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COPRIME GRAPH OF A VECTOR SPACE

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Abstract Let $\mathbb{V}$ be a finite dimensional vector space. In this paper, we introduce a new graph called the coprime graph $\Gamma_c(\mathbb{V})$ of $\mathbb{V}$ and study its properties. Actually, the coprime graph $\Gamma_c(\mathbb{V})$ of $\mathbb{V}$ is a simple undirected graph with the set of all nontrivial proper subspaces of $\mathbb{V}$ as the vertex set and two distinct nontrivial subspaces W_1 and W_2 are adjacent in $\Gamma_c(\mathbb{V})$ if and only if $\gcd(\dim(W_1), \dim(W_2)) = 1$. Having introduced this new graph, we study the structure and graph theoretical properties like connectivity, hamiltonicity, diameter, girth etc of the coprime graph of a finite dimensional vector space.

1 Introduction

The study about algebraic structures using properties of graphs, has become an exciting research topic in the last three decades, leading to many fascinating results and questions. There are many papers assigning a graph to a group or a ring or a vector space and investigating algebraic properties using the associated graph. Some interesting constructions are available in [9, 16, 17]. For the entire literature on graphs from rings, one can refer[1]. Throughout this paper, $\mathbb{V}$ is a finite dimensional vector space over a finite field $\mathbb{F}$. The dimension of any vector space $\mathbb{V}$ is denoted by $\dim(\mathbb{V})$. In this paper, we assign a graph to a finite dimensional vector space $\mathbb{V}$ and investigate algebraic properties of $\mathbb{V}$ using properties of the derived graph.

Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$ with $\{m_1, m_2, \ldots, m_k\}$ as a basis. Then any vector $a \in \mathbb{V}$ can be expressed uniquely as a linear combination of the form $a = a_1m_1 + a_2m_2 + \cdots + a_km_k$. The non-zero component graph of $\mathbb{V}$ with respect to the basis $\{m_1, \ldots, m_k\}$, denoted by $\Gamma(\mathbb{V})$, is a simple undirected graph with non-zero vectors of $\mathbb{V}$ as the vertex set and such that there is an edge between two distinct vertices x and y if and only if there exists at least one m_i along which both x and y have non-zero scalars. Das[3] proved that the graph $\Gamma(\mathbb{V})$ is independent of choice of basis (i.e., for two different bases of $\mathbb{V}$, the nonzero component graphs $\Gamma(\mathbb{V})$ are isomorphic). Also it was proved that the graph $\Gamma(\mathbb{V})$ is connected and properties of graph theoretical parameters such as domination number, independence number and degree of vertices of this graph are investigated in [3]. In the case of finite fields, Eulerian and Hamiltonicity of $\Gamma(\mathbb{V})$ were discussed in [5]. In 2017, Nikandish, Maimani etal.[14] studied the coloring of non-zero component graphs associated with finite dimensional vector spaces. Further genus characterizations of the non-zero component graph is obtained by Tamizh Chelvam and Prabha Ananthi [18]. Other graphs viz non-zero component union graph[6], subspace inclusion graph[4] and subspace inclusion graph [7] are also well studied. Further graphs corresponding to free semi-modules were studied in [19, 15]. The *coprime graph* $\Gamma_c(\mathbb{V})$ of $\mathbb{V}$ is a simple undirected graph with the collection of proper subspaces of $\mathbb{V}$ as the vertex set and two vertices W_1 and W_2 are adjacent if and only if $\gcd(\dim(W_1), \dim(W_2)) = 1$. In this paper, we investigate the structure and graph theoretical properties like connectivity, hamiltonicity, diameter, girth etc of $\Gamma_c(\mathbb{V})$. For terms in algebra, we refer Dummit and Foote [8].

