.

Case 1: Consider $r \ne s$. Let $X = (P^te_r)e_i^t + (Q^te_s)e_j^t$. Then $X \prec_{WCH} X$ but $T(X) \not\prec_{WCH} T(X)$, since in the k^{th} column $T(X)$ has two non zero component.

Case 2: Consider $r = s$. Let $X = (P^te_r)e_i^t - a_rb_s^{-1}(Q^te_r)e_j^t$ and $Y = (P^te_r)e_i^t$. Then $Y \prec_{WCH} X$. Also the k^{th} column of $T(Y)$ is not zero but the k^{th} column of $T(X)$ is zero. Hence $T(Y)$ is not weak column Hadamard majorized by $T(X)$.

Therefore for each i $(1 \le i \le n)$ there exist k_i such that T_{k_ii} is not zero. Hence for each i, we have $T_{k_ii}(x) = (P_ix) \odot a_i$, P_i is a permutation matrix of order m and $a_i \in \mathbb{R}^m$ for all $1 \le i \le n$. Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, then $T(X) = \begin{bmatrix} P_1x_{k_1} & P_2x_{k_2} & \cdots & P_nx_{k_n} \end{bmatrix} \odot A$.

Conversely assume $T(X)$ is of the form (3.1). Let $X, Y \in M_{m,n}$ be such that $X \prec_{WCH} Y$. Then their exists a column stochastic matrix C such that $X = C \odot Y$. Thus for each i $(1 \le i \le n)$, we have $x_i = c_i \odot y_i$, where $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$. Now

$$\begin{aligned} T(X) &= \begin{bmatrix} P_1x_{k_1} & P_2x_{k_2} & \cdots & P_nx_{k_n} \end{bmatrix} \odot A \\ &= \begin{bmatrix} P_1(c_{k_1} \odot y_{k_1}) & P_2(c_{k_2} \odot y_{k_2}) & \cdots & P_n(c_{k_n} \odot y_{k_n}) \end{bmatrix} \odot A \\ &= C' \odot \begin{bmatrix} P_1y_{k_1} & P_2y_{k_2} & \cdots & P_ny_{k_n} \end{bmatrix} \odot A \\ &= C' \odot T(Y), \end{aligned}$$

where $C' = \begin{bmatrix} P_1c_{k_1} & P_2c_{k_2} & \cdots & P_nc_{k_n} \end{bmatrix}$. Clearly C' is a column stochastic matrix and hence $T(X) \prec_{WCH} T(Y)$. Thus T preserves weak column Hadamard majorization. □

4 Strong Linear Preservers of Weak Column Hadamard Majorization

Let T be a linear operator on $M_{m,n}$. We say that T strongly preserves weak column Hadamard majorization if the following holds:

$$T(X) \prec_{WCH} T(Y) \text{ if and only if } X \prec_{WCH} Y.$$

Let T be a linear operator on $M_{m,n}$ defined by $T(X) = X \odot A$, where $A = [a_{ij}] \in M_{m,n}$ with $a_{ij} \ne 0$. Clearly T strongly preserves weak column Hadamard majorization. Suppose T is a linear operator on $M_{m,n}$ that strongly preserves weak column Hadamard majorization. Let $T(X) = \mathbf{0}$ for some $X \in M_{m,n}$. Then $T(X) \prec_{WCH} T(\mathbf{0})$ implies $X \prec_{WCH} \mathbf{0}$. Thus we have the following.

Lemma 4.1. *Let T be a linear operator on $M_{m,n}$. If T strongly preserves weak column Hadamard majorization then T is invertible.*

Lemma 4.2. *Let T be an invertible linear operator on $M_{m,n}$. If T preserves weak column Hadamard majorization then so does T^{-1}.*

Proof. Since T preserves weak column Hadamard majorization, we have

$$T(X) = \begin{bmatrix} P_1 x_{k_1} & P_2 x_{k_2} & \cdots & P_n x_{k_n} \end{bmatrix} \odot A \text{ for all } X \in M_{m,n} \tag{4.1}$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_n}$ are some columns of X (not necessarily distinct) and $P_1, P_2, \ldots, P_n$ are permutation matrices of order m. As T is invertible then $x_{k_1}, x_{k_2}, \ldots, x_{k_n}$ are all distinct and A do not have any zero entries. Therefore

$$T(X) = \left(\begin{bmatrix} \hat{P}_1 x_1 & \hat{P}_2 x_2 & \cdots & \hat{P}_n x_n \end{bmatrix} P \right) \odot A$$

for some permutation matrix P of order n and $\hat{P_{k_i}} = P_i$ for all $1 \leq i \leq n$. Thus $T^{-1}(X) = \left(\begin{bmatrix} \hat{P_1}^{-1} x_1 & \hat{P_2}^{-1} x_2 & \cdots & \hat{P_n}^{-1} x_n \end{bmatrix} P^{-1} \right) \odot B$, where $B = [a_{ij}^{-1}]$, $A = [a_{ij}]$. By Theorem 3.5 it is clear that T^{-1} preserves $\prec_{WCH}$. This completes the proof. □

As a consequence of Lemma 4.1 and Lemma 4.2 we have the following characterization of strong linear preserver of weak column Hadamard majorization.

Theorem 4.3. *Let T be a linear operator on $M_{m,n}$. Then T strongly preserves weak column Hadamard majorization if and only if there exist $A \in M_{m,n}$ with non-zero entries and permutation matrices $P_1, P_2, \ldots, P_n$ of order m and a permutation matrix P of order n such that*

$$T(X) = \left(\begin{bmatrix} P_1 x_1 & P_2 x_2 & \cdots & P_n x_n \end{bmatrix} P \right) \odot A.$$

5 Declaration

Conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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