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k-TYPE SLANT HELICES DUE TO GENERALIZED BISHOP FRAME (GBF) OF TYPE C AND TYPE D IN E^4

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Abstract The present article deals with the study of k-type slant helices with two equivalence classes of generalized Bishop frames (GBF) called type C frame and type D frame in E^4 and obtain characterizations for such curves with these frames.

1 Introduction

The geometry of space curves has been an area of central importance due to its significance and rich applications to various engineering, computer aided geometric design, DNA Analysis, etc. One of the most fascinating curves in geometry is general helix, the arc of which can be accurately expressed as a combination of trigonometric functions and polynomials ([11], [17]). In any study via classical differential geometry, a general helix (or a constant angle curve) is defined as a curve whose tangent vector field maintains a constant angle with a fixed direction at each point. General helices are characterized by constant ratio of their torsion and curvature ([4],[12]). Helical structures can be observed widely in nature e.g. in nano-springs, vines, carbon nano-tubes, screws, DNA double, springs, sea shells, etc which shows the importance of this structure to be considered for the study.
On the other hand, the notion of slant helices was first established in 2004 by Izumiya and Takeuchi [16]. They defined them as space curves whose principal normal vector inclines at a constant angle with a fixed direction. Several characterizations have been proved for slant helices in the recent literature which can be found in [2, 3, 5, 9, 14, 19, 20, 21]. The natural extension of slant helices, called V_n-type slant helices (commonly known as k-type slant helices) was introduced by I. Gök et al. in [7] in 2009 using new harmonic curvature functions which were characterized in terms of these harmonic functions.
Majorly all space curves are studied due to the Frenet-Serret frame which is the most fundamental frame to study them. In 1975, R.L. Bishop introduced an another possible way to define a frame on a curve. This alternative frame is called the Bishop frame [15]. The most important advantage of the Bishop Frame over the Frenet frame is that it is admitted by all regular curves whereas some Frenet frame does not exist on some regular curves. Several mathematicians considered space curves with Bishop frame and obtained related interesting results and characterizations. For more details, we refer to [8, 10, 13]. On the other hand, in [6], H. Nozawa and S. Nomoto introduced and studied the generalization of the Frenet frame and Bishop frame which are called generalized Bishop frame (GBF, in short) with four equivalence classes called type B, type C, type D and type F frames. They proved that if the derivative of the tangent vector is nowhere vanishing for a regular curve , then there exist all three types of GBF except a frame of type F which is, in some sense, a Frenet Frame. Recently, canal hypersurfaces have also been studied according to GBF in Euclidean 4-space in [1].

In the present paper, our objective is to study k-type slant helices due to two equivalence classes (type C and type D) of generalized Bishop frames in E^4. We exclude to study type B frame and type F frame of generalized Bishop frames since type B and type F are ordinary Bishop frame and Frenet frame respectively for which vast literature is available.

2 Preliminaries

We start this section by recalling some basic definitions along with a little idea about four equivalence classes of GBF known as type B, type C, type D and type F frames. To begin with, we first have

Definition 2.1. [6] An orthonormal frame on a curve $I \to E^4$ is a matrix valued function $M : I \to O(n)$ such that the frame consists of the row vector of M. For a frame on a regular curve, we will call the matrix valued function S such that $M' = SM$, the coefficient matrix of the frame.

Definition 2.2. [6] An orthonormal frame on a curve $I \to E^4$ is a generalized Bishop frame if its coefficient matrix has at most three non zero entries above the main diagonal.