By a graph $G = (V, E)$, we mean a simple undirected graph with nonempty vertex set V and edge set E. A graph G is said to be complete if every pair of distinct vertices are adjacent and a complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is said to be bipartite if the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. A complete bipartite graph is the bipartite graph in which all possible edges are included and if $|V_1| = m$ and $|V_2| = n$, then it is denoted by $K_{m,n}$. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \to V_2$ such that $u, v \in V(G_1)$ are adjacent in G_1 if and only if $f(u), f(v) \in V(G_2)$ are adjacent in G_2. If a graph G has a $u-v$ path, then the distance from u to v, written as $d(u, v)$ is the length of a shortest $u - v$ path. If G has no such path, then $d(u, v) = \infty$. The diameter $diam(G)$ of G is nothing but $\max\limits_{u,v \in V(G)} d(u, v)$. The girth $gr(G)$ of a graph G with a cycle is the length of a shortest cycle in G. If G has no cylce, then $gr(G) = \infty$. A graph is said to be triangulated if for any vertex u in $V(G)$, there exist v, w in $V(G)$, such that (u, v, w) is a triangle. A clique in a graph G is a complete subgraph of G. The order of the largest clique in G is called the clique number and the same is denoted by $\omega(G)$. The smallest number of colors in any coloring of a graph G is called the chromatic number of G, and is denoted by $\chi(G)$. A graph for which the clique number equals the chromatic number is called weakly perfect graph. If a graph can be drawn in the plane without crossing edges except at vertices, then it is called a planar graph. A planar graph is said to be outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. For a real number x, $\lfloor x \rfloor$ is the greatest integer not exceeding x and $\lceil x \rceil$ is the smallest integer not less than x. For terminology in graph theory we refer Chartrand [2] and West [20].

2 Basic Properties of $\Gamma_c(\mathbb{V})$

First let us see an example of coprime graph of a vector space.

Example 2.1. Consider the vector space $\mathbb{V} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ over the Galois field $\mathbb{Z}_2$. Then proper 1-dimensional subspaces of $\mathbb{V}$ are $W_{1,1} = \langle(0,0,1)\rangle, W_{1,2} = \langle(0,1,0)\rangle, W_{1,3} = \langle(1,0,0)\rangle$, $W_{1,4} = \langle(0,1,1)\rangle, W_{1,5} = \langle(1,0,1)\rangle, W_{1,6} = \langle(1,1,0)\rangle$, $W_{1,7} = \langle(1,1,1)\rangle$ and proper 2-dimensional subspaces of $\mathbb{V}$ are $W_{2,1} = \langle(0,0,1),(0,1,0)\rangle, W_{2,2} = \langle(0,0,1),(1,0,0)\rangle$, $W_{2,3} = \langle(0,1,0),(1,0,0)\rangle, W_{2,4} = \langle(0,1,1),(1,0,1)\rangle, W_{2,5} = \langle(0,0,1),(1,1,0)\rangle$, $W_{2,6} = \langle(0,1,0),(1,0,1)\rangle, W_{2,7} = \langle(1,0,0),(0,1,1)\rangle$. The graph $\Gamma_c(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is given in Figure 2.1.

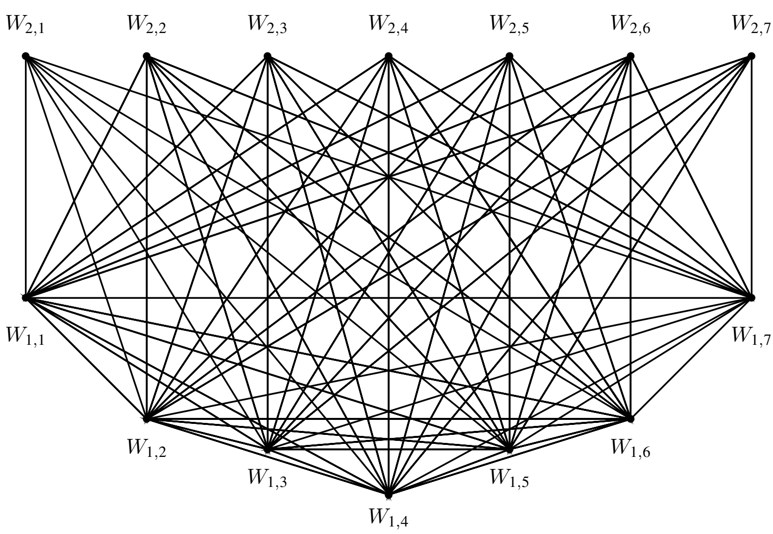

Figure 2.1: $\Gamma_c(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Lemma 2.2. *Let $\mathbb{V}$ be a finite dimensional vector space. Then the following can be observed about the coprime graph $\Gamma_c(\mathbb{V})$ of $\mathbb{V}$.*

