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Generalization of Fuzzy Ostrowski Like Inequalities for $(m, \alpha, \beta, \gamma, \mu)$-Convex Functions

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Abstract In the present article, the generalized notion of $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) function in mixed kind is introduced 1st time, which is the generalization of 15 functions. Our fundamental objective is to develop inequalities of Ostrowski like for $(m, \alpha, \beta, \gamma, \mu)$-convex functions via Fuzzy Riemann Integrals by implementing several techniques in which Hölder's and power mean inequalities are included. Moreover, we would obtain various consequences with respect to the convexity of function as special cases and also capture several established results of different authors of different articles.

1 Introduction

About the features of convex functions, we code some lines from [24] "Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool that is second-derivative test for recognizing convexity".

We would generalize the idea of convex functions in order to generalize Ostrowski's inequality. In this way, we may quickly identify the generalizations and specific instances of the inequality. We recall several definitions from the literature [2, 17, 18, 19] for various convex functions. For some related study see [8, 26]. Throughout the article K and J are convex subsets of $\mathbb{R}$ and $[0, \infty)$ respectively. Further we will use the convention that $0^0 = 1$ (see [5]).

Definition 1.1. A function $g : K \to \mathbb{R}$ is known as convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) tg(y) + (1-t)g(z), \tag{1.1}$$

$\forall\, y, z \in K, t \in [0, 1]$.

We recall term of $P-$convex(concave) function (see [9]).

Definition 1.2. A function $g : K \to \mathbb{R}$ is known as P-convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) g(y) + g(z),\ g \geq 0 \tag{1.2}$$

$\forall\, y, z \in K, t \in [0, 1]$.

We recall definition of quasi-convex function (see [13]).

Definition 1.3. A function $g : K \to \mathbb{R}$ is said to be quasi-convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) \max\{g(y), g(z)\} \tag{1.3}$$

$\forall\, y, z \in K, t \in [0,1]$.

Now we present term of s-convex(concave) functions in the 1st and 2nd kinds respectively (see [22]).

Definition 1.4. Suppose $s \in (0,1]$. A function $g : J \to [0,\infty)$ is said to be s-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^s g(y) + (1-t^s) g(z), \tag{1.4}$$

$\forall\, y, z \in J, t \in [0,1]$.

Remark 1.5. If we include $s = 0$ in the above inequality then we obtain the refinement of quasi-convex function.

Definition 1.6. Let $s \in (0,1]$. A function $g : J \to [0,\infty)$ is said to be s-convex(concave) in the 2nd kind, if

$$g(ty + (1-t)z) \leq (\geq) t^s g(y) + (1-t)^s g(z), \tag{1.5}$$

$\forall\, y, z \in J, t \in [0,1]$.

Remark 1.7. If we include $s = 0$ in the above inequality then we obtain the P-convex function.

Now we present term of m-convex(concave) functions (see [14]).

Definition 1.8. Let $m \in [0,1]$. A function $g : J \to \mathbb{R}$ is known as m-convex(concave), if

$$g(ty + m(1-t)z) \leq (\geq) t g(y) + m(1-t) g(z) \tag{1.6}$$

$\forall\, y, z \in J,\, t \in [0,1]$.

Remark 1.9. The terms of standard convex(concave) functions and star-shaped functions are obtained if we choose $m = 1$ and $m = 0$ in the above inequality respectively.

Mihesan stated (α, m)-convexity (see [20]).

Definition 1.10. Suppose $(\alpha, m) \in [0,1]^2$. A function $g : J \to \mathbb{R}$ is said to be (α, m)-convex(concave), if

$$g(ty + m(1-t)z) \leq (\geq) t^\alpha g(y) + m(1-t^\alpha) g(z) \tag{1.7}$$

$\forall\, y, z \in J,\, t \in [0,1]$. Another representation of above function is (m, s)-convex(concave) in the 1st kind.