Apart from degenerate ones which contain a zero column vector, there are a total of 16 types of such frames. Following are the four equivalence classes of these 16 frames which are known as generalized Bishop frame of type B, C, D and F(see [6]).

$$\underset{type\ B}{\begin{bmatrix} 0 & y_1 & y_2 & y_3 \\ -y_1 & 0 & 0 & 0 \\ -y_2 & 0 & 0 & 0 \\ -y_3 & 0 & 0 & 0 \end{bmatrix}}, \quad \underset{type\ C}{\begin{bmatrix} 0 & y_1 & y_2 & 0 \\ -y_1 & 0 & 0 & y_3 \\ -y_2 & 0 & 0 & 0 \\ 0 & -y_3 & 0 & 0 \end{bmatrix}}, \quad \underset{type\ D}{\begin{bmatrix} 0 & y_1 & 0 & 0 \\ -y_1 & 0 & y_2 & y_3 \\ 0 & -y_2 & 0 & 0 \\ 0 & -y_3 & 0 & 0 \end{bmatrix}}, \quad \underset{type\ F}{\begin{bmatrix} 0 & y_1 & 0 & 0 \\ -y_1 & 0 & y_2 & 0 \\ 0 & -y_2 & 0 & y_3 \\ 0 & 0 & -y_3 & 0 \end{bmatrix}}$$

Generalized Bishop frame of type C and type D equations are expressed as

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & y_1 & y_2 & 0 \\ -y_1 & 0 & 0 & y_3 \\ -y_2 & 0 & 0 & 0 \\ 0 & -y_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{2.1}$$

and

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & y_1 & 0 & 0 \\ -y_1 & 0 & y_2 & y_3 \\ 0 & -y_2 & 0 & 0 \\ 0 & -y_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{2.2}$$

respectively, where y_1, y_2, y_3 are nonzero functions on an interval I.

Remark 2.3. It can be noticed that the matrix of GBF of type B and type F are nothing but the matrix corresponding to ordinary Bishop frame and Frenet frame, respectively. Due to this reason, throughout the paper, we shall discuss results based on GBF of type C and type D.

Definition 2.4. [18] Let $x = x(s)$ be a curve parametrized by arc-length with $\{T, M_1, M_2, M_3\}$ a Bishop frame in E^4. If there exists a nonzero constant vector field U in E^4 such that $< M_k, U > \neq 0$ is a constant for all $s \in \mathfrak{I}$, where $M_0 = T$, then X is said to be k-type $(k \in \{0, 1, 2, 3\})$ slant helix due to Bishop frame, and U is called axis of x.

3 k-Type Slant Helix due to GBF of type C in Euclidean 4-space

In this section, we prove characterizations of k-type slant helix, $k = 0, 1, 2, 3$ due to the GBF of type C in Euclidean 4-space in the form of a integro-differential equation or differential equation. Let U be any nonzero constant vector in E^4. Then U can be represented according to GBF of type C as

$$U = c_o(s)T(s) + c_1(s)M_1(s) + c_2(s)M_2(s) + c_3(s)M_3(s), \tag{3.1}$$

where $c_i(s)$, $i \in \{0, 1, 2, 3\}$, are differentiable functions. If we differentiate (3.1) and use (2.1), then we have

$$\begin{aligned}&(c_o'(s) - c_1(s)y_1(s) - c_2(s)y_2(s))T(s) + (c_o(s)y_1(s) + c_1'(s) - c_3(s)y_3(s))M_1(s)\\&+ (c_o(s)y_2(s) + c_2'(s))M_2(s) + (c_1(s)y_3(s) + c_3'(s))M_3(s) = 0\end{aligned}$$

which implies that

$$\left.\begin{aligned}&c_o'(s) - c_1(s)y_1(s) - c_2(s)y_2(s) = 0,\\&c_o(s)y_1(s) + c_1'(s) - c_3(s)y_3(s) = 0,\\&c_o(s)y_2(s) + c_2'(s) = 0,\\&c_1(s)y_3(s) + c_3'(s) = 0.\end{aligned}\right\} \tag{3.2}$$

Here, with the aid of (3.1) and (3.2), we will obtain the necessary and sufficient conditions for a smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 to be k-type ($k = 0, 1, 2, 3$) slant helix due to GBF of type C.

Theorem 3.1. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 0-type slant helix due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if the integro-differential equation*

$$y_1(s) + \left(\frac{y_2(s)}{y_1(s)}\int y_2(s)ds\right)' + y_3(s)\int\left(\frac{y_2(s)y_3(s)}{y_1(s)}\int y_2(s)ds\right)ds = 0 \tag{3.3}$$

holds.