(1) *If W_1 is a 1-dimensional subspace and W_2 be any proper subspace of $\mathbb{V}$, then W_1 and W_2 are adjacent in $\Gamma_c(\mathbb{V})$.*

(2) *Let S be the set of all one dimensional subspaces of $\mathbb{V}$. Then the induced subgraph $\Gamma_c(\langle S\rangle)$ of $\mathbb{V}$ is complete.*

(3) *If W_1 and W_2 are two distinct proper m-dimensional subspaces of $\mathbb{V}$ and $m \geq 2$, then W_1 is not adjacent to W_2 in $\Gamma_c(\mathbb{V})$.*

(4) *If* $\dim(\mathbb{V}) = 2$, *then $\Gamma_c(\mathbb{V})$ is a complete graph and hence $\Gamma_c(\mathbb{V})$ is hamiltonian.*

(5) *If* $\dim(\mathbb{V}) \geq 3$, *then $\Gamma_c(\mathbb{V})$ is not complete.*

(6) *If W is a subspace of $\mathbb{V}$ with dimension greater than 1, then $\Gamma_c(W)$ is a subgraph of $\Gamma_c(\mathbb{V})$.*

(7) *If* $\dim(\mathbb{V}) \geq 2$, *then $\Gamma_c(\mathbb{V})$ is connected.*

In the following lemma, we obtain the diameter of the coprime graph $\Gamma_c(\mathbb{V})$.

Lemma 2.3. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then the diameter,* $diam(\Gamma_c(\mathbb{V})) = \begin{cases} 1 & \text{if } \dim(\mathbb{V}) = 2; \\ 2 & \text{if } \dim(\mathbb{V}) > 2. \end{cases}$

Proof. If $\dim(\mathbb{V}) = 2$, then $\Gamma_c(\mathbb{V})$ is complete and so $diam(\Gamma_c(\mathbb{V})) = 1$. If $\dim(\mathbb{V}) > 2$, then then there exists at least two proper subspaces W_1 and W_2 of $\mathbb{V}$ whose dimensions are same and greater than or equal to 2. Hence W_1 and W_2 of $\mathbb{V}$ are not adjacent and so $d(W_1, W_2) > 1$. Also both W_1 and W_2 are adjacent to a subspace of dimension 1. Hence $diam(\Gamma_c(\mathbb{V})) = 2$. □

Lemma 2.4. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then the girth $gr(\Gamma_c(\mathbb{V})) = 3$.*

Proof. If $\dim(\mathbb{V}) = 2$, then the coprime graph is complete and hence result is trivially true. If $\dim(\mathbb{V}) \geq 3$, then there exists two linearly independent vectors $\alpha, \beta \in \mathbb{V}$. Then the subspaces $W_1 = \langle\alpha\rangle$, $W_2 = \langle\beta\rangle$ and $W_3 = \langle\alpha, \beta\rangle$ are of dimensions 1, 1 and 2 respectively. Therefore $W_1 - W_2 - W_3 - W_1$ is a triangle and hence the girth of $\Gamma_c(\mathbb{V})$ is 3. □

Lemma 2.5. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. If* $\dim(\mathbb{V}) \geq 2$, *then $\Gamma_c(\mathbb{V})$ is triangulated.*

Proof. Let $\mathbb{V}$ be a finite dimensional vector space.

Case 1. If $\dim(\mathbb{V}) = 2$, then $\Gamma_c(\mathbb{V})$ is complete and hence triangulated.

Case 2. Assume that $\dim(V) > 2$. Let W be a proper subspace of $\mathbb{V}$. Clearly there exists two subspaces W_1 & W_2 of dimension 1. Then $W - W_1 - W_2 - W$ forms a triangle. □

A remarkable simple characterization for planar graphs was given by Kuratowski.

Theorem 2.6. [20] *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Lemma 2.7. [8] *Let $\mathbb{V}$ be a finite dimensional vector space of dimension n over a finite field $\mathbb{F}$ of q elements. Then the number of k-dimensional subspaces of $\mathbb{V}$ is*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\dots(q - 1)}.$$

Now, we obtain a characterization for $\Gamma_c(\mathbb{V})$ to be planar.