Definition 1.11. [11] Let $(s, r) \in [0,1]^2$. A function $g : J \to [0,\infty)$ is said to be (s, r)-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{rs} g(y) + (1-t^r)^s g(z), \tag{1.8}$$

$\forall\, y, z \in J, t \in [0,1]$.

Definition 1.12. [10] Suppose $(\alpha, \beta) \in [0,1]^2$. A function $g : J \to [0,\infty)$ is said to be (α, β)-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^\alpha g(y) + (1-t^\beta) g(z), \tag{1.9}$$

$\forall\, y, z \in J, t \in [0,1]$.

Definition 1.13. [10] Suppose $(\alpha, \beta) \in [0,1]^2$. A function $g : J \to [0,\infty)$ is said to be (α, β)-convex(concave) in the 2nd kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha} g(y) + (1-t)^{\beta} g(z), \tag{1.10}$$

$\forall\, y, z \in J, t \in [0,1]$.

The following definition is extracted from [4] (see also [3]).

Definition 1.14. Let $(m, s, r) \in [0,1]^3$. A function $g : J \to [0,\infty)$ is said to be (m, s, r)-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{rs} g(y) + m(1-t^r)^s g(z), \tag{1.11}$$

$\forall\, y, z \in J, t \in [0,1]$.

Upcoming definition is $(\alpha, \beta, \gamma, \mu)$-convex(concave) function (see [11]).

Definition 1.15. Suppose $(\alpha, \beta, \gamma, \mu) \in [0,1]^4$. A function $g : J \to [0,\infty)$ is known as $(\alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + (1-t^{\beta})^{\mu} g(z), \tag{1.12}$$

$\forall\, y, z \in J, t \in [0,1]$.

We introduce a new class of function which we will call as class of $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) functions in the mixed kind and having whole previously said classes of functions. This terminology is applied sequentially in the article.

Definition 1.16. Suppose $(m, \alpha, \beta, \gamma, \mu) \in [0,1]^5$. A function $g : J \to [0,\infty)$ is known as $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + m(1-t^{\beta})^{\mu} g(z), \tag{1.13}$$

$\forall\, y, z \in J, t \in [0,1]$.

We can capture all above-stated definitions as our special cases by putting different values of $m, \alpha, \beta, \gamma, \mu$. Further we can also introduce some new classes as well for example when $\alpha = 1, \beta = s, \gamma = s, \mu = 1$.

Definition 1.17. Let $(m, s) \in (0,1]^2$. A function $g : J \to [0,\infty)$ is said to be (m, s)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{s} g(y) + m(1-t)^{s} g(z) \tag{1.14}$$

$\forall\, y, z \in J,\, t \in [0,1]$.

Similarly when we put $\gamma = 1, \mu = 1$.

Definition 1.18. Let $(m, \alpha, \beta) \in [0,1]^3$. A function $g : J \to [0,\infty)$ is known as (m, α, β)-convex(concave) in the 1st kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1-t^{\beta}) g(z), \tag{1.15}$$

$\forall\, y, z \in J, t \in [0,1]$.

Again similarly when we put $\beta = 1, \gamma = 1, \mu = \beta$.

Definition 1.19. Let $(m, \alpha, \beta) \in [0,1]^3$. A function $g : J \to [0,\infty)$ is known as (m, α, β)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1-t)^{\beta} g(z), \tag{1.16}$$

$\forall\, y, z \in J, t \in [0,1]$.

Remark 1.20. The following scenarios are found in Definition 1.16 as special cases.

S.No.	m	α	β	γ	μ	Definition obtained	Kind
1	1	–	–	–	–	$(\alpha, \beta, \gamma, \mu)$-convex(concave) function	mixed
2	1	–	1	1	β	(α, β)-convex(concave) function	2nd
3	1	–	–	1	1	(α, β)-convex(concave) function	1st
4	1	r	r	s	s	(s, r)-convex(concave) function	mixed
5	–	1	s	s	1	(m, s)-convex(concave) function	1st
6	–	1	1	1	1	m-convex(concave) function	–
7	1	s	1	1	s	s-convex(concave) function	2nd
8	1	s	s	1	1	s-convex(concave) function	1st
9	1	0	0	1	1	quasi-convex(concave) function	–
10	1	0	1	1	0	P-convex(concave) function	–
11	1	1	1	1	1	ordinary convex(concave) function	–

Inequalities play a important role in all scientific fields practically. Our primary goal is to generalize Ostrowski's like inequalities for various type of convex functions.