Proof. Let $x(s)$ be a 0-type slant helix due to GBF of type C in E^4. In this situation, there will be a constant nonzero vector field U given by (3.1) in E^4 which satisfies

$$< T(s), U >= c_o \neq 0 \ (constant).$$

Combining the above equation with (3.2), we obtain

$$\left.\begin{aligned}&c_1(s) = c_o\frac{y_2(s)}{y_1(s)}\int y_2(s)ds,\\&c_2(s) = -c_o\int y_2(s)ds,\\&c_3(s) = -c_o\int\left(\frac{y_2(s)y_3(s)}{y_1(s)}\int y_2(s)ds\right)ds,\\&c_3(s) = c_o\frac{y_1(s)}{y_3(s)} + \frac{\left(c_o\frac{y_2(s)}{y_1(s)}\int y_2(s)ds\right)'}{y_3(s)}.\end{aligned}\right\} \tag{3.4}$$

From last two equations of the equation (3.4), we have (3.3).

For converse, let us assume that equation (3.3) holds. If we consider the axis that

$$\begin{aligned}U = c_o\Bigg(T(s) + \frac{y_2(s)}{y_1(s)}\int y_2(s)dsM_1(s) - \int y_2(s)dsM_2(s) -\\ \int\left(\frac{y_2(s)y_3(s)}{y_1(s)}\int y_2(s)ds\right)dsM_3(s)\Bigg),\end{aligned} \tag{3.5}$$

where c_0 is a nonzero constant, then by differentiating (3.5) and using (2.1) and (3.3), we have $U' = 0$. Thus, the proof is completed. □

From Theorem 3.1, following corollary can be easily concluded:

Corollary 3.2. *Let $x(s)$ be a 0-type slant helix with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (3.5).*

Theorem 3.3. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 1-type slant helix due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if the integro-differential equation*

$$y_1(s) + y_2(s) \int \left(\frac{y_2(s)y_3(s)}{y_1(s)} \int y_3(s)ds \right) ds + \left(\frac{y_3(s)}{y_1(s)} \int y_3(s)ds \right)' = 0 \tag{3.6}$$

holds.

Proof. Let $x(s)$ be a 1-type slant helix due to GBF of type C in E^4. Hence there exists a constant nonzero vector field U given by (3.1) in E^4 which satisfies

$$< M_1(s), U >= c_1 \neq 0 \ (constant).$$

Combining the above equation with (3.2), we obtain

$$\left. \begin{aligned} c_o(s) &= -c_1 \frac{y_3(s)}{y_1(s)} \int y_3(s)ds, \\ c_2(s) &= c_1 \int \left(\frac{y_2(s)y_3(s)}{y_1(s)} \int y_3(s)ds \right) ds, \\ c_3(s) &= -c_1 \int y_3(s)ds. \end{aligned} \right\}$$

Using the above equations in the first equation of the equation (3.2), we have (3.6).

Conversely, let us assume that (3.6) holds. If we consider the axis that

$$U = c_1 \left(-\frac{y_3(s)}{y_1(s)} \int y_3(s)dsT(s) + M_1(s) + \int \left(\frac{y_2(s)y_3(s)}{y_1(s)} \int y_3(s)ds \right) dsM_2(s) - \int y_3(s)dsM_3(s) \right), \tag{3.7}$$

where c_1 is a nonzero constant, then by differentiating (3.7) and using (2.1) and (3.6), we have $U' = 0$. Hence the proof. □

Thus, we have

Corollary 3.4. *Let $x(s)$ be a 1-type slant helix with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (3.7).*

Theorem 3.5. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 2-type slant helix due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if the following differential equation*

$$\frac{y_2(s)y_3(s)}{y_1(s)} + \left(\frac{1}{y_3(s)} \left(\frac{y_2(s)}{y_1(s)} \right)' \right)' = 0 \tag{3.8}$$

is satisfied.

