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ON n-DIMENSIONAL INTEGRAL TRANSFORM FOR TIME SCALES

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Abstract In this paper, we define a new integral transform for time scales, namely the n-dimensional integral transform. This transform is a generalization of finite-dimensional integral transforms resembling the Laplace transform on time scales. Further existence theorem and some essential properties are given. Integral transforms generalized on time scales, such as the Laplace, Sumudu, Shehu, double Laplace, and Laplace-Sumudu integral transforms, are particular cases of our proposed transform. Finally, we describe applications of this transform for solving partial dynamic, integro dynamic, and integral equations.

1 Introduction

Integral transforms execute an important role of transforming a function in one domain (frequently the time domain) to another domain (frequently the frequency domain). The properties of the transformed function are often revealed with the original function and are easier to work with or analyze. Because of this remarkable quality, integral transforms are specifically used for solving differential, partial differential, and integral equations with specified initial conditions. Fascinating work on integral transforms has been done in [2, 12, 13, 20, 21, 22, 23, 24, 25, 26, 27]. In [9], Jafari introduced a general integral transform for a continuous function $f(t)$ defined as

$$\mathcal{T}_S\{f(t)\} = \mathcal{F}(s) = p_1(s) \int_0^\infty e^{-q_1(s)t} f(t)dt, \;\; t > 0 \tag{1.1}$$

provided the integral exists. Further in 2022 [11] M. Meddhai et al. introduced a new general integral transform for a continuous function $f(t, x)$ of two variables as

$$\mathcal{T}_D\{f(t,x)\} = \mathcal{F}(s_1, s_2) = p_1(s_1)p_2(s_2) \int_0^\infty \int_0^\infty e^{-q_1(s_1)t-q_2(s_2)x} f(t,x)dtdx, \;\; x,t > 0 \tag{1.2}$$

provided the integral exists. The above-defined transforms specifically cover a large number of single and double integral transforms, respectively. The n-dimensional integral transform is an extension of the concept of integral transforms into functions of n variables. The main concern is to apply an integral transform n times once concerning each variable.

The time scale is assigned as a nonempty closed subset of the set of real numbers. It is a generalization of the usual notion of a timeline, which may contain continuous numbers, integers, or both so it constitutes a more general set of numbers. Time scale calculus consolidates differential calculus into a unified theory, thus allowing formulation, analysis, and finding solutions to dynamic equations over continuous, discrete, and miscellaneous types of domains. Similar to classical transforms, integral transforms on time scales offer an advantage for converting complex dynamic equations in the original time scale domain to a simple algebraic form in the transformed domain, making them easier to solve. As yet Laplace, Fourier, Sumudu, and Shehu transforms are generalized for time scales [1, 4, 10, 14]. The accompanying double-Laplace, Laplace-Sumudu, and double-Shehu transforms are generalized for time scales [8, 15, 19].

The main intend of our present work is to introduce a general integral transform on time scales having the speciality of generalizing the aforesaid integral transforms of dimensions one and two and also most of the integral transforms that have duality with the Laplace transform of higher dimensions that are still to be introduced on time scales. We also present new formulae for the transform of partial derivatives and n-dimensional convolution for functions with n independent variables.

The configuration of this paper is as follows, Section 2 consists of some preliminary results that are the principal constituents of our work. In Section 3 the n-dimensional integral transform is defined and existence conditions are given. Section 4 demonstrates some salient properties and theorems of our established transform. In Section 5, partial dynamic, integral equation, and integro-dynamic equation consisting of functions of n-independent variables are solved using the proposed transform of a specific dimension. Finally, Section 6 highlights the outcomes of our work.

2 Preliminaries

We recall results from [3, 5, 6, 15, 16, 17, 18] in this section. For a given time scale $\mathbb{T}$ and $t \in \mathbb{T}$, the forward jump operator is a function $\sigma : \mathbb{T} \to \mathbb{T}$ defined as $\sigma(t) = inf\{\tau \in \mathbb{T} : \tau > t\}$. Further, the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given as $\mu(t) = \sigma(t) - t$. Let $n \in \mathbb{N}$ be a fixed. For each $i \in \{1, 2, \ldots, n\}$ let a time scale $\mathbb{T}_i$ is unbounded above.

Definition 2.1. For each time scale $\mathbb{T}_i,\ i \in \{1, 2, \ldots, n\}$ the set

$$T^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \{(t_1, t_2, \ldots, t_n) : t_i \in \mathbb{T}_i, i = 1, 2, \ldots, n\}$$

is called n-dimensional time scale.

It is important to note that the set T^n equipped with the metric $d(t, \tau) = \left(\sum_{i=1}^{n} |t_i - \tau_i|^2\right)^{1/2}$ for $t, \tau \in T^n$ is a complete metric space. Therefore the fundamental concepts such as open balls, neighbourhood of points, open sets, compact sets, etc. follow for T^n. Also, we have for functions $f : T^n \to \mathbb{C}$, the concepts of limit, continuity, and properties of continuous functions on general metric spaces.

For an n-dimensional time scale T^n, we define $\sup T^n = (\sup(\mathbb{T}_1), \sup(\mathbb{T}_2), \ldots, \sup(\mathbb{T}_n))$. And in this study, we denote $\bar{t} = (t_1, t_2, \ldots, t_n)$ as an element of T^n and $\bar{z} = (z_1, z_2, \ldots, z_n)$ as an element of $\mathbb{C}^n$.

Definition 2.2. The operator $\sigma : T^n \to T^n$ with

$$\sigma(\bar{t}) = (\sigma_1(t_1), \sigma_2(t_2), \ldots, \sigma_n(t_n)), \quad \bar{t} = (t_1, t_2, \ldots, t_n) \in T^n$$

is said to be the forward jump operator in T^n, where $\sigma_i,\ i = 1, 2, \ldots, n$ is the forward jump operator in $\mathbb{T}_i$.

Definition 2.3. The operator $\rho : T^n \to T^n$ with

$$\rho(\bar{t}) = (\rho_1(t_1), \rho_2(t_2), \ldots, \rho_n(t_n)), \quad \bar{t} = (t_1, t_2, \ldots, t_n) \in T^n$$

is said to be the backward jump operator in T^n, where $\rho_i,\ i = 1, 2, \ldots, n$ is backward jump operator in $\mathbb{T}_i$.

Definition 2.4. For $\bar{t} = (t_1, t_2, \ldots, t_n) \in T^n$.

(1) $\sigma(\bar{t}) > \bar{t}$, then $\bar{t}$ is called strictly right scattered.

(2) If $\sigma(\bar{t}) \geq \bar{t}$ and there are $j, l \in \{1, 2, \ldots, n\}$ such that $\sigma_j(t_j) > t_j$ and $\sigma_l(t_l) = t_l$, then $\bar{t}$ is called right scattered.

(3) If $\bar{t} < \sup T^n$ and $\sigma(\bar{t}) = \bar{t}$, then $\bar{t}$ is called right dense.

(4) If $\rho(\bar{t}) < \bar{t}$, then $\bar{t}$ is called strictly left scattered.

(5) If $\rho(\bar{t}) \leq \bar{t}$ and there are $j, l \in \{1, 2, \ldots, n\}$ such that $\rho_j(t_j) < t_j$ and $\rho_l(t_l) = t_l$, then $\bar{t}$ is called left scattered.

(6) If $\bar{t} > \inf T^n$ and $\rho(\bar{t}) = \bar{t}$, then $\bar{t}$ is called left dense.

(7) If $\bar{t}$ is strictly right scattered and strictly left scattered, then $\bar{t}$ is said to be strictly isolated.

(8) If $\bar{t}$ is right dense and left dense, then $\bar{t}$ is said to be dense.

(9) If $\bar{t}$ is right scattered and left scattered, then $\bar{t}$ is said to be isolated.

Definition 2.5. The graininess function $\mu : T^n \to [0, \infty)^n$ is defined by

$$\mu(\bar{t}) = (\mu_1(t_1), \mu_2(t_2), \ldots, \mu_n(t_n)),$$

where $\mu_i,\ i \in \{1, 2, \ldots, n\}$ is the graininess function for $\mathbb{T}_i$.

Definition 2.6. For $f : T^n \to \mathbb{R}$, the forward shift $f^\sigma : T^n \to \mathbb{R}$ is defined as

$$\begin{aligned} f^\sigma(\bar{t}) &= (f \circ \sigma)(\bar{t}) \\ &= f(\sigma(\bar{t})) \\ &= f(\sigma_1(t_1), \sigma_2(t_2), \ldots \sigma_n(t_n)). \end{aligned}$$

Further

$$f_i^{\sigma_i}(\bar{t}) = f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n),$$

and

$$f_{i_1, i_2, \ldots, i_l}^{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_l}}(\bar{t}) = f(\ldots, \sigma_{i_1}(t_{i_1}), \ldots, \sigma_{i_2}(t_{i_2}), \ldots, \sigma_{i_l}(t_{i_l}), \ldots)$$

where $1 \leq i_1 < i_2 < \cdots < i_l \leq n,\ i_m \in \mathbb{N},\ m = 1, 2, \ldots, l,\ l \in \mathbb{N}$.

Definition 2.7. For a time scale $\mathbb{T}$, $\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise.} \end{cases}$

Definition 2.8. For a n-dimensional time scale, T^n the non-maximal sets are defined as

(1) $T^{\kappa n} = \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \cdots \times \mathbb{T}_n^\kappa$.

(2) $T_i^{\kappa_i n} = \mathbb{T}_1 \times \cdots \times \mathbb{T}_{i-1} \times \mathbb{T}_i^\kappa \times \mathbb{T}_{i+1} \times \cdots \times \mathbb{T}_n$.

(3) $T_{i_1 i_2 \ldots i_l}^{\kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_l} n} = \cdots \times \mathbb{T}_{i_1}^\kappa \times \cdots \times \mathbb{T}_{i_2}^\kappa \times \cdots \times \mathbb{T}_{i_l}^\kappa \times \ldots$
where $1 \leq i_1 < i_2 < \cdots < i_l \leq n,\quad i_m \in \mathbb{N},\ m = 1, 2, \ldots, l, l \in \mathbb{N}$.