The convexity theory is closely related to the inequalities theory. In literature, many well-known inequalities are direct consequence of convex functions. The Ostrowski's inequality is a remarkable inequality for convexity that has been thoroughly researched in recent decades. In 1938, Ostrowski [23] proved the below interesting inequality for differentiable mappings with bounded derivatives. It is sometimes referred to as "Ostrowski inequality" in the literature.

Proposition 1.21. *Let $g : K \to \mathbb{R}$ be differentiable mapping in the interior K^o of K, here $j, k \in K^o$ along $k > j$. If $\mathfrak{M} \geq |g'(y)|$, each $y \in [j, k]$ where $0 \leq \mathfrak{M}$ is constant, then*

$$\left| g(y) - \frac{1}{k-j} \int_j^k g(\tau) d\tau \right| \leq \mathfrak{M}(k-j) \left[\frac{1}{4} + \frac{\left(y - \frac{j+k}{2}\right)^2}{(k-j)^2} \right]. \tag{1.17}$$

The value $\frac{1}{4}$ is the best possible constant that this cannot be replaced by the smaller one.

In 2003, Anastassiou [1] extended Ostrowski's like inequalities into the Fuzzy environment, recognizing that fuzziness is a natural reality different than randomness and determinism. The idea of fuzzy Riemann Integrals was introduced By Congxin and Ming in [6]. Fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

2 Preliminaries

Under this heading, we recall some basic definitions and notations that would help us in the sequel manner.

Definition 2.1. [6] $\rho : \mathbb{R} \to [0, 1]$ is called a fuzzy number if satisfies below points

(i) ρ is normal (i.e, $\exists$ an $y_0 \in \mathbb{R}$ such that $1 = \rho(y_0)$).

(ii) ρ is a convex fuzzy set, i.e., $yt + (1-t)z) \geq \min\{\rho(y), \rho(z)\}$, each $z, y \in \mathbb{R}$, $t \in [0, 1]$. (ρ is said to be convex fuzzy subset)

(iii) ρ is upper semi continuous in the interval $\mathbb{R}$, i.e., every $y_0 \in \mathbb{R}$ & $\forall\, \epsilon > 0$, $\exists$ Neighborhood $V(y_0) : \rho(y) \leq \rho(y_0) + \epsilon, \forall\, y \in V(y_0)$.

(iv) The set $[\rho]^0 = \overline{\{y \in \mathbb{R} : \rho(y) > 0\}}$ is compact where $\overline{A}$ denotes the closure of A.

$\mathbb{R}^F$ denotes the set of all fuzzy numbers. For $\alpha \in (0, 1]$ and $\rho \in \mathbb{R}^F$, $[\rho]^\alpha = \{y \in \mathbb{R} : \rho(y) \geq \alpha\}$. Then, from (1) to (4) it follows that the $\alpha-$level set $[\rho]^\alpha$ is a closed interval $\forall\, \alpha \in [0, 1]$. Moreover, $[\rho]^\alpha = [\rho_-^{(\alpha)}, \rho_+^{(\alpha)}]\ \forall\, \alpha \in [0, 1]$, where $\rho_-^{(\alpha)} \leq \rho_+^{(\alpha)}$ and $\rho_-^{(\alpha)}, \rho_+^{(\alpha)} \in \mathbb{R}$, i.e., $\rho_-^{(\alpha)}$ and $\rho_+^{(\alpha)}$ are the endpoints of $[\rho]^\alpha$.

Definition 2.2. [7] Let $\rho, \varrho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the addition and scalar multiplication are defined by the equations, respectively.