Proof. Let $x(s)$ be a 2-type slant helix due to GBF of type C in E^4. Therefore by definition, there exists a constant nonzero vector field U given by (3.1) in E^4 such that

$$< M_2(s), U >= c_2 \neq 0 \ (constant).$$

Combining the above equation with (3.2), we obtain

$$\left.\begin{aligned} c_o(s) &= 0, \\ c_1(s) &= -c_2\frac{y_2(s)}{y_1(s)}, \\ c_3(s) &= -\frac{c_2}{y_3(s)}\left(\frac{y_2(s)}{y_1(s)}\right)'. \end{aligned}\right\}$$

Using the above equations in fourth equation of the equation (3.2), we have (3.8).

Conversely, let us assume that (3.8) holds. If we consider the axis that

$$U = c_2\left(-\frac{y_2(s)}{y_1(s)}M_1(s) + M_2(s) - \frac{1}{y_3(s)}\left(\frac{y_2(s)}{y_1(s)}\right)' M_3(s)\right), \tag{3.9}$$

where c_2 is a nonzero constant, then by differentiating (3.9) and using (2.1) and (3.8), we have $U' = 0$. This completes the proof. □

Following corollary can be easily obtained from Theorem 3.5.

Corollary 3.6. *Assume that $x(s)$ is a 2-type slant helix with nonzero functions $y_1(s), y_2(s), y_3(s)$ due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (3.9).*

Theorem 3.7. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 3-type slant helix due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if the differential equation*

$$\frac{y_2(s)y_3(s)}{y_1(s)} + \left(\frac{1}{y_2(s)}\left(\frac{y_3(s)}{y_1(s)}\right)'\right)' = 0 \tag{3.10}$$

is satisfied.

Proof. Let us assume that $x(s)$ be a 3-type slant helix due to GBF of type C in E^4. In this situation, there exists a constant nonzero vector field U given by (3.1) in E^4 which satisfies

$$< M_3(s), U >= c_3 \neq 0\ (constant).$$

Combining the above equation with (3.2), we obtain

$$\left.\begin{aligned} c_1(s) &= 0, \\ c_o(s) &= c_3\frac{y_3(s)}{y_1(s)}, \\ c_2(s) &= \frac{c_3}{y_2(s)}\left(\frac{y_3(s)}{y_1(s)}\right)'. \end{aligned}\right\}$$

If we use the above equations in the third equation of the equation (3.2), then we have (3.10).

For converse, let us assume that (3.10) holds. If we consider

$$U = c_3\left(\frac{y_3(s)}{y_1(s)}T(s) + \frac{1}{y_2(s)}\left(\frac{y_3(s)}{y_1(s)}\right)' M_2(s) + M_3(s)\right), \tag{3.11}$$

where c_3 is a nonzero constant, then by differentiating (3.11) and using (2.1) and (3.10), we have $U' = 0$. Thus, we completes the proof. □

From Theorem 3.7, we can state the following corollary:

Corollary 3.8. *Let $x(s)$ be a 3-type slant helix with nonzero functions $y_1(s), y_2(s), y_3(s)$ due to GBF of type C $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (3.11).*

4 k-Type Slant Helix due to GBF of Type D in Euclidean 4-space

In this last section, we consider k-type slant helix due to the GBF of type D in E^4 and discuss the characterizations.

Let U be any nonzero constant vector in E^4. Then U can be represented due to the generalized Bishop frame of type D as

$$U = c_o(s)T(s) + c_1(s)M_1(s) + c_2(s)M_2(s) + c_3(s)M_3(s), \tag{4.1}$$

where $c_i(s)$, $i \in \{0, 1, 2, 3\}$, are differentiable functions. By differentiating (4.1) and using (2.2), we get

$$\begin{aligned}&(c_o'(s) - c_1(s)y_1(s))T(s) + (c_o(s)y_1(s) + c_1'(s) - c_2(s)y_2(s) - c_3(s)y_3(s))M_1(s)\\&+ (c_1(s)y_2(s) + c_2'(s))M_2(s) + (c_1(s)y_3(s) + c_3'(s))M_3(s) = 0\end{aligned}$$

which implies that

$$\left.\begin{aligned}&c_o'(s) - c_1(s)y_1(s) = 0,\\&c_o(s)y_1(s) + c_1'(s) - c_2(s)y_2(s) - c_3(s)y_3(s) = 0,\\&c_1(s)y_2(s) + c_2'(s) = 0,\\&c_1(s)y_3(s) + c_3'(s) = 0.\end{aligned}\right\} \tag{4.2}$$

Here, with the aid of (4.1) and (4.2), we will give theorems which contain the necessary and sufficient conditions for a smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 to be k-type ($k = 0, 1, 2, 3$) slant helix due to GBF of type D.