Lemma 2.8. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then $\Gamma_c(\mathbb{V})$ is planar if and only if* $\dim(\mathbb{V}) = 2$ *and* $|\mathbb{F}| = 2$ *or* 3.

Proof. If $\dim(\mathbb{V}) = 2$ and $|\mathbb{F}| = 2$, then $\Gamma_c(\mathbb{V}) \cong K_3$ and so $\Gamma_c(\mathbb{V})$ is planar.

If $\dim(\mathbb{V}) = 2$ and $|\mathbb{F}| = 3$, then $\Gamma_c(\mathbb{V}) \cong K_4$ and so $\Gamma_c(\mathbb{V})$ is planar.

Conversely assume that $\Gamma_c(\mathbb{V})$ is planar. Suppose $\dim(\mathbb{V}) \geq 3$. Then as shown in Figure 2.1, $K_{7,7}$ is a subgraph of $\Gamma_c(\mathbb{V})$ and hence $\Gamma_c(\mathbb{V})$ is not planar, a contradiction. Hence $\dim(\mathbb{V}) = 2$.

If $|\mathbb{F}| \geq 4$, then, by Lemma 2.7, the number of one dimensional subspaces of $\mathbb{V}$ is at least 5. Hence $\Gamma_c(\mathbb{V})$ contains K_5 as a subgraph and so $\Gamma_c(\mathbb{V})$ is not planar, again a contradiction. Hence $|\mathbb{F}| \leq 3$. □

Now, we present another characterization of coprime graph in terms of dimension of the underlying vector space and cardinality of the base field. The following lemma identifies outer planar coprime graphs.

Lemma 2.9. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then $\Gamma_c(\mathbb{V})$ is outerplanar if and only if* $\dim(\mathbb{V}) = 2$ *and* $|\mathbb{F}| = 2$.

Proof. Since $K_{7,7}$ is not outer planar, the result follows from Lemma 2.8. □

Now, we obtain the clique number of the coprime graph.

Lemma 2.10. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then the clique number $\omega(\Gamma_c(\mathbb{V})) = |S_1| + m$ where S_1 is the set of all 1-dimensional subspaces of $\mathbb{V}$ and m is the number of prime numbers less than* $\dim(\mathbb{V})$.

Proof. Let S_1 be the set of all one dimensional subspaces of $\mathbb{V}$. Let M be the set of prime numbers greater than 2 and less than $\dim(\mathbb{V})$. Let T be the set containing exactly one $i-$dimensional subspace for each $i \in M$ and $S = S_1 \cup T$. Note that S_1 and T are disjoint and the subgraph induced by S is complete. Hence S is a clique and so $\omega(\Gamma_c(\mathbb{V})) \geq |S| = |S_1| + |T|$.

Suppose there exists a subspace U of $\mathbb{V}$ such that $\dim(U)$ is not prime. Then there is an element $W \in T$ such that $\gcd(\dim(W), \dim(U)) \neq 1$ and hence W and U are not adjacent. Hence S is a maximum clique and so $\omega(\Gamma_c(\mathbb{V})) = |S_1| + m$. □

In the following theorem, we prove that the coprime graph is weakly perfect.

Theorem 2.11. *The coprime graph $\Gamma_c(\mathbb{V})$ of a finite dimensional vector space $\mathbb{V}$ is weakly perfect.*

Proof. In view of Lemma 2.10, it is enough to prove that the chromatic number $\chi(\Gamma_c(\mathbb{V})) \geq |S_1| + m$, where S_1 is the set of all one dimensional subspaces of $\mathbb{V}$, M is be the set of prime numbers greater than 2 and less than $\dim(\mathbb{V})$ and $m = |M|$.

For, let us colour each one dimensional subspace W of $\mathbb{V}$ by a different colour using $|S_1|$ colours. Now colour subspaces of even dimension by a colour different from colours used so far. Colour the subspaces whose dimension is a multiple of 3 except those that are already coloured by some colour that was not already used. Continuing this process, we get $\chi(\Gamma_c(\mathbb{V})) \leq |S_1| + m$. Hence $\chi(\Gamma_c(\mathbb{V})) = |S_1| + m$ and hence $\Gamma_c(\mathbb{V})$ is weakly perfect. □

Theorem 2.12. *The coprime graph $\Gamma_c(\mathbb{V})$ of a finite dimensional vector space $\mathbb{V}$ is regular if and only if* $\dim(\mathbb{V}) = 2$.