Definition 2.9. A function $f : T^n \to \mathbb{R}$ is called regulated provided its right sided limit exists (finite) at all right dense points in T^n and its left sided limit exists (finite) at all left dense points in T^n.

Definition 2.10. A function $f : T^n \to \mathbb{R}$ is called rd-continuous provided it is continuous at right dense points in T^n and its left sided limit exists (finite) at left dense points in T^n. The set of rd-continuous functions $f : T^n \to \mathbb{R}$ is denoted by

$$C_{\text{rd}} = C_{\text{rd}}(T^n, \mathbb{R}).$$

The set of functions $f : T^n \to \mathbb{R}$ that are differentiable and whose derivatives is rd-continuous is denoted by

$$C_{\text{rd}}^1 = C_{\text{rd}}^1(T^n, \mathbb{R}).$$

Definition 2.11. A function $f \in C_{\text{rd}}(T^n, \mathbb{R})$ is called regressive if $1+\mu f \neq 0$. For $1+\mu f > 0$, f is positively regressive and for $1 + \mu f < 0$, f is negatively regressive. The set of regressive, positively regressive, and negatively regressive functions is denoted by

$$R(T^n, \mathbb{R}), R^+(T^n, \mathbb{R}), \text{ and } \quad R^-(T^n, \mathbb{R})$$

resepectively.

Definition 2.12. For $a, b \in \mathbb{R},\ a < b$ the interval $[a, b]$ in $\mathbb{T}$ is defined as

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Using the concept of intervals in time scales, we define the time scale generalized parallelopiped as follows.

Definition 2.13. For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ the time scale generalized parallelopiped is defined as

$$\begin{aligned} &[a_1, \infty)_{\mathbb{T}_1} \times [a_2, \infty)_{\mathbb{T}_2} \times \ldots [a_n, \infty)_{\mathbb{T}_n} \\ &= \{(t_1, t_2, \ldots, t_n) \in T^n : a_1 \leq t_1 < \infty, a_2 \leq t_2 < \infty, \ldots, a_n \leq t_n < \infty\}. \end{aligned}$$

Throughout this work we will assume that for all $\mathbb{T}_i$, the respective graininess function μ_i is bounded i.e. $0 < \mu_{i_{\text{min}}} \leq \mu_i \leq \mu_{i_{\text{max}}} < \infty$. We set $\mu_{\text{max}} = \mu^* = (\mu_{1_{\text{max}}}, \mu_{2_{\text{max}}}, \ldots, \mu_{n_{\text{max}}})$ and $\mu_{\text{min}} = \mu_* = (\mu_{1_{\text{min}}}, \mu_{2_{\text{min}}}, \ldots, \mu_{n_{\text{min}}})$.

For $f_1, f_2 \in R^+(T^n, \mathbb{R})$ we define the circle plus addition $\oplus$ by,

$$(f_1 \oplus f_2)(\bar{t}) = f_1(\bar{t}) + f_2(\bar{t}) + \mu(\bar{t}) f_1(\bar{t}) f_2(\bar{t}) \text{ for all } \bar{t} \in T^n.$$

Using properties of the exponential function, we generalize [4, Lemma 3.1] given below, which is useful for proving theorems from upcoming sections.

Lemma 2.14. *For $x_1, x_2, \ldots, x_n \in \mathbb{R}$ regressive,*

$$\begin{aligned} &e^{\sigma_1\sigma_2\ldots\sigma_n}_{\ominus x_1 \ominus x_2 \cdots \ominus x_n}(t_1, t_2, \ldots, t_n, t_1^0, t_2^0, \ldots, t_n^0) \\ &= \frac{e_{\ominus x_1 \ominus x_2 \cdots \ominus x_n}(t_1, t_2, \ldots, t_n, t_1^0, t_2^0, \ldots, t_n^0)}{(1 + \mu_1 x_1)(1 + \mu_2 x_2) \ldots (1 + \mu_n x_n)}. \end{aligned}$$

In this paper, we follow the notation,

$$e_{\ominus x_1 \ominus x_2 \cdots \ominus x_n}(t_1, t_2, \ldots, t_n, t_1^0, t_2^0, \ldots, t_n^0) = e_{\ominus x_1}(t_1, t_1^0) \cdot e_{\ominus x_2}(t_2, t_2^0) \ldots e_{\ominus x_n}(t_n, t_n^0).$$

We extend the definition for a multivariable function of exponential type II for n-dimensional time scales as follows.

Definition 2.15. A function $f \in C_{\text{rd}}(T^n, \mathbb{R})$ is said to be of exponential type II, provided there exists constants $C > 0$, and $k_i \in R^+(\mathbb{T}_i, \mathbb{R})$ for $i = 1, 2, \ldots, n$ such that

$$|f(\bar{t})| \leq C e_{k_1 \oplus k_1 \oplus \cdots \oplus k_n}(t_1, t_2, \ldots, t_n, t_1^0, t_2^0, \ldots, t_n^0)$$

for all $\bar{t} \in [t_1^0, \infty)_{\mathbb{T}_1} \times [t_2^0, \infty)_{\mathbb{T}_2} \times \ldots [t_n^0, \infty)_{\mathbb{T}_n}$.

With this knowledge, we proceed to the next section.

3 Main Result

In this section, we define the n-dimensional integral transform on time scales and discuss its existence conditions.

Definition 3.1 (n-Dimensional integral transform on time scales). Let T^n is an n-dimensional time scale such that $\sup\{\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n\} = \infty$ and $\bar{t^0} = (t_1^0, t_2^0, \dots, t_n^0) \in T^n$ is fixed. Let $f : T^n \to \mathbb{C}$ is rd-continuous function then the n-dimensional integral transforn of $f(\bar{t})$ is

$$\begin{aligned} &\mathcal{N}_n\{f(\bar{t})\} \\ &= \mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_n}\{f(\bar{t})\} \\ &= F_n(\bar{z}) \\ &= p_1(z_1)p_2(z_2)\dots p_n(z_n)\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,\dots,t_n,t_1^0,\dots,t_n^0) \\ &\qquad \cdot f(\bar{t})\ \Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n, \end{aligned} \tag{3.1}$$

provided the integral exists with $p_1(z_1) \neq 0, p_2(z_2) \neq 0, \dots, p_n(z_n) \neq 0$ and $q_1(z_1), \dots, q_n(z_n)$ are positively regressive transform functions for all $t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2, \cdots, t_n \in \mathbb{T}_n$ respectively.

Now, we see existence criteria for our proposed transform.

Theorem 3.2 (Existence theorem). *Let $\bar{t^0} = (t_1^0, t_2^0, \dots, t_n^0) \in T^n$ and $f : T^n \to \mathbb{C}$ is a rd-continuous function on time scale generalized rectangular parallelopiped $[t_1^0, \infty)_{\mathbb{T}_1} \times [t_2^0, \infty)_{\mathbb{T}_2} \times \cdots \times [t_n^0, \infty)_{\mathbb{T}_n}$ and is of exponential type II, then n-dimensional integral transform of $f(\bar{t})$ exists for all positively regressive functions $q_1(z_1), q_2(z_2), \dots, q_n(z_n)$, provided $\lim_{t_1\to\infty} e_{k_1\ominus q_1(z_1)}(t_1, t_1^0) \to 0,\ \dots,\ \lim_{t_n\to\infty} e_{k_n\ominus q_n(z_n)}(t_n, t_n^0) \to 0$ with $\mathcal{R}e_h(q_1(z_1)) > k_1$, $\dots, \mathcal{R}e_h(q_n(z_n)) > k_n$.*

Proof.

$$\begin{aligned} &\big|\mathcal{N}_n\{f(\bar{t})\}\big| \\ &= \Bigg| p_1(z_1)p_2(z_2)\dots p_n(z_n) \\ &\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0) f(\bar{t})\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n \Bigg| \\ &\le p_1(z_1)p_2(z_2)\dots p_n(z_n) \\ &\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)\big|f(\bar{t})\big|\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n \\ &\le Cp_1(z_1)p_2(z_2)\dots p_n(z_n) \\ &\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0) \\ &\qquad \cdot e_{k_1\oplus k_2\cdots\oplus k_n}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n \\ &\le Cp_1(z_1)p_2(z_2)\dots p_n(z_n) \\ &\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} \frac{e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)}{[1+\mu_1(t_1)q_1(z_1)][1+\mu_2(t_2)q_2(z_2)]\dots[1+\mu_n(t_n)q_n(z_n)]} \\ &\qquad \cdot e_{k_1\oplus k_2\cdots\oplus k_n}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)\Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n \\ &= Cp_1(z_1)p_2(z_2)\dots p_n(z_n)\int_{t_1^0}^{\infty}\frac{e_{k_1\ominus q_1(z_1)}(t_1,t_1^0)}{1+\mu_1(t_1)q_1(z_1)} \end{aligned}$$

$$\cdot\left[\int_{t_2^0}^{\infty}\frac{e_{k_2\ominus q_2(z_2)}(t_2,t_2^0)}{1+\mu_2(t_2)q_2(z_2)}\cdots\left[\int_{t_n^0}^{\infty}\frac{e_{k_n\ominus q_n(z_n)}(t_n,t_n^0)}{1+\mu_n(t_n)q_n(z_n)}\Delta_n t_n\right]\Delta_{n-1}t_{n-1}\cdots\right]\Delta_1 t_1$$

$$= C\frac{p_1(z_1)}{k_1-q_1(z_1)}\int_{t_1^0}^{\infty}k_1\ominus q_1(z_1)e_{k_1\ominus q_1(z_1)}(t_1,t_1^0)$$

$$\cdot\left[\frac{p_2(z_2)}{k_2-q_2(z_2)}\int_{t_2^0}^{\infty}k_2\ominus q_2(z_2)e_{k_2\ominus q_2(z_2)}(t_2,t_2^0)\right.$$

$$\left.\cdots\left[\frac{p_n(z_n)}{k_n-q_n(z_n)}\int_{t_n^0}^{\infty}k_n\ominus q_n(z_n)e_{k_n\ominus q_n(z_n)}(t_n,t_n^0)\Delta_n t_n\right]\Delta_{n-1}t_{n-1}\right]\cdots\Bigg]\Delta_1 t_1$$

$$= \frac{Cp_1(z_1)p_2(z_2)\ldots p_n(z_n)}{(k_1-q_1(z_1))(k_2-q_2(z_2))\ldots(k_n-q_n(z_n))}\int_{t_1^0}^{\infty}e^{\Delta_1^1}_{k_1\ominus q_1(z_1)}(t_1,t_1^0)$$

$$\cdot\left[\int_{t_2^0}^{\infty}e^{\Delta_2^1}_{k_2\ominus q_2(z_2)}(t_2,t_2^0)\ldots\left[\int_{t_n^0}^{\infty}e^{\Delta_n^1}_{k_n\ominus q_n(z_n)}(t_n,t_n^0)\Delta_n t_n\right]\Delta_{n-1}t_{n-1}\ldots\right]\Delta_1 t_1$$

$$= \frac{Cp_1(z_1)p_2(z_2)\ldots p_n(z_n)}{(q_1(z_1)-k_1)(q_2(z_2)-k_2)\ldots(q_n(z_n)-k_n)}.$$