Theorem 4.1. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 0-type slant helix due GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if*

$$c_2\frac{y_2(s)}{y_1(s)} + c_3\frac{y_3(s)}{y_1(s)} \tag{4.3}$$

is a nonzero constant, where c_2 and c_3 are constants.

Proof. Let $x(s)$ be a 0-type slant helix due to generalized Bishop frame of type D in E^4. In this situation, there exists a nonzero constant vector field U given by (4.1) in E^4 which satisfies

$$< T(s), U >= c_o \neq 0 \ (constant).$$

Therefore, from first equation of (4.2), we have $c_1 = 0$ and so, we get the following equations:

$$\left.\begin{aligned}&c_oy_1(s) - c_2(s)y_2(s) - c_3(s)y_3(s) = 0,\\&c_2(s) = constant,\\&c_3(s) = constant.\end{aligned}\right\} \tag{4.4}$$

From first equation of the equation (4.4), we have

$$c_o = c_2\frac{y_2(s)}{y_1(s)} + c_3\frac{y_3(s)}{y_1(s)}. \tag{4.5}$$

Conversely, let us assume that $c_2\frac{y_2(s)}{y_1(s)} + c_3\frac{y_3(s)}{y_1(s)} \neq 0$ (constant), where c_2 and c_3 are constants. If we consider the axis that

$$U = \left(c_2\frac{y_2(s)}{y_1(s)} + c_3\frac{y_3(s)}{y_1(s)}\right)T(s) + c_2M_2(s) + c_3M_3(s), \tag{4.6}$$

then by differentiating (4.6) and using (2.2), we have $U' = 0$.
Thus, we complete the proof. □

We can easily deduce the following corollary.

Corollary 4.2. *Let $x(s)$ be a 0-type slant helix with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (4.6).*

Theorem 4.3. *Any smooth curve $x(s)$ with nonzero functions $y_1(s), y_2(s), y_3(s)$ in E^4 is a 1-type slant helix due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if the integral equation*

$$\sum_{i=1}^{3}\left(y_i(s)\int y_i(s)ds\right)=0 \tag{4.7}$$

holds.

Proof. Let $x(s)$ be a 1-type slant helix due to GBF of type D in E^4. In this case, there exists a constant nonzero vector field U given by (4.1) in E^4 which satisfies

$$< M_1, U >= c_1 \neq 0\ (constant).$$

Combining the above equation with (4.2), we obtain

$$\left.\begin{aligned} &c_0(s)y_1(s)-c_2(s)y_2(s)-c_3(s)y_3(s)=0,\\ &c_o(s)=c_1\int y_1(s)ds,\\ &c_2(s)=-c_1\int y_2(s)ds,\\ &c_3(s)=-c_1\int y_3(s)ds \end{aligned}\right\} \tag{4.8}$$

and so, using the last three equations of (4.8) in the first equation of (4.8), we have

$$y_1(s)\int y_1(s)ds+y_2(s)\int y_2(s)ds+y_3(s)\int y_3(s)ds=0. \tag{4.9}$$

Conversely, let us assume that (4.9) holds. If we consider the axis as

$$U=c_1\left(\int y_1(s)dsT(s)+M_1(s)-\int y_2(s)dsM_2(s)-\int y_3(s)dsM_3(s)\right), \tag{4.10}$$

where c_1 is a nonzero constant, then by differentiating (4.10) and using (2.2) and (4.7), we have $U'=0$. Thus, the proof is completes. □

Thus,

Corollary 4.4. *Let $x(s)$ be a 1-type slant helix with nonzero functions $y_1(s), y_2(s), y_3(s)$ due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (4.10).*

Theorem 4.5. *Any smooth curve $x(s)$ with nonzero functions $y_1(s), y_2(s), y_3(s)$ in E^4 is a 2-type slant helix due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if*

$$c_o\frac{y_1(s)}{y_2(s)}-c_3\frac{y_3(s)}{y_2(s)} \tag{4.11}$$

is a nonzero constant, where c_o and c_3 are constants.