Proof. If $\dim(\mathbb{V}) = 2$, then the result is obvious by Lemma2.2(4). Conversely assume that $\Gamma_c(\mathbb{V})$ is regular. Suppose $\dim(\mathbb{V}) \geq 3$. Let $W_1,\ W_2$ be subspaces of dimensions 1 and 2 respectively. Let W be another subspace of dimension 2. Clearly W_1 is adjacent to all other subspaces of dimension both 1 and 2. But W_2 and W are not adjacent. Therefore $deg(W_1) \neq deg(W_2)$ and $deg(W_1) \neq deg(W)$ which is a contradiction. Hence $\dim(\mathbb{V}) = 2$. □

Now we obtain the number of vertices and number of edges in $\Gamma_c(\mathbb{V})$ for some particular cases. Actually, we obtain these parameters through the Galois number. For a positive integer n and a prime power q, the Galois number $G(n, q)$ is defined as $G(n, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$ where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1)\ldots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\ldots(q - 1)}$$

Proposition 2.13. [3] *Let $\mathbb{V}$ be an n-dimensional vector space over a finite field of order $q = p^r$, where p is prime and r is a positive integer. Then $\Gamma_c(\mathbb{V})$ is a graph of order $G(n, q) - 2$, where $G(n, q)$ is the Galois number.*

Lemma 2.14. *Let $\mathbb{V}$ be a two dimensional vector space over a finite field of order q. Then the order and size of $\Gamma_c(\mathbb{V})$ are $(q + 1)$ and $\dfrac{q^2 + 3q + 2}{2}$ respectively.*

Proof. Let $\dim(\mathbb{V}) = 2$ and the underlying field has q elements. Then the number of 1-dimensional subspaces is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \left(\dfrac{q^2-1}{q-1}\right) = (q+1)$ and so the number of vertices in $\Gamma_c(\mathbb{V})$ is $q+1$. By Lemma 2.2 (4), the graph $\Gamma_c(\mathbb{V})$ is complete. Hence the number of edges is $\binom{q+1}{2} = \dfrac{q^2+3q+2}{2}$. □

Theorem 2.15. *Let $\mathbb{V}$ be a three dimensional vector space over a finite field of order q. Then the order and size of $\Gamma_c(\mathbb{V})$ are $2(q^2+q+1)$ and $\dfrac{3q^4+6q^3+10q^2+7q+4}{2}$ respectively.*

Proof. Note that the number of 1-dimensional and 2-dimensional subspaces are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$ respectively. Hence the number of vertices in $\Gamma_c(\mathbb{V})$ is

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 2(q^2+q+1).$$

Further, the number of edges of $\Gamma_c(\mathbb{V})$ is

$$\begin{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \\ 2 \end{pmatrix} + \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q\right) = \frac{(q^2+q+1)(q^2+q+2)}{2} + (q^2+q+1)^2$$
$$= \frac{3q^4+6q^3+10q^2+7q+4}{2}.$$ □

Theorem 2.16. *Let $\mathbb{V}$ be a four dimensional vector space over a finite field of order q. Then the order and size of $\Gamma_c(\mathbb{V})$ are $(q^4+3q^3+4q^2+3q+3)$ and $\dfrac{4q^7+11q^6+22q^5+29q^4+33q^3+26q^2+15q+8}{2}$ respectively.*

Proof. Note that the number of 1-dimensional, 2-dimensional and 3-dimensional subspaces are $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}_q$ respectively. Hence the number of vertices of $\Gamma_c(\mathbb{V})$ is

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4+3q^3+4q^2+3q+3.$$

Also the number of edges in $\Gamma_c(\mathbb{V})$ is

$$\begin{pmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \\ 2 \end{pmatrix} + \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \times \left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q\right\}\right) + \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \times \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q\right)$$
$$= \begin{pmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \\ 2 \end{pmatrix} + \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \times 2\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q\right\}\right) + \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q\right)^2$$
$$= \frac{4q^7+11q^6+22q^5+29q^4+33q^3+26q^2+15q+8}{2}.$$ □

3 Hamiltonian Characterization

In this section, we give some sufficient conditions for the existence of a hamiltonian cycle in $\Gamma_c(\mathbb{V})$.