□

3.1 The n-dimensional integral transform of some elementary functions

Using Definition 3.1 the n-dimensional integral transform of some elementary functions is given in Table 1.

3.2 Relationship of n-dimensional integral transform with other transforms

In this subsection we have given the relationship of our transform using Definition 3.1 with some familiar 1 and 2-dimensional integral transforms in tabular form.

4 Some properties and theorems

The following properties of n-dimensional integral transform are easily proved using Definition 3.1.

Property 4.1 (Linearity property). If $f(\bar{t}) : T^n \to \mathbb{C}$ and $g(\bar{t}) : T^n \to \mathbb{C}$ are rd-continuous functions with n-dimensional integral transforms $\mathcal{N}_n\{f(\bar{t})\}$ and $\mathcal{N}_n\{g(\bar{t})\}$ respectively. Then for $a_1, a_2 \in \mathbb{R}$,

$$\mathcal{N}_n\{a_1 f(\bar{t}) + a_2 g(\bar{t})\} = a_1\mathcal{N}_n\{f(\bar{t})\} + a_2\mathcal{N}_n\{g(\bar{t})\}.$$

Property 4.2. Let $f : T^n \to \mathbb{C}$ is rd-continuous function such that, $f(t_1, t_2, \ldots, t_n) = f_1(t_1)\cdot f_2(t_2)\ldots f_n(t_n),\ \ t_1 > t_1^0, t_2 > t_2^0, \ldots, t_n > t_n^0$. Then

$$\mathcal{N}_n\{f(t_1,t_2,\ldots,t_n)\} = F(\bar{z}) = \mathcal{N}_{t_1}\{f(t_1)\}\cdot\mathcal{N}_{t_2}\{f(t_2)\}\ldots\mathcal{N}_{t_n}\{f(t_n)\}$$

where $\mathcal{N}_{t_1}\{f(t_1)\}, \mathcal{N}_{t_2}\{f(t_2)\}, \ldots, \mathcal{N}_{t_n}\{f(t_n)\}$ are one-dimensional integral transforms for $f_1(t_1), f_2(t_2), \ldots, f_n(t_n)$ respectively.

Table 1. Transform of some elementary functions

$f(\bar{t})$	$\mathcal{N}_n\{f(\bar{t})\}$
1	$\frac{p_1(z_1)p_2(z_2)\dots p_n(z_n)}{q_1(z_1)q_2(z_2)\dots q_n(z_n)}$
$e_{a_1\oplus\cdots\oplus a_n}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-a_1)\dots(q_n(z_n)-a_n)}$
$e_{i(a_1\oplus\cdots\oplus a_n)}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-ia_1)\dots(q_n(z_n)-ia_n)}$
$sin_{a_1\oplus\cdots\oplus a_n}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{1}{2i}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-ia_1)\dots(q_n(z_n)-ia_n)}$ $-\frac{1}{2i}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)+ia_1)\dots(q_n(z_n)+ia_n)}$
$cos_{a_1\oplus\cdots\oplus a_n}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-ia_1)\dots(q_n(z_n)-ia_n)}$ $+\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)+ia_1)\dots(q_n(z_n)+ia_n)}$
$sinh_{a_1\oplus\cdots\oplus a_n}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-a_1)\dots(q_n(z_n)-a_n)}$ $-\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)+a_1)\dots(q_n(z_n)+a_n)}$
$cosh_{a_1\oplus\cdots\oplus a_n}(t_1,\dots,t_n,t_1^0,\dots,t_n^0)$	$\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)-a_1)\dots(q_n(z_n)-a_n)}$ $+\frac{1}{2}\frac{p_1(z_1)\dots p_n(z_n)}{(q_1(z_1)+a_1)\dots(q_n(z_n)+a_n)}$
$h_{m_1}(t_1,t_1^0)\dots h_{m_n}(t_n,t_n^0)$	$\frac{p_1(z_1)\dots p_n(z_n)}{q_1(z_1)^{m_1+1}\dots q_n(z_n)^{m_n+1}}$

Theorem 4.3 (Shifting theorem). *For* $\lambda_1\in\mathbb{T}_1,\lambda_2\in\mathbb{T}_2,\dots,\lambda_n\in\mathbb{T}_n$ *with* $\lambda_1>0$, $\lambda_2>0,\dots,\lambda_n>0$, *we have*

$$H_{\lambda_1,\lambda_2,\dots,\lambda_n}(t_1,t_2,\dots,t_n)=\begin{cases}0 & t_1\in\mathbb{T}_1,\dots,t_n\in\mathbb{T}_n \text{ and } t_1<\lambda_1,\dots,t_n<\lambda_n\\ 1 & t_1\in\mathbb{T}_1,\dots,t_n\in\mathbb{T}_n \text{ and } t_1\geq\lambda_1,\dots,t_n\geq\lambda_n.\end{cases}$$

Then for rd-continuous $f:T^n\to\mathbb{C}$,

$$\mathcal{N}_n\{H_{\lambda_1,\lambda_2,\dots,\lambda_n}(t_1,t_2,\dots,t_n)f(\bar{t})\}$$
$$=e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\lambda_1,\lambda_2,\dots,\lambda_n,t_1^0,t_2^0,\dots,t_n^0)\,\mathcal{N}_n\{f(\bar{t})\}.$$

Proof.

$$\mathcal{N}_n\{H_{\lambda_1,\lambda_2,\dots,\lambda_n}(t_1,t_2,\dots,t_n)f(\bar{t})\}$$

$$=p_1(z_1)p_2(z_2)\dots p_n(z_n)$$
$$\cdot\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty}e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)$$
$$\cdot H_{\lambda_1,\lambda_2,\dots,\lambda_n}(t_1,t_2,\dots,t_n)f(\bar{t})\;\Delta_1t_1\Delta_2t_2\dots\Delta_nt_n$$

$$=p_1(z_1)p_2(z_2)\dots p_n(z_n)$$
$$\cdot\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty}\left[\frac{e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,t_1^0,t_2^0,\dots,t_n^0)}{[1+\mu_1(t_1)q_1(z_1)][1+\mu_2(t_2)q_2(z_2)]\dots[1+\mu_n(t_n)q_n(z_n)]}\right.$$
$$\left.\cdot H_{\lambda_1,\lambda_2,\dots,\lambda_n}(t_1,t_2,\dots,t_n)\right]f(\bar{t})\;\Delta_1t_1\Delta_2t_2\dots\Delta_nt_n$$

Table 2. Relationship of n-dimensional integral transform with other transforms

2*n	n-dimensional time scale	Transform functions	Integral transform on time scale
3* 1	$T^1 = \mathbb{T}_1$	$p_1(z_1) = 1,$ $q_1(z_1) = z_1$	Laplace transform $L\{f(t_1)\}$ $= \int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus z_1}(t_1, t_1^0) f(t_1)\Delta_1 t_1$
3* 1	$T^1 = \mathbb{T}_1$	$p_1(z_1) = \frac{1}{z_1},$ $q_1(z_1) = \frac{1}{z_1}$	Sumudu transform $S\{f(t_1)\}$ $= \frac{1}{z_1} \int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus \frac{1}{z_1}}(t_1, t_1^0) \cdot f(t_1)\Delta_1 t_1$
3* 1	$T^1 = \mathbb{T}_1$	$p_1(z_1) = 1,$ $q_1(z_1) = \frac{z_1}{u}$	Shehu transform $Sh\{f(t_1)\}$ $= \int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus \frac{z_1}{u}}(t_1, t_1^0) \cdot f(t_1)\Delta_1 t_1$
4* 2	$T^2 = \mathbb{T}_1 \times \mathbb{T}_2$	$p_1(z_1) = 1, p_2(z_2) = 1,$ $q_1(z_1) = z_1, q_2(z_2) = z_2$	Double Laplace transform $L_{t_1}L_{t_2}\{f(t_1, t_2)\}$ $= \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}(t_1, t_2, t_1^0, t_2^0)$ $\cdot f(t_1, t_2)\Delta_1 t_1 \Delta_2 t_2$
4* 2	$T^2 = \mathbb{T}_1 \times \mathbb{T}_2$	$p_1(z_1) = 1, p_2(z_2) = 1,$ $q_1(z_1) = \frac{z_1}{u_1}, q_2(z_2) = \frac{z_2}{u_2}$	Double Shehu transform $Sh_{t_1}Sh_{t_2}\{f(t_1, t_2)\}$ $= \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus \frac{z_1}{u_1} \ominus \frac{z_2}{u_2}}(t_1, t_2, t_1^0, t_2^0)$ $\cdot f(t_1, t_2)\Delta_1 t_1 \Delta_2 t_2$
4* 2	$T^2 = \mathbb{T}_1 \times \mathbb{T}_2$	$p_1(z_1) = 1, p_2(z_2) = \frac{1}{z_2}$ $q_1(z_1) = z_1, q_2(z_2) = \frac{1}{z_2}$	Laplace-Sumudu transform $L_{t_1}S_{t_2}\{f(t_1, t_2)\}$ $= \frac{1}{z_2} \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus z_1 \ominus \frac{1}{z_2}}(t_1, t_2, t_1^0, t_2^0)$ $\cdot f(t_1, t_2)\Delta_1 t_1 \Delta_2 t_2$