(i) $[\rho \oplus \varrho]^\alpha = [\rho]^\alpha + [\varrho]^\alpha$

(ii) $[a \odot \rho]^\alpha = a[\rho]^\alpha$

$\forall\ \alpha \in [0,1]$ where $[\rho]^\alpha + [\varrho]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $a[\rho]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Proposition 2.3. *[15] Suppose $\rho, \varrho \in \mathbb{R}^F$ & $a \in \mathbb{R}$, then below points hold:*

(i) $1 \odot \rho = \rho$

(ii) $\rho \oplus \varrho = \varrho \oplus \rho$

(iii) $a \odot \rho = \rho \odot a$

(iv) $[\rho]^{\alpha_1} \subseteq [\rho]^{\alpha_2}$ *whenever* $0 \leq \alpha_2 \leq \alpha_1 \leq 1$

(v) For any α_n converging increasingly to $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty}[\rho]^{\alpha_n} = [\rho]^\alpha$.

Definition 2.4. [6] Assume $D : \mathbb{R}^F \times \mathbb{R}^F \to \mathbb{R}_+ \cup \{0\}$ is a function, represented as

$$D(\rho, \varrho) = \sup_{\alpha \in [0,1]} \max \left\{ \left| \rho_-^{(\alpha)}, \varrho_-^{(\alpha)} \right|, \left| \rho_+^{(\alpha)}, \varrho_+^{(\alpha)} \right| \right\}$$

$\forall\ \rho, \varrho \in \mathbb{R}^F$, then D is Metric on $\mathbb{R}^F$.

Proposition 2.5. *[6] Suppose $\varrho, \rho, e, \sigma \in \mathbb{R}^F$ & $a \in \mathbb{R}$, then*

(i) $(\mathbb{R}^F, D)$ *is a Complete Metric space*

(ii) $D(\rho \oplus \sigma, \varrho \oplus \sigma) = D(\rho, \varrho)$

(iii) $D(a \odot \rho, a \odot \varrho) = |a| D(\rho, \varrho)$

(iv) $D(\rho \oplus \varrho, \sigma \oplus e) = D(\rho, \sigma) + D(\varrho, e)$

(v) $D(\rho \oplus \varrho, \widetilde{0}) \leq D(\rho, \widetilde{0}) + D(\varrho, \widetilde{0})$

(vi) $D(\rho \oplus \varrho, \sigma) \leq D(\rho, \sigma) + D(\varrho, \widetilde{0})$

where $\widetilde{0} \in \mathbb{R}^F$ is expressed as $\widetilde{0}(y) = 0$ for every $y \in \mathbb{R}$.

Definition 2.6. [7] Suppose $z, y \in \mathbb{R}^F$ if $\exists\ \theta \in \mathbb{R}^F$, $\ni y = \theta \oplus z$, then θ is H-Difference of y and z stated as $\theta = y \ominus z$.

Definition 2.7. [7] Suppose $T := [y_0, y_0 + \gamma] \subseteq \mathbb{R}$, with $0 < \gamma$. A function $g : T \to \mathbb{R}^F$ is said to be H-Differentiable at $y \in T$ If $\exists\ g'(y) \in \mathbb{R}^F$ i.e., both limits (with respect to the Metric D)

$$\lim_{h \to 0^+} \frac{g(y+h) \ominus g(y)}{h}, \quad \lim_{h \to 0^+} \frac{g(y) \ominus g(y-h)}{h}$$

exists and are equal to $g'(y)$. We call g' the derivative or $H-$Derivative of g at y. If g is $H-$Differentiable at any $y \in T$, we call g differentiable or $H-$Differentiable and it has $H-$Derivative over T the function g'.