Lemma 3.1. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. If $\dim(\mathbb{V}) = 6$ and $|\mathbb{F}| = 2$, then $\Gamma_c(\mathbb{V})$ is hamiltonian.*

Proof. Let $\dim(\mathbb{V}) = 6$ and $|\mathbb{F}| = 2$. By Lemma 2.7, the number of 1-dimensional as well as 5-dimensional subspaces of $\mathbb{V}$ is 63, the number of 2-dimensional as well as 4-dimensional subspaces of $\mathbb{V}$ is 651 and the number of 3-dimensional subspaces of $\mathbb{V}$ is 1395. Let $S_i = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$ be the set of all i-dimensional subspaces of $\mathbb{V}$ for $1 \leq i \leq 5$ where n_i is the number of i-dimensional subspaces of $\mathbb{V}$. Now, let us obtain a hamiltonian cycle of $\Gamma_c(\mathbb{V})$ as follows:

1. Construct the path P_1 using the vertices of S_3 and S_2 as $P : u_{31} - u_{21} - u_{32} - u_{22} - \cdots - u_{3\ 651} - u_{2\ 651} - u_{3\ 652}$.

2. Continue the path P_1 with vertices from S_4 and let newly formed path be $P_2 : u_{3\ 652} - u_{41} - u_{3\ 653} - u_{42} - \cdots - u_{3\ 1302} - u_{42} - u_{3\ 1303}$.

3. Continue the path P_2 with vertices from S_5 and let newly formed path be $P_3 : u_{3\ 1303} - u_{51} - u_{3\ 1304} - u_{52} - \cdots - u_{3\ 1365} - u_{5\ 63} - u_{3\ 1366}$.

4. Continue the path P_3 with vertices from S_1 and let newly formed path be $P_4 : u_{3\ 1366} - u_{11} - u_{3\ 1367} - u_{12} - \cdots - u_{3\ 1395} - u_{1\ 30}$.

5. Continue the path P_4 with vertices of S_1 and let newly formed path be $P_5 : u_{1\ 30} - u_{1\ 31} - u_{1\ 32} - \cdots - u_{1\ 63} - u_{31}$.

The entire construction of the path P_5 is exhibited in Figure 3.1. Note that P_5 is a hamiltonian cycle in $\Gamma_c(\mathbb{V})$ and so $\Gamma_c(\mathbb{V})$ is hamiltonian.

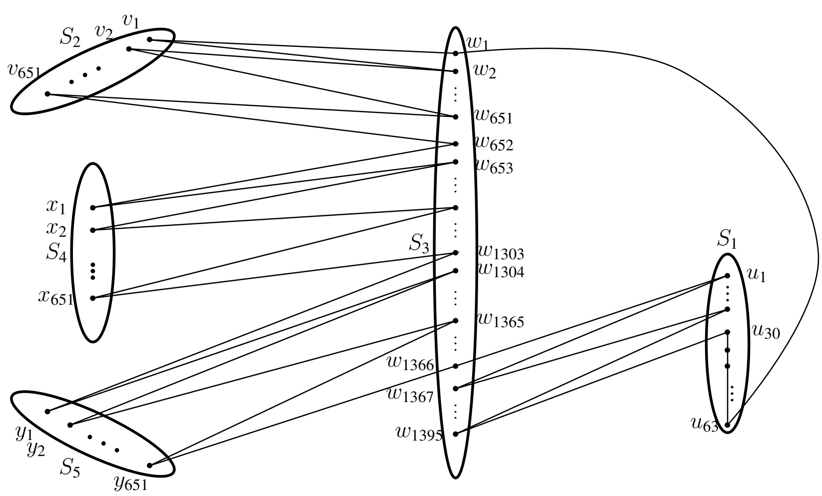

Figure 3.1: Hamiltonian cycle in $\Gamma_c(\mathbb{V})$ where $\dim(\mathbb{V}) = 6$ and $|\mathbb{F}| = 2$ □

We need the following notation to prove Lemma 3.2.
Notation: If P is a $u - v$ path then we denote P^{-1} is the reverse of P which is a $v - u$ path.