$$= p_1(z_1)p_2(z_2)\dots p_n(z_n) \cdot \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty}\Bigg[\Big(e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\lambda_1,\lambda_2,\dots,\lambda_n,t_1^0,t_2^0,\dots,t_n^0) \frac{\cdot\, e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,\lambda_1,\lambda_2,\dots,\lambda_n)\Big)}{[1+\mu_1(t_1)q_1(z_1)][1+\mu_2(t_2)q_2(z_2)]\dots[1+\mu_n(t_n)q_n(z_n)]}\Bigg] f(\bar{t})\ \Delta_1 t_1\dots\Delta_n t_n$$

$$= p_1(z_1)p_2(z_2)\dots p_n(z_n)\ e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\lambda_1,\lambda_2,\dots,\lambda_n,t_1^0,t_2^0,\dots,t_n^0) \cdot \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty}\frac{e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,\lambda_1,\lambda_2,\dots,\lambda_n)}{[1+\mu_1(t_1)q_1(z_1)][1+\mu_2(t_2)q_2(z_2)]\dots[1+\mu_n(t_n)q_n(z_n)]} \cdot f(\bar{t})\ \Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n$$

$$= e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\lambda_1,\lambda_2,\dots,\lambda_n,t_1^0,t_2^0,\dots,t_n^0)\ p_1(z_1)p_2(z_2)\dots p_n(z_n) \cdot \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\cdots\int_{\lambda_n}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1,t_2,\dots,t_n,\lambda_1,\lambda_2,\dots,\lambda_n) \cdot f(\bar{t})\ \Delta_1 t_1\Delta_2 t_2\dots\Delta_n t_n$$

$$= e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\lambda_1,\lambda_2,\dots,\lambda_n,t_1^0,t_2^0,\dots,t_n^0)\mathcal{N}_n\{f(\bar{t})\}.$$

□

Theorem 4.4 (Transform of partial derivatives). *Let $f : T^n \to \mathbb{C}$ is rd-continuous function such that*
(i) $f^{\Delta_1^1}(t_1,t_2,\dots,t_n) = \frac{\partial f}{\Delta_1 t_1}(t_1,t_2,\dots,t_n)$, *(ii)* $f^{\Delta_1^2}(t_1,t_2,\dots,t_n) = \frac{\partial^2 f}{\Delta_1 t_1^2}(t_1,t_2,\dots,t_n)$,
(iii) $f^{\Delta_1^3}(t_1,t_2,\dots,t_n) = \frac{\partial^3 f}{\Delta_1 t_1^3}(t_1,t_2,\dots,t_n)$, *(iv)* $f^{\Delta_1^1\Delta_2^1}(t_1,t_2,\dots,t_n) = \frac{\partial^2 f}{\Delta_1 t_1^1 \Delta_2 t_2^1}(t_1,t_2,\dots,t_n)$,
are also rd-continuous. Then

$$\begin{aligned}
(1)\ & \mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\}\\
&= q_1(z_1)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- p_1(z_1)\mathcal{N}_{t_2}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\dots,t_n)\},\\
(2)\ & \mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^2}(t_1,t_2,\dots,t_n)\}\\
&= q_1^2(z_1)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_1(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\dots,t_n)\}\\
&- p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n)\},\\
(3)\ & \mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^3}(t_1,t_2,\dots,t_n)\}\\
&= q_1^3(z_1)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_1^2(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\dots,t_n)\}\\
&- q_1(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n)\}\\
&- p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f^{\Delta_1^2}(t_1^0,t_2,\dots,t_n)\},\\
(4)\ & \mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1\Delta_2^1}(t_1,t_2,\dots,t_n)\}\\
&= q_1(z_1)q_2(z_2)\ \mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}
\end{aligned}$$

$$
\begin{aligned}
&- q_1(z_1)p_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1,t_2^0,\ldots,t_n)\}\\
&- q_2(z_2)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\ldots,t_n)\}\\
&+ p_1(z_1)p_2(z_2)\mathcal{N}_{t_3}\mathcal{N}_{t_4}\ldots\mathcal{N}_{t_n}\{f(t_1^0,t_2^0,\ldots,t_n)\},
\end{aligned}
$$

provided,
$\lim_{t_1\to\infty} e_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\to 0$, $\lim_{t_1\to\infty} e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)\to 0$,
$\lim_{t_1\to\infty} e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^2}(t_1,t_2,\ldots,t_n)\to 0$, $\lim_{t_1\to\infty} e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_2,\ldots,t_n)\to 0$,
$\lim_{t_2\to\infty} e_{\ominus q_2(z_2)}(t_2,t_2^0)f(t_1,t_2,\ldots,t_n)\to 0$.

Proof. (1) Applying 1-dimensional integral transform with respect to t_1 to $f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)$, we get

$$
\begin{aligned}
&\mathcal{N}_{t_1}\{f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)\}\\
&= p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)\Delta_1 t_1\\
&= p_1(z_1)\int_{t_1^0}^{\infty}\Big[\Big(e_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\Big)^{\Delta_1^1} - f(t_1,t_2,\ldots,t_n)e^{\Delta_1^1}_{\ominus q_1(z_1)}(t_1,t_1^0)\Big]\Delta_1 t_1\\
&= p_1(z_1)\Big[-f(t_1^0,t_2,\ldots,t_n) - \int_{t_1^0}^{\infty}\ominus q_1(z_1)e_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\Delta_1 t_1\Big]\\
&= p_1(z_1)\Big[-f(t_1^0,t_2,\ldots,t_n) + q_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\Delta_1 t_1\Big]\\
&= p_1(z_1)q_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\Delta_1 t_1 - p_1(z_1)f(t_1^0,t_2,\ldots,t_n).
\end{aligned}
$$

Now applying 1-dimensional integral transform with respect to t_2, we get

$$
\begin{aligned}
&\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)\}\\
&= q_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\ldots,t_n)\Delta_1 t_1\Delta_2 t_2\\
&- p_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f(t_1^0,t_2,\ldots,t_n)\Delta_2 t_2\\
&= q_1(z_1)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus q_1(z_1)\ominus q_2(z_2)}(t_1,t_2,t_1^0,t_2^0)f(t_1,t_2,\ldots,t_n)\ \Delta_1 t_1\Delta_2 t_2\\
&- p_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f(t_1^0,t_2,\ldots,t_n)\Delta_2 t_2\\
&= q_1(z_1)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\{f(t_1,t_2,\ldots,t_n)\} - p_1(z_1)\mathcal{N}_{t_2}\{f(t_1^0,t_2,\ldots,t_n)\}.
\end{aligned}
$$

Applying 1-dimensional integral transforms with respect to $t_3,t_4,\ldots,t_n$ successively, we get

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1}(t_1,t_2,\ldots,t_n)\}$$

$$= q_1(z_1)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\} - p_1(z_1)\mathcal{N}_{t_2}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\dots,t_n)\}.$$

(2) Applying 1-dimensional integral transform with respect to t_1 to $f^{\Delta_1^2}(t_1,t_2,\dots,t_n)$, we get

$$\mathcal{N}_{t_1}\{f^{\Delta_1^2}(t_1,t_2,\dots,t_n)\}$$

$$= p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^2}(t_1,t_2,\dots,t_n)\Delta_1 t_1$$

$$= p_1(z_1)\int_{t_1^0}^{\infty}\Big[\Big(e_{\ominus q_1(z_1)}(t_1.t_1^0)f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\Big)^{\Delta_1^1} - f^{\Delta_1^1}(t_1,t_2,\dots,t_n)e^{\Delta_1^1}_{\ominus q_1(z_1)}(t_1,t_1^0)\Big]\Delta_1 t_1$$

$$= p_1(z_1)\Big[-f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n) - \int_{t_1^0}^{\infty}\ominus q_1(z_1)e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\Delta_1 t_1\Big]$$

$$= -p_1(z_1)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n) - p_1(z_1)\int_{t_1^0}^{\infty}\ominus q_1(z_1)e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\Delta_1 t_1$$

$$= -p_1(z_1)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n) + q_1(z_1)p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\Delta_1 t_1$$

$$= q_1(z_1)\mathcal{N}_{t_1}\{f^{\Delta_1^1}(t_1,t_2,\dots,t_n)\} - p_1(z_1)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n)$$

$$= q_1(z_1)\Big[q_1(z_1)\mathcal{N}_{t_1}\{f(t_1,t_2,\dots,t_n)\} - p_1(z_1)f(t_1^0,t_2,\dots,t_n)\Big] - p_1(z_1)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n)$$

$$= q_1^2(z_1)\mathcal{N}_{t_1}\{f(t_1,t_2,\dots,t_n)\} - p_1(z_1)q_1(z_1)f\{(t_1^0,t_2,\dots,t_n)\} - p_1(z_1)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n).$$

Now applying 1-dimensional integral transform with respect to t_2.