Definition 2.8. [12] Let $g : [j,k] \to \mathbb{R}^F$ if every $\zeta > 0$, $\exists\ 0 < \eta$, for any partition $P = \{[\rho, \varrho]; \delta\}$ of $[j,k]$ along norm $\Delta(P) < \eta$, we have

$$D\left(\sum_P^{*} (\varrho - \rho) \odot g(\delta, K) \right) < \zeta,$$

then we say that g is Fuzzy-Riemann integrable to the interval $K \in \mathbb{R}^F$,it can be written as

$$K := (FR) \int_j^k g(y)dy.$$

Regarding some recent results related to Fuzzy-Riemann integrals, refer to [16].

Well-known classical Hölder's integral inequality in its general integral form is defined as follows [21]:

Proposition 2.9. *If* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *where* $g \in L_p$, $z \in L_q$, *then* $gz \in L_1$ *and*

$$\int |g(y)z(y)|dy \leq \|g\|_p \|z\|_q,$$

if $\|g\|_p = (\int |g(y)|^p dy)^{\frac{1}{p}} < \infty$. L *represents class of Lebesgue integral functions.*

Remark 2.10. Note that if we put $p = q = 2$, the above inequality becomes Cauchy-Schwarz inequality. Also, if we put $q = 1$ and let $p \to \infty$, then we attain,

$$\int |g(y)z(y)|dy \leq ||g||_\infty ||z||_1,$$

where $||g||_\infty$ stands for the essential supremum of $|g|$, i.e.,

$$||g||_\infty = ess \sup_{\forall y} |g(y)|.$$

The following inequality is known in literature as Power mean integral inequality (see [25]).

Proposition 2.11. *If* g *and* z *are real valued functions defined on* K *with* $|g|$ *and* $|g||z|^q$ *are integrable on* K *then for* $q \geq 1$ *we have:*

$$\int_a^b |g(y)||z(y)|dy \leq \left(\int_a^b |g(y)|dy \right)^{1-\frac{1}{q}} \left(\int_a^b |g(y)||z(y)|^q dy \right)^{\frac{1}{q}}.$$

This research primarily aims to develop generalized Fuzzy Ostrowski's like inequalities for $(m, \alpha, \beta, \gamma, \mu)-$convex function in the mixed kind. As special cases, we derive several findings concerning the convexity of function and also capture several previous established consequences of different authors of different articles [18] and [27].

3 Generalized Fuzzy Ostrowski's Like Inequalities for $(m, \alpha, \beta, \gamma, \mu)$-Convex Functions

Regarding to prove our main results, we require the following Lemma.

Lemma 3.1. *Suppose* $g : K \to \mathbb{R}^F$ *is differentiable mapping in interval* K^o *where* $mj, mk \in K$ *along* $mj < mk$, *if* $g' \in C^F[mj, mk] \bigcap L^F[mj, mk]$, *then*

$$\begin{aligned} &\frac{1}{k-j} \odot (FR) \int_{mj}^{mk} g(u)du \\ &\oplus \frac{(y-mj)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(ty + m(1-t)j)dt \\ &= m \odot g(y) \oplus \frac{(mk-y)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(ty + m(1-t)k)dt \end{aligned} \tag{3.1}$$

$\forall\, y \in (mj, mk)$.

Proof. See proof Lemma 3.1 of [27]. □

Remark 3.2. If we choose $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$, we capture Lemma 3.1 of [27].

Remark 3.3. If we choose $m = 1$ and $\mu = \delta$ in Theorem 3.4, we capture Lemma 3.1 of [18].

Theorem 3.4. *Let all the suppositions of Lemma 3.1 be true and assuming* $D(g'(y),\widetilde{0})$ *is* $(m,\alpha,\beta,\gamma,\mu)$*-convex function in the interval* $[mj,mk]$ *and* $D(g'(y),\widetilde{0})\leq M$*. Then*

$$D\left(m\odot g(y),\frac{1}{k-j}\odot(FR)\int_{mj}^{mk}g(u)du\right)$$
$$\leq M\left(\frac{1}{\alpha\gamma+2}+\frac{m}{\beta}B\left(\frac{2}{\beta},\mu+1\right)\right)I(y), \tag{3.2}$$

$\forall\, y\in(mj,mk)$ *and* $\beta>0$*, where* $I(y)=\frac{(y-mj)^2+(mk-y)^2}{k-j}$.