Lemma 3.2. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. If* $\dim(\mathbb{V})$ *is odd, then $\Gamma_c(\mathbb{V})$ is hamiltonian.*

Proof. Let $\dim(\mathbb{V}) = 2m + 1$ for some positive integer $n \geq 1$. Let $S_i = \{u_{i_1}, u_{i_2}, u_{i_3}, \ldots, u_{i_{m_i}}\}$ be the set of all i-dimensional subspaces of $\mathbb{V}$ for $1 \leq i \leq 2m$ and $m_i = |S_i|$. Also we observe that for $1 \leq k \leq n$, $|S_k| = |S_{(2n+1)=k}|$. Now, let us construct a Hamiltonian cycle as given below:

1. Construct a path P_1 between vertices of S_1 and S_2 as $P_1 : u = u_{11} - u_{21} - u_{12} - u_{22} - \cdots - u_{2\ m_1-1} - u_{1\ m_1} - u_{2\ m_1}$.

2. Continue the path P_1 with vertices from S_3 and let newly formed path be namely P_2 : $u_{2\ m_1} - u_{3\ 1} - u_{2\ m_1+1} - u_{3\ 2} - \cdots - u_{3\ j_1-1} - u_{2\ m_2} - u_{3\ j_1}$ where $j_1 = m_2 - m_1$.

3. Continue the path P_2 with vertices from S_4 and let newly formed path be namely P_3 : $u_{3\ j_1} - u_{4\ 1} - u_{3\ j_1+1} - u_{4\ 2} - \cdots - u_{4\ j_2-1} - u_{3\ m_3} - u_{4\ j_2}$ where $j_2 = m_3 - j_1$.

Continue this process until all the vertices of S_{n-1} are covered and this last path is denoted by P_{n-1}.

4. Similar to the construction of P_{n-1}, construct a path Q_1 between vertices of S_{2n} and S_{2n-1} namely $Q_1 : v = u_{2n\ 1} - u_{(2n-1)\ 1} - u_{2n\ 2} - u_{(2n-1)\ 2} - \cdots - u_{(2n-1)\ m_1-1} - u_{2n\ m_1} - u_{(2n-1)\ m_1}$, since $m_1 = m_{2n}$.

5. Continue the path Q_1 with vertices from S_{2n-2} and let newly formed path be namely $Q_2 : u_{(2n-1)\ m_1} - u_{(2n-2)\ 1} - u_{(2n-1)\ m_1+1} - u_{(2n-2)\ 2} - \cdots - u_{(2n-2)\ j_1-1} - u_{(2n-1)\ m_2} - u_{(2n-2)\ j_1}$ where $j_1 = m_2 - m_1$, since $m_2 = m_{(2n-1)}$.

6. Continue the path Q_2 with vertices from S_{2n-3} and let newly formed path be Q_3 : $u_{(2n-2)\ j_1} - u_{(2n-3)\ 1} - u_{(2n-2)\ j_1+1} - u_{(2n-3)\ 2} - \cdots - u_{(2n-3)\ j_2-1} - u_{(2n-2)\ m_3} - u_{(2n-3)\ j_2}$ where $j_2 = m_3 - j_1$, since $m_3 = m_{(2n-2)}$.

Continue this process until all the elements of the set S_{n+2} are coveredand this last path is denoted by Q_{n-1}.

The number of uncovered vertices in S_n and S_{n+1} remains the same.
7. Now construct a path R between vertices of S_n and S_{n+1} namely $R : u_{n\ k} - u_{(n+1)\ k+1} - u_{n\ k+1} - u_{(n+1)\ k+2} - \cdots - u_{n\ (m_n-1)} - u_{(n+1)\ m_n} - u_{n\ m_n} - u_{(n+1)\ k}$ where $k = j_{n-2}$.

The entire construction of the paths P_{n-1}, Q_{n-1} and R is indicated in Figure 3.2. Note that $P_{n-1} - R - Q_{n-1}^{-1}$ followed by the edge vu is a hamiltonian cycle.