$$\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^2}(t_1,t_2,\dots,t_n)\}$$

$$= q_1^2(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_1 t_2$$
$$- p_1(z_1)q_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f(t_1^0,t_2,\dots,t_n)\Delta_2 t_2$$
$$- p_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f^{\Delta_1^1}(t_1^0,t_2,\dots,t_n)\Delta_2 t_2$$

$$= q_1^2(z_1)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus q_1(z_1)\ominus q_2(z_2)}(t_1,t_2,t_1^0,t_2^0)f(t_1,t_2,\dots,t_n)\Delta_1 t_1\Delta_2 t_2$$
$$- p_1(z_1)q_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f(t_1^0,t_2,\dots,t_n)\Delta_2 t_2$$

$$-p_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f^{\Delta_1^1}(t_1^0,t_2,\ldots,t_n)\Delta_2 t_2$$

$$= q_1^2(z_1)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\{f(t_1,t_2,\ldots,t_n)\} - p_1(z_1)q_1(z_1)\mathcal{N}_{t_2}\{f(t_1^0,t_2,\ldots,t_n)\}$$
$$- p_1(z_1)\mathcal{N}_{t_2}\{f^{\Delta_1^1}(t_1^0,t_2,\ldots,t_n)\}.$$

Applying 1-dimensional integral transforms with respect to $t_3,t_4,\ldots,t_n$ successively, we get

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^2}(t_1,t_2,\ldots,t_n)\}$$

$$= q_1^2(z_1)\mathcal{N}_n\{f(t_1,t_2,\ldots,t_n)\} - q_1(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\ldots,t_n)\}$$
$$- p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f^{\Delta_1^1}(t_1^0,t_2,\ldots,t_n)\}.$$

(3) Following similar steps we can prove that,

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^3}(t_1,t_2,\ldots,t_n)\}$$

$$= q_1^3(z_1)\mathcal{N}_n\{f(t_1,t_2,\ldots,t_n)\} - q_1^2(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\ldots,t_n)\}$$
$$- q_1(z_1)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f^{\Delta_1^1}(t_1^0,t_2,\ldots,t_n)\} - p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f^{\Delta_1^2}(t_1^0,t_2,\ldots,t_n)\}.$$

(4) Applying 1-dimensional integral transform with respect to t_1, we get

$$\mathcal{N}_{t_1}\{f^{\Delta_1^1\Delta_2^1}(t_1,t_2,\ldots,t_n)\}$$

$$= p_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_1^1\Delta_2^1}(t_1,t_2,\ldots,t_n)\Delta_1 t_1$$

$$= p_1(z_1)\int_{t_1^0}^{\infty}\Big[\Big(e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_2,\ldots,t_n)\Big)^{\Delta_1^1} - \Big(e^{\Delta_1^1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_2,\ldots,t_n)\Big)\Big]\Delta_1 t_1$$

$$= -p_1(z_1)f^{\Delta_2^1}(t_1^0,t_2,\ldots,t_n) - p_1(z_1)\int_{t_1^0}^{\infty}\ominus q_1(z_1)e_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_2,\ldots,t_n)\Delta_1 t_1$$

$$= -p_1(z_1)f^{\Delta_2^1}(t_1^0,t_2,\ldots,t_n) + p_1(z_1)q_1(z_1)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_2,\ldots,t_n)\Delta_1 t_1.$$

Applying 1-dimensional integral transform with respect to t_2, we get

$$\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1\Delta_2^1}(t_1,t_1,\ldots,t_n)\}$$

$$= -p_1(z_1)p_2(z_2)\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)f^{\Delta_2^1}(t_1^0,t_2,\ldots,t_n)\Delta_2 t_2 + q_1(z_1)p_1(z_1)p_2(z_2)$$
$$\cdot\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2,t_2^0)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1,t_1^0)f^{\Delta_2^1}(t_1,t_1,\ldots,t_n)\Delta_1 t_1\Delta_2 t_2$$

$$= -p_1(z_1)\mathcal{N}_{t_2}\{f^{\Delta_2^1}(t_1^0, t_2, \ldots, t_n)\} + q_1(z_1)p_1(z_1)p_2(z_2)$$
$$\cdot \int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1, t_1^0)\Big[\int_{t_2^0}^{\infty} e^{\sigma_2}_{\ominus q_2(z_2)}(t_2, t_2^0) f^{\Delta_2^1}(t_1, t_1, \ldots, t_n)\Delta_2 t_2\Big]\Delta_1 t_1$$

$$= -p_1(z_1)\Big[q_2(z_2)\mathcal{N}_{t_2}\{f(t_1^0, t_2, \ldots, t_n)\} - p_2(z_2)f(t_1^0, t_2^0, \ldots, t_n)\Big]$$
$$+ q_1(z_1)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1, t_1^0)\Big[\int_{t_2^0}^{\infty}\Big(e_{\ominus q_2(z_2)}(t_2, t_2^0)f(t_1, t_2, \ldots, t_n)\Big)^{\Delta_2^1}$$
$$- \Big(e^{\Delta_2^1}_{\ominus q_2(z_2)}(t_2, t_2^0)f(t_1, t_2, \ldots, t_n)\Big)\Delta_2 t_2\Big]\Delta_1 t_1$$

$$= -p_1(z_1)q_2(z_2)\mathcal{N}_{t_2}\{f(t_1^0, t_2, \ldots, t_n)\} + p_1(z_1)p_2(z_2)f(t_1^0, t_2^0, \ldots, t_n)$$
$$+ q_1(z_1)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1, t_1^0)\Big[-f(t_1, t_2^0, \ldots, t_n)$$
$$- \int_{t_2^0}^{\infty} \ominus q_2(z_2)e_{\ominus q_2(z_2)}(t_2, t_2^0)f(t_1, t_2, \ldots, t_n)\Delta_2 t_2\Big]\Delta_1 t_1$$

$$= -p_1(z_1)q_2(z_2)\mathcal{N}_{t_2}\{f(t_1^0, t_2, \ldots, t_n)\} + p_1(z_1)p_2(z_2)f(t_1^0, t_2^0, \ldots, t_n)$$
$$- q_1(z_1)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty} e^{\sigma_1}_{\ominus q_1(z_1)}(t_1, t_1^0)f(t_1, t_2^0, \ldots, t_n)\Delta_1 t_1$$
$$+ q_1(z_1)q_2(z_2)p_1(z_1)p_2(z_2)\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus q_1(z_1)\ominus q_2(z_2)}(t_1, t_2, t_1^0, t_2^0)f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2$$

$$= q_1(z_1)q_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\{f(t_1, t_2, \ldots, t_n)\} - q_1(z_1)p_2(z_2)\mathcal{N}_{t_1}\{f(t_1, t_2^0, \ldots, t_n)\}$$
$$- q_2(z_2)p_1(z_1)\mathcal{N}_{t_2}\{f(t_1^0, t_2, \ldots, t_n)\} + p_1(z_1)p_2(z_2)f(t_1^0, t_2^0, \ldots, t_n).$$

Applying 1-dimensional integral transforms with respect to $t_3, t_4, \ldots, t_n$, we get

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_1^1\Delta_2^1}(t_1, t_2, \ldots, t_n)\}$$

$$= q_1(z_1)q_2(z_2)\mathcal{N}_n\{f(t_1, t_2, \ldots, t_n)\}$$
$$- q_1(z_1)p_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1, t_2^0, \ldots, t_n)\}$$
$$- q_2(z_2)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\ldots\mathcal{N}_{t_n}\{f(t_1^0, t_2, \ldots, t_n)\}$$
$$+ p_1(z_1)p_2(z_2)\mathcal{N}_{t_3}\mathcal{N}_{t_4}\ldots\mathcal{N}_{t_n}\{f(t_1^0, t_2^0, \ldots, t_n)\}.$$

□

In general we have following formulae for transform of partial derivatives of higher order and mixed partial derivatives.

(1) $\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_i^1}(t_1, t_2, \ldots, t_n)\}$

$$= q_i(z_i)\mathcal{N}_n\{f(t_1, t_2, \ldots, t_n)\}$$
$$- p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\ldots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\ldots\mathcal{N}_{t_n}\{f(t_1, t_2, \ldots, t_i^0, \ldots, t_n)\}.$$

(2) $\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_i^2}(t_1,t_2,\dots,t_n)\}$

$$\begin{aligned}
&= q_i^2(z_i)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_i(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^1}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}.
\end{aligned}$$

(3) $\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_i^3}(t_1,t_2,\dots,t_n)\}$

$$\begin{aligned}
&= q_i^3(z_i)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_i^2(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- q_i(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^1}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- p_i(z_i)\mathcal{N}_{t_1}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^2}(t_1,t_2,\dots,t_n,t_1^0,\dots,t_n^0)\}.
\end{aligned}$$

(4) $\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_i^m}(t_1,t_2,\dots,t_n)\}$

$$\begin{aligned}
&= q_i^m(z_i)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_i^{m-1}(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^1}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- q_i^{m-2}(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^2}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- \dots - q_i(z_i)p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^{m-2}}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}\\
&- p_i(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^{m-1}}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}
\end{aligned}$$

$$\begin{aligned}
&= q_i^m(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_n}\{f(t_1,t_2,\dots,t_n)\}\\
&- p_i(z_i)\sum_{k=0}^{m-1} q_i^{m-1-k}(z_i)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f^{\Delta_i^k}(t_1,t_2,\dots,t_i^0,\dots,t_n)\}.
\end{aligned}$$