Proof. Utilizing Lemma 3.1 and using Proposition 2.5, then

$$\begin{aligned}
&D\left(m\odot g(y),\frac{1}{k-j}\odot(FR)\int_{mj}^{mk}g(u)du\right)\\
&\leq D\left(\frac{(y-mj)^2}{k-j}\odot(FR)\int_0^1 tg'(ty+m(1-t)j)dt,\widetilde{0}\right)\\
&+D\left(\frac{(mk-y)^2}{k-j}\odot(FR)\int_0^1 tg'(ty+m(1-t)k)dt,\widetilde{0}\right)\\
&=\frac{(y-mj)^2}{k-j}D\left((FR)\int_0^1 tg'(ty+m(1-t)j)dt,\widetilde{0}\right)\\
&+\frac{(mk-y)^2}{k-j}D\left((FR)\int_0^1 tg'(ty+m(1-t)k)dt,\widetilde{0}\right)\\
&\leq\frac{(y-mj)^2}{k-j}\int_0^1 tD\left(g'(ty+m(1-t)j),\widetilde{0}\right)dt\\
&+\frac{(mk-y)^2}{k-j}\int_0^1 tD\left(g'(ty+m(1-t)k),\widetilde{0}\right)dt,
\end{aligned} \tag{3.3}$$

as $D(g'(y),\widetilde{0})$ be $(m,\alpha,\beta,\gamma,\mu)-$convex functon and $D(g'(y),\widetilde{0})\leq M$, then

$$\begin{aligned}
D\left(g'(ty+m(1-t)j),\widetilde{0}\right)&\leq t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m\left(1-t^{\beta}\right)^{\mu}D\left(g'(j),\widetilde{0}\right)\\
&\leq M\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]
\end{aligned} \tag{3.4}$$

Also

$$\begin{aligned}
D\left(g'(ty+m(1-t)k),\widetilde{0}\right)&\leq t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m\left(1-t^{\beta}\right)^{\mu}D\left(g'(k),\widetilde{0}\right)\\
&\leq M\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]
\end{aligned} \tag{3.5}$$

Now using (3.4) & (3.5) in (3.3) we get (3.2). □

Note: Where B is Beta function and represented as $B(l,n)=\int_0^1 t^{l-1}(1-t)^{n-1}dt=\dfrac{\lceil(l)\lceil(n)}{\lceil(l+n)}$. Since $\lceil(l)=\int_0^{\infty}e^{-u}u^{l-1}du.$

Remark 3.5. Some remarks regarding Theorem 3.4 are below as special cases.

S.No.	m	α	β	γ	μ	Fuzzy Ostrowski Like Inequality for	Kind
1	–	–	1	1	β	(m, α, β)-convex function	2nd
2	–	–	–	1	1	(m, α, β)-convex function	1st
3	–	r	r	s	s	(m, s, r)-convex function	mixed
4	1	–	1	1	β	(α, β)-convex function	2nd
5	1	–	–	1	1	(α, β)-convex function	1st
6	1	r	r	s	s	(s, r)-convex function	mixed
7	–	s	1	1	s	(m, s)-convex function	2nd
8	–	s	s	1	1	(m, s)-convex function	1st
9	–	1	1	1	1	m-convex function	–
10	1	s	1	1	s	s-convex function	2nd
11	1	s	s	1	1	s-convex function	1st
12	1	0	0	1	1	quasi-convex function	–
13	1	0	1	1	0	P-convex function	–
14	1	1	1	1	1	ordinary convex function	–

Remark 3.6. If we choose $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.4, we capture the main Theorem 3.2 of [27].

Remark 3.7. If choose $m = 1$ and $\mu = \delta$ in Theorem 3.4, we capture the main Theorem 3.1 of [18].

Remark 3.8. By choosing suitable values of $m, \alpha, \beta, \gamma, \mu$ in Theorem 3.4, we capture all results of Corollary 3.1 of [18].