Proof. Let $x(s)$ be a 2-type slant helix due to GBF of type D in E^4. In this situation, there exists a constant nonzero vector field U given by (4.1) in E^4 which satisfies

$$< M_2(s), U >= c_2 \neq 0\ (constant).$$

Therefore, from third equation of (4.2), we have $c_1=0$ and so , we get the following equations:

$$\left.\begin{aligned} &c_o(s)y_1(s)-c_2y_2(s)-c_3(s)y_3(s)=0,\\ &c_o(s)=constant,\\ &c_3(s)=constant. \end{aligned}\right\} \tag{4.12}$$

From first equation of the equation (4.12), we have

$$c_2 = c_o \frac{y_1(s)}{y_2(s)} - c_3 \frac{y_3(s)}{y_2(s)}. \tag{4.13}$$

Conversely, let us assume that $c_o \frac{y_1(s)}{y_2(s)} - c_3 \frac{y_3(s)}{y_2(s)} \neq 0$ (constant), where c_o and c_3 are constants. If we consider the axis as

$$U = c_o T(s) + \left(c_o \frac{y_1(s)}{y_2(s)} - c_3 \frac{y_3(s)}{y_2(s)} \right) M_2(s) + c_3 M_3(s), \tag{4.14}$$

then by differentiating (4.14) and using (2.2), we have $U' = 0$ and this completes the proof. □

Corollary 4.6. *Let $x(s)$ be a 2-type slant helix with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (4.14).*

Theorem 4.7. *Any smooth curve $x(s)$ with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ in E^4 is a 3-type slant helix due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ if and only if*

$$c_o \frac{y_1(s)}{y_3(s)} - c_2 \frac{y_2(s)}{y_3(s)} \tag{4.15}$$

is a nonzero constant, where c_o and c_2 are constants.

Proof. Let $x(s)$ be a 3-type slant helix due to GBF of type D in E^4. In this situation, there exists a nonzero constant vector field U given by (4.1) in E^4 which satisfies

$$< M_3(s), U >= c_3 \neq 0 \ (constant).$$

Therefore, from fourth equation of (4.2), we have $c_1 = 0$ and so, we get the following equations:

$$\left.\begin{aligned} & c_o(s) y_1(s) - c_2(s) y_2(s) - c_3(s) y_3(s) = 0, \\ & c_o(s) = constant, \\ & c_2(s) = constant. \end{aligned}\right\} \tag{4.16}$$

From first equation of the equation (4.16), we have

$$c_3 = c_o \frac{y_1(s)}{y_3(s)} - c_2 \frac{y_2(s)}{y_3(s)}.$$

Conversely, let us assume that $c_o \frac{y_1(s)}{y_3(s)} - c_2 \frac{y_2(s)}{y_3(s)} \neq 0$ (constant), where c_o and c_2 are constants. If we consider the axis as

$$U = c_o T(s) + c_2 M_2(s) + \left(c_o \frac{y_1(s)}{y_3(s)} - c_2 \frac{y_2(s)}{y_3(s)} \right) M_3, \tag{4.17}$$

then by differentiating (4.17) and using (2.2), we have $U' = 0$.
This completes the proof. □

Corollary 4.8. *Let $x(s)$ be a 3-type slant helix with nonzero functions $y_1(s)$, $y_2(s)$, $y_3(s)$ due to GBF of type D $\{T(s), M_1(s), M_2(s), M_3(s)\}$ in E^4. Then the axis of $x(s)$ is given by (4.17).*

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