(5) $\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f^{\Delta_{t_1}^1\Delta_{t_2}^1\dots\Delta_{t_n}^1}(t_1,t_2,\dots,t_n)\}$

$$\begin{aligned}
&= q_1(z_1)q_2(z_2)\dots q_n(z_n)\mathcal{N}_n\{f(t_1,t_2,\dots,t_n)\}\\
&- q_2(z_2)q_3(z_3)\dots q_n(z_n)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2,\dots,t_n)\}\\
&- q_1(z_1)q_3(z_3)\dots q_n(z_n)p_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{f(t_1,t_2^0,\dots,t_n)\}\\
&\ \ \vdots\\
&- q_1(z_1)q_2(z_2)\dots q_{i-1}(z_{i-1})q_{i+1}(z_{i+1})\dots q_n(z_n)p_i(z_i)\\
&\cdot\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{f(t_1,t_2,\dots,t_{i-1},t_i^0,t_{i+1},\dots,t_n)\}\\
&\ \ \vdots\\
&- q_1(z_1)q_2(z_2)\dots q_{n-1}(z_{n-1})p_n(z_n)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-1}}\{f(t_1,t_2,\dots,t_{n-1},t_n^0)\}\\
&+ q_3(z_3)q_4(z_4)\dots q_n(z_n)p_1(z_1)p_2(z_2)\mathcal{N}_{t_3}\mathcal{N}_{t_4}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2^0,\dots,t_n)\}
\end{aligned}$$

$$+ q_1(z_1)q_4(z_4)\dots q_n(z_n)p_2(z_2)p_3(z_3)\mathcal{N}_{t_1}\mathcal{N}_{t_4}\dots\mathcal{N}_{t_n}\{f(t_1,t_2^0,t_3^0,\dots,t_n)\}$$
$$\vdots$$
$$+ q_1(z_1)q_2(z_2)\dots q_{n-2}(z_{n-2})p_{n-1}(z_{n-1})p_n(z_n)$$
$$\cdot \mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-2}}\{f(t_1,t_2,\dots,t_{n-1}^0,t_n^0)\}$$

$$- q_4(z_4)q_5(z_5)\dots q_n(z_n)p_1(z_1)p_2(z_2)p_3(z_3)\mathcal{N}_{t_4}\mathcal{N}_{t_5}\dots\mathcal{N}_{t_n}\{f(t_1^0,t_2^0,t_3^0,\dots,t_n)\}$$
$$- q_1(z_1)q_5(z_5)\dots q_n(z_n)p_2(z_2)p_3(z_3)p_4(z_4)\mathcal{N}_{t_1}\mathcal{N}_{t_5}\dots\mathcal{N}_{t_n}\{f(t_1,t_2^0,t_3^0,t_4^0,\dots,t_n)\}$$
$$\vdots$$
$$- q_1(z_1)q_2(z_2)\dots q_{n-3}(z_{n-3})p_{n-2}(z_{n-2})p_{n-1}(z_{n-1})p_n(z_n)$$
$$\cdot \mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-3}}\{f(t_1,t_2,\dots,t_{n-3},t_{n-2}^0,t_{n-1}^0,t_n^0)\}$$
$$\vdots$$
$$+ (-1)^n q_n(z_n)p_1(z_1)p_2(z_2)\dots p_{n-1}(z_{n-1})\mathcal{N}_{t_n}\{f(t_1^0,t_2^0,\dots,t_{n-1}^0,t_n)\}$$
$$+ (-1)^n q_{n-1}(z_{n-1})p_1(z_1)p_2(z_2)\dots p_{n-2}(z_{n-2})p_n(z_n)\mathcal{N}_{t_{n-1}}\{f(t_1^0,t_2^0,\dots,t_{n-1},t_n^0)\}$$
$$\vdots$$
$$+ (-1)^n q_1(z_1)p_2(z_2)p_3(z_3)\dots p_{n-1}(z_{n-1})\mathcal{N}_{t_1}\{f(t_1^0,t_2^0,\dots,t_{n-1}^0,t_n)\}$$
$$+ (-1)^n p_1(z_1)p_2(z_2)\dots p_n(z_n)\{f(t_1^0,t_2^0,\dots,t_n^0)\}.$$

Theorem 4.5 (Transform of integral). *Let $f : T^n \to \mathbb{C}$ is regulated. Then*

$$\mathcal{N}_n\left\{\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2}\cdots\int_{t_n^0}^{t_n} f(\alpha_1,\alpha_2,\dots,\alpha_n)\Delta_1\alpha_1\Delta_2\alpha_2\dots\Delta_n\alpha_n\right\}$$
$$= \frac{1}{q_1(z_1)q_2(z_2)\dots q_n(z_n)}\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f(t_1,t_2,\dots,t_n)\}.$$

Proof. Let

$$F(t_1,t_2,\dots,t_n) = \int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2}\cdots\int_{t_n^0}^{t_n} f(\alpha_1,\alpha_2,\dots,\alpha_n)\ \Delta_1\alpha_1\Delta_2\alpha_2\dots\Delta_n\alpha_n.$$

Then

$$F^{\Delta_1^1\Delta_2^1\dots\Delta_n^1}(t_1,t_2,\dots,t_n) = f(t_1,t_2,\dots,t_n),$$

and

$$F(t_1^0,t_2^0,\dots,t_n^0) = F(t_1^0,t_2,\dots,t_n) = \dots = F(t_1,t_2,\dots,t_n^0) = 0,$$
$$F(t_1^0,t_2^0,\dots,t_n) = F(t_1,t_2^0,t_3^0,\dots,t_n) = \dots F(t_1,t_2,\dots,t_{n-1}^0,t_n^0) = 0,$$
$$\vdots$$
$$F(t_1^0,\dots,t_{n-1}^0,t_n)\dots = F(t_1,t_2^0,\dots,t_n^0) = 0.$$

Applying n-dimensional integral transform and using general case (5) of Theorem 4.4, we get

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{F^{\Delta_1^1\Delta_2^1\dots\Delta_n^1}(t_1,t_2,\dots,t_n)\}$$
$$= q_1(z_1)q_2(z_2)\dots q_n(z_n)\mathcal{N}_n\{F(t_1,t_2,\dots,t_n)\}$$
$$- q_2(z_2)q_3(z_3)\dots q_n(z_n)p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{F(t_1^0,t_2,\dots,t_n)\}$$

$$
\begin{aligned}
&- q_1(z_1)q_3(z_3)\dots q_n(z_n)p_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\dots\mathcal{N}_{t_n}\{F(t_1,t_2^0,\dots,t_n)\}\\
&\ \vdots\\
&- q_1(z_1)q_2(z_2)\dots q_{i-1}(z_{i-1})q_{i+1}(z_{i+1})\dots q_n(z_n)p_i(z_i)\\
&\cdot \mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{i-1}}\mathcal{N}_{t_{i+1}}\dots\mathcal{N}_{t_n}\{F(t_1,t_2,\dots,t_{i-1},t_i^0,t_{i+1},\dots,t_n)\}\\
&\ \vdots\\
&- q_1(z_1)q_2(z_2)\dots q_{n-1}(z_{n-1})p_n(z_n)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-1}}\{F(t_1,t_2,\dots,t_{n-1},t_n^0)\}\\
&\\
&+ q_3(z_3)q_4(z_4)\dots q_n(z_n)p_1(z_1)p_2(z_2)\mathcal{N}_{t_3}\mathcal{N}_{t_4}\dots\mathcal{N}_{t_n}\{F(t_1^0,t_2^0,\dots,t_n)\}\\
&+ q_1(z_1)q_4(z_4)\dots q_n(z_n)p_2(z_2)p_3(z_3)\mathcal{N}_{t_1}\mathcal{N}_{t_4}\dots\mathcal{N}_{t_n}\{F(t_1,t_2^0,t_3^0,\dots,t_n)\}\\
&\ \vdots\\
&+ q_1(z_1)q_2(z_2)\dots q_{n-2}(z_{n-2})p_{n-1}(z_{n-1})p_n(z_n)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-2}}\{F(t_1,t_2,\dots,t_{n-1}^0,t_n^0)\}\\
&\\
&- q_4(z_4)q_5(z_5)\dots q_n(z_n)p_1(z_1)p_2(z_2)p_3(z_3)\mathcal{N}_{t_4}\mathcal{N}_{t_5}\dots\mathcal{N}_{t_n}\{F(t_1^0,t_2^0,t_3^0,\dots,t_n)\}\\
&- q_1(z_1)q_5(z_5)\dots q_n(z_n)p_2(z_2)p_3(z_3)p_4(z_4)\mathcal{N}_{t_1}\mathcal{N}_{t_5}\dots\mathcal{N}_{t_n}\{F(t_1,t_2^0,t_3^0,t_4^0,\dots,t_n)\}\\
&\ \vdots\\
&- q_1(z_1)q_2(z_2)\dots q_{n-3}(z_{n-3})p_{n-2}(z_{n-2})p_{n-1}(z_{n-1})p_n(z_n)\\
&\quad\cdot \mathcal{N}_{t_1}\mathcal{N}_{t_2}\dots\mathcal{N}_{t_{n-3}}\{F(t_1,t_2,\dots,t_{n-3},t_{n-2}^0,t_{n-1}^0,t_n^0)\}\\
&\ \vdots\\
&\\
&+ (-1)^n q_n(z_n)p_1(z_1)p_2(z_2)\dots p_{n-1}(z_{n-1})\mathcal{N}_{t_n}\{F(t_1^0,t_2^0,\dots,t_{n-1}^0,t_n)\}\\
&+ (-1)^n q_{n-1}(z_{n-1})p_1(z_1)p_2(z_2)\dots p_{n-2}(z_{n-2})p_n(z_n)\mathcal{N}_{t_{n-1}}\{F(t_1^0,t_2^0,\dots,t_{n-1},t_n^0)\}\\
&\ \vdots\\
&+ (-1)^n q_1(z_1)p_2(z_2)p_3(z_3)\dots p_{n-1}(z_{n-1})\mathcal{N}_{t_1}\{F(t_1^0,t_2^0,\dots,t_{n-1}^0,t_n)\}\\
&+ (-1)^n p_1(z_1)p_2(z_2)\dots p_n(z_n)F(t_1^0,t_2^0,\dots,t_n^0).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{N}_n&\left\{\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2}\cdots\int_{t_n^0}^{t_n} f(\alpha_1,\alpha_2,\dots,\alpha_n)\Delta_1\alpha_1\Delta_2\alpha_2\dots\Delta_n\alpha_n\right\}\\
&= \frac{1}{q_1(z_1)q_2(z_2)\dots q_n(z_n)}\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\dots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f(t_1,t_2,\dots,t_n)\}.
\end{aligned}
$$

□

Next, we prove the convolution theorem for the n-dimensional integral transform. Firstly, we define the n-dimensional convolution of a multivariable function on time scales using the concept of delay of a 1-dimensional function from [7] as follows.