Theorem 3.9. *Let all the suppositions of Lemma 3.1 be true and assuming $[D(g'(y), \widetilde{0})]^q$ is $(m, \alpha, \beta, \gamma, \mu)$-convex function in the interval $[mj, mk]$, $q \geq 1$ and $D(g'(y), \widetilde{0}) \leq M$. Then*

$$D\left(m \odot g(y), \frac{1}{k-j} \odot (FR) \int_{mj}^{mk} g(u)du\right)$$
$$\leq \frac{M}{(2)^{1-\frac{1}{q}}} \left(\frac{1}{\alpha\gamma + 2} + \frac{m}{\beta} B\left(\frac{2}{\beta}, \mu + 1\right)\right)^{\frac{1}{q}} I(y). \tag{3.6}$$

$\forall\, y \in (mj, mk)$ *and* $\beta > 0$.

Proof. Utilizing (3.3) and using power mean inequality, then

$$D\left(m \odot g(y), \frac{1}{k-j} \odot (FR) \int_{mj}^{mk} g(u)du\right)$$
$$\leq \frac{(y-mj)^2}{k-j} \int_0^1 tD\left(g'(ty + m(1-t)j), \widetilde{0}\right) dt$$
$$+ \frac{(mk-y)^2}{k-j} \int_0^1 tD\left(g'(ty + m(1-t)k), \widetilde{0}\right) dt$$
$$\leq \frac{(y-mj)^2}{k-j} \left(\int_0^1 tdt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left[D\left(g'(ty + m(1-t)j), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(mk-y)^2}{k-j} \left(\int_0^1 tdt\right)^{1-\frac{1}{q}} \left(\int_0^1 t\left[D\left(g'(ty + m(1-t)k), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \tag{3.7}$$

As $[D(g'(y),\widetilde{0})]^q$ is $(m,\alpha,\beta,\gamma,\mu)-$convex function and $D(g'(y),\widetilde{0}) \leq M$, then

$$\begin{aligned}
&\left[D\Big(g'(ty+m(1-t)j),\widetilde{0}\Big)\right]^q \\
&\leq t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q + m\left(1-t^{\beta}\right)^{\mu}\left[D\left(g'(j),\widetilde{0}\right)\right]^q \\
&\leq M^q\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]
\end{aligned} \tag{3.8}$$

Also

$$\begin{aligned}
&\left[D\Big(g'(ty+m(1-t)k),\widetilde{0}\Big)\right]^q \\
&\leq t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q + m\left(1-t^{\beta}\right)^{\mu}\left[D\left(g'(k),\widetilde{0}\right)\right]^q \\
&\leq M^q\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]
\end{aligned} \tag{3.9}$$

Now using (3.8) and (3.9) in (3.7) we get (3.6). □

Remark 3.10. Since we have provided remarks (1) to (14) for Theorem 3.4, all of the remarks employ to Theorem 3.9.

Remark 3.11. If we choose $q = 1$ in Theorem 3.9, we capture our main Theorem 3.4.

Remark 3.12. If we choose $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.9, we capture the Theorem 3.4 of [27].

Remark 3.13. If we choose $m = 1$ and $\mu = \delta$ in Theorem 3.9, we capture the Theorem 3.2 of [18].

Remark 3.14. By choosing suitable values of $m,\alpha,\beta,\gamma,\mu$ in Theorem 3.9, we capture all results of Corollary 3.2 and Remarks 3.1 of [18].

Theorem 3.15. *Let all the suppositions of Lemma 3.1 be true and assuming* $[D(g'(y),\widetilde{0})]^q$ *is* $(m,\alpha,\beta,\gamma,\mu)$*-convex function in the interval* $[mj,mk]$*,* $p,q>1$ *and* $D(g'(y),\widetilde{0}) \leq M$*. Then*

$$\begin{aligned}
&D\Big(m\odot g(y),\frac{1}{k-j}\odot(FR)\int_{mj}^{mk}g(u)du\Big) \\
&\leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{m}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}} I(y),
\end{aligned} \tag{3.10}$$

$\forall\, y \in (mj,mk)$ *and* $\beta > 0$*, where* $p^{-1}+q^{-1}=1$.