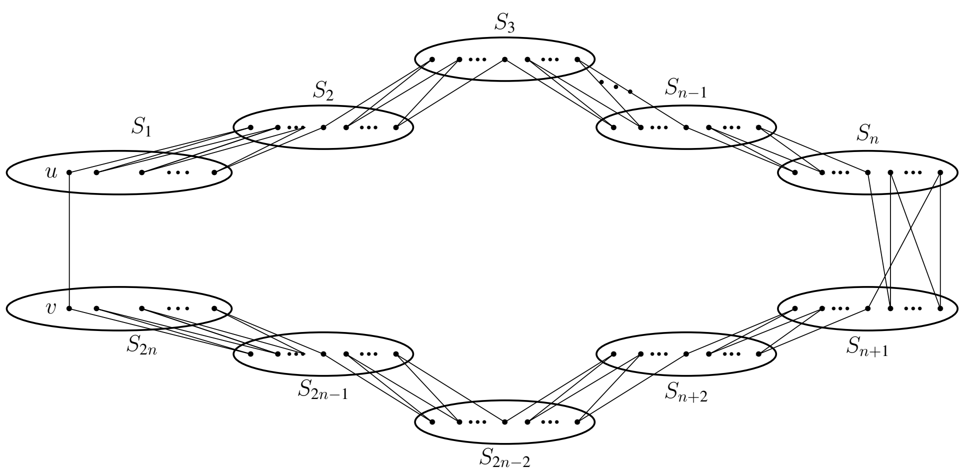

Figure 3.2: Hamiltonian cycle

□

Now, we obtain a necessary and sufficient condition $\Gamma_c(\mathbb{V})$ to be hamiltonian in terms of the dimension of $\mathbb{V}$. For this purpose, we make use of the following sufficient condition for hamiltonian graphs.

Theorem 3.3. [2, Theorem] *Let G be a connected graph. If G is a Hamiltonian graph, then for every nonempty proper subset S of vertices of G, the number of connected components of $G \setminus S$ is less than or equal to the cardinality of S.*

Theorem 3.4. *Let $\Gamma_c(\mathbb{V})$ be the coprime graph of a finite dimensional vector space $\mathbb{V}$. Then $\Gamma_c(\mathbb{V})$ is hamiltonian if and only if anyone of the following conditions hold*

(1) $\dim(\mathbb{V})$ *is odd;*

(2) $\dim(\mathbb{V}) = 2$;

(3) $\dim(\mathbb{V}) = 6$ *and* $|\mathbb{F}| = 2$.

Proof. Proof for sufficient part follows from Lemma 2.2 (4), Lemmas 3.1 and 3.2.
Conversely, assume that $\Gamma_c(\mathbb{V})$ is Hamiltonian. Suppose the condition (1) is not true. Then the dimension of $\mathbb{V}$ is even. If $\dim(\mathbb{V}) = 2$, then nothing to prove. Assume that $\dim(\mathbb{V}) = 4$.

Let S be the set of all 1-dimensional and 3-dimensional subspaces of $\mathbb{V}$ and T be the set of all 2-dimensional subspaces of $\mathbb{V}$. Clearly the number of elements in T is greater than the number of elements of S. i.e., $|T| > |S|$. Now the removal of elements of S from $\Gamma_c(\mathbb{V})$ results in a totally disconnected graph. Thus the number of connected components in $\Gamma_c(\mathbb{V}) \setminus S$ is $|T| > |S|$. By Theorem 3.3, $\Gamma_c(\mathbb{V})$ is not hamiltonian which is a contradiction. Hence $\dim(\mathbb{V}) > 4$.

Suppose $\dim(\mathbb{V}) > 6$ and $\dim(\mathbb{V}) = 2m$. Now let T be the set of all m-dimensional subspaces of $\mathbb{V}$ and $S = V\Gamma_c(\mathbb{V}) \setminus T$. Clearly the number of elements in T is greater than the number of elements of S. Now the removal of elements in S from $\Gamma_c(\mathbb{V})$ results in a totally disconnected. Also the number of connected components in $\Gamma_c(\mathbb{V}) \setminus S$ is $|T| > |S|$. By Theorem 3.3, $\Gamma_c(\mathbb{V})$ is not hamiltonian which is a contradiction. Hence $\dim(\mathbb{V}) = 6$.

Suppose $\dim(\mathbb{V}) = 6$ and $|\mathbb{F}| \geq 3$. Let T be the set of all 3-dimensional subspaces of $\mathbb{V}$ and $S = V\Gamma_c(\mathbb{V}) \setminus T$. Here again we get a contradiction as observed in the previous case. Hence $\dim(\mathbb{V}) = 6$ and $|\mathbb{F}| = 2$. □

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