Definition 4.6. Let $f : T^n \to \mathbb{C}$ is rd-continuous and $g : T^n \to \mathbb{C}$ is piecewise rd-continuous, and of exponential type II. Then the n-dimensional convolution of f and g is given by

$$\begin{aligned}&\big(f *^n g\big)(t_1, \dots, t_n)\\ &= \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \cdots \int_{t_n^0}^{t_n} f(t_1, \dots, t_n, \sigma_1(\tau_1), \dots, \sigma_n(\tau_n)) \cdot g(\tau_1, \dots, \tau_n) \Delta_1\tau_1 \dots \Delta_n\tau_n,\end{aligned}$$

where $f(t_1, t_2, \dots, t_n, \sigma_1(\tau_1), \sigma_2(\tau_2), \dots, \sigma_n(\tau_n))$ is delay of a function $f(t_1, t_2, \dots, t_n)$ by $\sigma_1(\tau_1) \in \mathbb{T}_1$, $\sigma_2(\tau_2) \in \mathbb{T}_2$, and $\sigma_n(\tau_n) \in \mathbb{T}_n$ respectively. And $*^n$ denotes n-dimensional convolution with respect to $t_1, t_2, \dots, t_n$.

Theorem 4.7 (Convolution theorem). *If $f : T^n \to \mathbb{C}$ and $g : T^n \to \mathbb{C}$ are rd-continuous functions of exponential type II having n-dimensional integral transforms $\mathcal{N}_n\{f(\bar{t})\}$ and $\mathcal{N}_n\{g(\bar{t})\}$ respectively. Then*

$$\mathcal{N}_n\big\{\big(f *^n g\big)(\bar{t})\big\} = \frac{1}{p_1(z_1)p_2(z_2)\dots p_n(z_n)} \mathcal{N}_n\{f(\bar{t})\} \cdot \mathcal{N}_n\{g(\bar{t})\}.$$

Proof. From Definition 3.1 of n-dimensional integral transform, and Definition 4.6 of n-dimensional convolution, we get

$$\begin{aligned}&\mathcal{N}_n\big\{\big(f *^n g\big)(t_1, t_2, \dots, t_n)\big\}\\
&= p_1(z_1)p_2(z_2)\dots p_n(z_n) \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} \cdots \int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1, t_2, \dots, t_n, t_1^0, t_2^0, \dots, t_n^0)\\
&\qquad \big(f *^n g\big)(t_1, t_2, \dots, t_n)\Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n\\
&= p_1(z_1)p_2(z_2)\dots p_n(z_n) \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} \cdots \int_{t_n^0}^{\infty} e^{\sigma_1\sigma_2\dots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(t_1, t_2, \dots, t_n, t_1^0, t_2^0, \dots, t_n^0)\\
&\quad \cdot \left[\int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \cdots \int_{t_n^0}^{t_n} f(t_1, \dots, t_n, \sigma_1(\tau_1), \dots, \sigma_n(\tau_n)) \cdot g(\tau_1, \dots, \tau_n) \Delta_1\tau_1 \dots \Delta_n\tau_n\right]\\
&\qquad \cdot \Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n\\
&= \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} \cdots \int_{t_n^0}^{\infty} g(\tau_1, \dots, \tau_n) p_1(z_1)p_2(z_2)\dots p_n(z_n)\\
&\quad \cdot \left[\int_{\sigma_1(\tau_1)}^{\infty} \int_{\sigma_2(\tau_2)}^{\infty} \cdots \int_{\sigma_n(\tau_n)}^{\infty} f(t_1, \dots, t_n, \sigma_1(\tau_1), \dots, \sigma_n(\tau_n))\right.\\
&\quad \left. \cdot e^{\sigma_1, \sigma_2, \dots, \sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\dots q_n(z_n)}(t_1, t_2, \dots, t_n, t_1^0, t_2^0, \dots, t_n^0)\Delta_1 t_1 \Delta_2 t_2 \dots \Delta_n t_n\right] \Delta_1\tau_1 \Delta_2\tau_2 \dots \Delta_n\tau_n\\
&= \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} \cdots \int_{t_n^0}^{\infty} g(\tau_1, \dots, \tau_n) \Bigg[p_1(z_1)p_2(z_2)\dots p_n(z_n)\\
&\quad \cdot \int_{t_1^0}^{\infty} \int_{t_2^0}^{\infty} \cdots \int_{t_n^0}^{\infty} f(t_1, \dots, t_n, \sigma_1(\tau_1), \dots, \sigma_n(\tau_n)) \;\; H_{\sigma_1(\tau_1)\sigma_2(\tau_2)\dots\sigma_n(\tau_n)}(t_1, \dots, t_n)\\
&\quad e^{\sigma_1, \sigma_2, \dots, \sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\dots q_n(z_n)}(t_1, t_2, \dots, t_n, t_1^0, t_2^0, \dots, t_n^0)\Delta_1 t_1 \dots \Delta_n t_n\Bigg] \cdot \Delta_1\tau_1 \dots \Delta_n\tau_n\end{aligned}$$

$$= \int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} g(\tau_1,\ldots,\tau_n)\mathcal{N}_n\{H_{\sigma_1(\tau_1)\sigma_2(\tau_2)\ldots\sigma_n(\tau_n)}(t_1,\ldots,t_n) \cdot f(t_1,\ldots,t_n,\sigma_1(\tau_1),\ldots,\sigma_n(\tau_n))\}\Delta_1\tau_1\ldots\Delta_n\tau_n$$

$$= \int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} g(\tau_1,\ldots,\tau_n)\Big[e_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\sigma_1(\tau_1),\ldots,\sigma_1(\tau_1),t_1^0,\ldots,t_n^0) \cdot \mathcal{N}_n\{f(t_1,\ldots,t_n)\}\Big]\Delta_1\tau_1\ldots\Delta_n\tau_n$$

$$= \frac{\mathcal{N}_n\{f(t_1,\ldots,t_n)\}}{p_1(z_1)p_2(z_2)\ldots p_n(z_n)}\cdot p_1(z_1)p_2(z_2)\ldots p_n(z_n)$$
$$\int_{t_1^0}^{\infty}\int_{t_2^0}^{\infty}\cdots\int_{t_n^0}^{\infty} e^{\sigma_1\sigma_1\ldots\sigma_n}_{\ominus q_1(z_1)\ominus q_2(z_2)\cdots\ominus q_n(z_n)}(\tau_1,\tau_2,\ldots,\tau_n,t_1^0,t_2^0,\ldots,t_n^0)g(\tau_1,\tau_2,\ldots,\tau_n) \cdot\Delta_1\tau_1\Delta_2\tau_2\ldots\Delta_n\tau_n$$

$$= \frac{1}{p_1(z_1)p_2(z_2)\ldots p_n(z_n)}\mathcal{N}_n\{f(t_1,t_2,\ldots,t_n)\}\cdot\mathcal{N}_n\{g(t_1,t_2,\ldots,t_n)\}.$$

□

5 Applications

In this section, some dynamic equations containing a fixed number of independent variables are solved using an n-dimensional integral transform considering the required dimension n. Here we have used the result. For $f : T^n \to \mathbb{C}$,

$$\mathcal{N}_{t_n}\mathcal{N}_{t_{n-1}}\ldots\mathcal{N}_{t_2}\mathcal{N}_{t_1}\{f(\bar{t})\} = \mathcal{N}_n\{f(\bar{t})\} = \mathcal{N}_{t_1}\mathcal{N}_{t_2}\ldots\mathcal{N}_{t_{n-1}}\mathcal{N}_{t_n}\{f(\bar{t})\}$$

which follows from Definition 3.1 and Lemma 2.14. In addition, we mention that the transform of some elementary functions and their inverses are referred from Table 1.

Example 5.1. Consider the following 3-dimensional Poisson equation for a 3-dimensional time scale $T^3 = \mathbb{T}_1\times\mathbb{T}_2\times\mathbb{T}_3$ such that $0\in\mathbb{T}_1, 0\in\mathbb{T}_2, 0\in\mathbb{T}_3$.

$$\frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_1 t_1^2} + \frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_2 t_2^2} + \frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_3 t_3^2} = 2sin_1(t_1,0)\cdot cos_1(t_2,0)\cdot sinh_1(t_3,0). \quad (5.1)$$

Subject to initial conditions

$$f(0,t_2,t_3) = 0,\ f(t_1,0,t_3) = sin_1(t_1,0)\cdot sinh_2(t_3,0),\ f(t_1,t_2,0) = 0,$$
$$\frac{\partial f(0,t_2,t_3)}{\Delta_1 t_1} = cos_1(t_2,0)\cdot sinh_2(t_3,0),\ \frac{\partial f(t_1,0,t_3)}{\Delta_2 t_2} = 0,$$
$$\frac{\partial f(t_1,t_2,0)}{\Delta_3 t_3} = 2\cdot sin_1(t_1,0)\cdot cos_1(t_2,0).$$

Applying the n-dimensional integral transform to Equation 5.1 and using Theorem 4.4 for $n = 3$, we get

$$\mathcal{N}_3\Big\{\frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_1 t_1^2}\Big\} + \mathcal{N}_3\Big\{\frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_2 t_2^2}\Big\} + \mathcal{N}_3\Big\{\frac{\partial^2 f(t_1,t_2,t_3)}{\Delta_3 t_3^2}\Big\}$$
$$= \mathcal{N}_3\{2sin_1(t_1,0)\cdot cos_1(t_2,0)\cdot sinh_2(t_3,0)\}.$$