Proof. By (3.3) and using Hölder's inequality, then

$$\begin{aligned}
&D\Big(m\odot g(y),\frac{1}{k-j}\odot(FR)\int_{mj}^{mk}g(u)du\Big) \\
&\leq \frac{(y-mj)^2}{k-j}\int_0^1 tD\Big(g'(ty+m(1-t)j),\widetilde{0}\Big)dt \\
&+\frac{(mk-y)^2}{k-j}\int_0^1 tD\Big(g'(ty+m(1-t)k),\widetilde{0}\Big)dt \\
&\leq \frac{(y-mj)^2}{k-j}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\Big(g'(ty+m(1-t)j),\widetilde{0}\Big)\right]^q dt\right)^{\frac{1}{q}} \\
&+\frac{(mk-y)^2}{k-j}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\Big(g'(ty+m(1-t)k),\widetilde{0}\Big)\right]^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.11}$$

As $[D(g'(y),\widetilde{0})]^q$ is $(m,\alpha,\beta,\gamma,\mu)$-convex function and $M \geq D(g'(y),\widetilde{0})$, then

$$\begin{aligned}&\left[D\left(g'(ty+m(1-t)j),\widetilde{0}\right)\right]^q\\&\leq t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q+m\left(1-t^{\beta}\right)^{\mu}\left[D\left(g'(j),\widetilde{0}\right)\right]^q\\&\leq M^q\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]\end{aligned}\tag{3.12}$$

Also

$$\begin{aligned}&\left[D\left(g'(ty+m(1-t)k),\widetilde{0}\right)\right]^q\\&\leq t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q+m\left(1-t^{\beta}\right)^{\mu}\left[D\left(g'(k),\widetilde{0}\right)\right]^q\\&\leq M^q\left[t^{\alpha\gamma}+m\left(1-t^{\beta}\right)^{\mu}\right]\end{aligned}\tag{3.13}$$

Now using (3.12) and (3.13) in (3.11) we get (3.10). □

Remark 3.16. Since we have provided remarks (1) to (14) for Theorem 3.4, all of the remarks employ to Theorem 3.15.

Remark 3.17. If we choose $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.15, we capture the Theorem 3.3 of [27].

Remark 3.18. If we choose $m = 1$ and $\mu = \delta$ in Theorem 3.15, we capture the Theorem 3.3 of [18].

Remark 3.19. By choosing suitable values of $m,\alpha,\beta,\gamma,\mu$ in Theorem 3.15, we capture all results of Corollary 3.3 and Remarks 3.2 of [18].

4 Conclusion

As we all know "Ostrowski inequality is one of the most celebrated inequalities". We introduced the generalized idea of $(m,\alpha,\beta,\gamma,\mu)$-convex function in the mixed kind for the first time in this current article, which contains the generalization of many functions that are convex, P-convex, quasi-convex, s-convex in the 1^{st} kind, s-convex in the 2^{nd} kind, m-convex, (m,s)-convex in the 1st kind, (m,s)-convex in the 2nd kind, (s,r)-convex in the mixed kind, (α,β)-convex in the 1^{st} kind, (α,β)-convex in the 2^{nd} kind, (m,s,r)-convex in the mixed kind, (m,α,β)-convex in the 1st kind, (m,α,β)-convex in the 2nd kind and $(\alpha,\beta,\gamma,\mu)$-convex in the mixed kind. We proved the generalized Ostrowski's like inequalities for $(m,\alpha,\beta,\gamma,\mu)$-convex functions via fuzzy Riemann integrals by implementing Hölder's and power mean inequalities. Further that we obtained several consequences with respect to the convexity of function as special cases and captured various established results of different authors of different articles [18] and [27].

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