Then

$$q_1^2(z_1)\mathcal{N}_3\{f(t_1,t_2,t_3)\} - p_1(z_1)q_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\{f(0,t_2,t_3)\} - p_1(z_1)\mathcal{N}_{t_2}\mathcal{N}_{t_3}\{f^{\Delta_1^1}(0,t_2,t_3)\}$$
$$+ q_2^2(z_2)\mathcal{N}_3\{f(t_1,t_2,t_3)\} - p_2(z_2)q_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\{f(t_1,0,t_3)\} - p_2(z_2)\mathcal{N}_{t_1}\mathcal{N}_{t_3}\{f^{\Delta_2^1}(t_1,0,t_3)\}$$
$$+ q_3^3(z_3)\mathcal{N}_3\{f(t_1,t_2,t_3)\} - p_3(z_3)q_3(z_3)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\{f(t_1,t_2,0)\} - p_3(z_3)\mathcal{N}_{t_1}\mathcal{N}_{t_2}\{f^{\Delta_3^1}(t_1,t_2,0)\}$$
$$= \frac{2}{2i}\Big[\frac{p_1(z_1)}{(q_1(z_1)-i)} - \frac{p_1(z_1)}{(q_1(z_1)+i)}\Big] \cdot \frac{1}{2}\Big[\frac{p_2(z_2)}{(q_2(z_2)-i)} - \frac{p_2(z_2)}{(q_2(z_2)+i)}\Big]$$
$$\cdot \frac{1}{2}\Big[\frac{p_3(z_3)}{(q_3(z_3)-2)} - \frac{p_3(z_3)}{(q_3(z_3)+2)}\Big].$$

Applying n-dimensional transform for $n = 2$ to the initial conditions and substiuting in above equation after simplification gives,

$$\mathcal{N}_3\{f(t_1,t_2,t_3) = \frac{p_1(z_1)}{q_1^2(z_1)+1} \cdot \frac{p_2(z_2)q_2(z_2)}{q_2^2(z_2)+1} \cdot \frac{2p_3(z_3)}{q_3^2(z_3)+4}.$$

Taking inverse transform, we get

$$f(t_1,t_2,t_3) = sin_1(t_1,0) \cdot cos_1(t_2,0) \cdot sinh_2(t_3,0).$$

Example 5.2. Consider the integro-dynamic equation for a 2-dimensional time scale $T^2 = \mathbb{T}_1 \times \mathbb{T}_2$ such that $0 \in \mathbb{T}_1, 0 \in \mathbb{T}_2$.

$$g^{\Delta_1^1}(t_1,t_2) + g^{\Delta_2^1}(t_1,t_2) + 1 - e_1(t_1,0) - e_1(t_2,0) - e_{1\oplus 1}(t_1,t_2,0,0) = \int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\Delta s_1 \Delta s_2. \tag{5.2}$$

Subject to initial conditions

$$g(t_1,0) = e_1(t_1,0),\ g(0,t_2) = e_1(t_2,0).$$

Applying n-dimensional integral transform to Equation 5.2 and using Theorem 4.4 for $n = 2$, we get

$$\mathcal{N}_2\{g^{\Delta_1^1}(t_1,t_2)\} + \mathcal{N}_2\{g^{\Delta_2^1}(t_1,t_2)\} + \mathcal{N}_2\{1\} - \mathcal{N}_2\{e_1(t_1,0)\}$$
$$-\mathcal{N}_2\{e_1(t_2,0)\} - \mathcal{N}_2\{e_{1\oplus 1}(t_1,t_2,0,0)\} = \mathcal{N}_2\Big\{\int_0^{t_1}\int_0^{t_2} g(s_1,s_2)\Delta s_1 \Delta s_2\Big\}.$$

Then

$$q_1(z_1)\mathcal{N}_2\{g(t_1,t_2)\} - p_1(z_1)\mathcal{N}_{t_2}\{g(t_1,0)\} + q_2(z_2)\mathcal{N}_2\{g(t_1,t_2)\}$$
$$- p_2(z_2)\mathcal{N}_{t_1}\{g(0,t_2)\} + \frac{p_1(z_1)p_2(z_2)}{q_1(z_1)q_2(z_2)} - \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1)-1)(q_2(z_2))}$$
$$- \frac{p_1(z_1)p_2(z_1)}{(q_1(z_1))(q_2(z_2)-1)} - \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1)-1)(q_2(z_2)-1)}$$
$$= \frac{1}{q_1(z_1)q_2(z_2)}\mathcal{N}_2\{g(t_1,t_2)\}.$$

Applying n-dimensional integral transform for $n = 1$ to initial conditions,

$$\mathcal{N}_{t_1}\{g(t_1,0)\} = \frac{p_1(z_1)}{(q_1(z_1)-1)},\ \mathcal{N}_{t_2}\{g(0,t_2)\} = \frac{p_2(z_2)}{(q_2(z_2)-1)}.$$

Substituting into above equation, we get

$$q_1(z_1)\mathcal{N}_2\{g(t_1,t_2)\} - p_1(z_1)\frac{p_2(z_2)}{(q_2(z_2)-1)} + q_2(z_2)\mathcal{N}_2\{g(t_1,t_2)\} - p_2(z_2)\frac{p_1(z_1)}{(q_1(z_1)-1)}$$

$$+\frac{p_1(z_1)p_2(z_2)}{q_1(z_1)q_2(z_2)} - \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1)-1)(q_2(z_2))} - \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1))(q_2(z_2)-1)} - \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1)-1)(q_2(z_2)-1)}$$
$$= \frac{1}{q_1(z_1)q_2(z_2)}\mathcal{N}_2\{g(t_1,t_2)\}.$$

Simplifying above expression,

$$\mathcal{N}_2\{g(t_1,t_2)\} = \frac{p_1(z_1)p_2(z_2)}{(q_1(z_1)-1)(q_2(z_2)-1)}.$$

Taking inverse transform, we get

$$g(t_1,t_2) = e_{1\oplus 1}(t_1,t_2,0,0).$$

Example 5.3. Consider the second order dynamic equation for a 1-dimensional time scale T such that $0 \in T$ and $\mu(t) \neq \frac{1}{2}$ for all $t \in T$.

$$y^{\Delta^2}(t_1) - 4y(t_1) = 0. \tag{5.3}$$

Subject to initial conditions

$$y(0) = 1, \;\; y^{\Delta^1}(0) = 0.$$

Applying n-dimensional integral transform to Equation 5.3 and using Theorem 4.4 for $n = 1$, we get

$$\mathcal{N}_1\{y^{\Delta^2}(t_1)\} \; - \; 4\mathcal{N}_1\{y(t_1)\} = 0,$$
$$q_1^2(z_1)\mathcal{N}_1\{y(t_1)\} - p_1(z_1)q_1(z_1)y(0) - p_1(z_1)y^{\Delta}(0) - 4\mathcal{N}_1\{y(t_1)\} = 0.$$

Using initial conditions,

$$q_1^2(z_1)\mathcal{N}_1\{y(t_1)\} - p_1(z_1)q_1(z_1) - 4\mathcal{N}_1\{y(t_1)\} = 0.$$

Simplifying,

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)q_1(z_1)}{q_1^2(z_1)-4},$$
$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)q_1(z_1)}{(q_1(z_1)-2)(q_1(z_1)+2)},$$
$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)}{2(q_1(z_1)-2)} \; + \; \frac{p_1(z_1)}{2(q_1(z_1)+2)}.$$

Taking inverse transform, we get

$$y(t_1) = \frac{1}{2}e_2(t_1,0) \; + \; \frac{1}{2}e_{-2}(t_1,0).$$

Example 5.4. Consider the Volterra integral equation for a 1-dimensional time scale T such that $0 \in T$ and $\mu(t) \neq 1$ for all $t \in T$.

$$y(t_1) = 1 + \int_0^{t_1} h_1(t,\sigma(\tau)) \cdot y(\tau)\Delta\tau.$$

Applying n-dimensional integral transform to given equation and using Theorem 4.7 for $n = 1$, we get

$$\mathcal{N}_1\{y(t_1)\} = \mathcal{N}_1\{1\} + \mathcal{N}_1\Big\{\int_0^{t_1} h_1(t_1,\sigma(\tau)) \cdot y(\tau)\Delta\tau\Big\},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)}{q_1(z_1)} + \mathcal{N}_1\{h_1(t_1,0) * y(t_1)\},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)}{q_1(z_1)} + \frac{1}{p_1(z_1)}\mathcal{N}_1\{h_1(t_1,0)\} \cdot \mathcal{N}_1\{y(t_1)\},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)}{q_1(z_1)} + \frac{1}{p_1(z_1)} \cdot \frac{p_1(z_1)}{q_1^2(z_1)}\mathcal{N}_1\{y(t_1)\},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)}{q_1(z_1)} + \frac{1}{q_1^2(z_1)}\mathcal{N}_1\{y(t_1)\},$$

$$\mathcal{N}_1\{y(t_1)\}\Big[1 - \frac{1}{q_1^2(z_1)}\Big] = \frac{p_1(z_1)}{q_1(z_1)},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{p_1(z_1)q_1(z_1)}{q_1^2(z_1) - 1},$$

$$\mathcal{N}_1\{y(t_1)\} = \frac{1}{2}\frac{p_1(z_1)}{q_1(z_1) - 1} + \frac{1}{2}\frac{p_1(z_1)}{q_1(z_1) + 1}.$$

Taking the inverse transform, we get

$$y(t_1) = \frac{1}{2}e_1(t_1,0) + \frac{1}{2}e_{-1}(t_1,0).$$

6 Conclusion

Motivated by the research related to integral transforms that exhibit duality with the Laplace transform, we introduced a novel n-dimensional integral transform on time scales. Due to their remarkable formulation, Laplace, Sumudu, Shehu, double Laplace, double Shehu, and Laplace-Sumudu transforms extended on time scales are special cases of our proposed transform. Furthermore, the transform can cover all integral transforms of finite dimensions resembling the Laplace transform that are still to be generalized on time scales. Hence, there is no need to form new formulas for integral transforms or to study their elementary properties.

In addition, we discuss its existence conditions and provide some fundamental properties, including the convolution theorem. Generalized formule for the transform of partial derivatives of the function of n-variables are given. We finally solved some partial dynamic and integral equations without converting them into dynamic equations using our transform.

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