Palestine Journal of Mathematics

Vol 14(2)(2025) , 820–826

On Nil-Symmetric Modules

Kh. Herachandra, K. Praminda and M. Rhoades

Communicated by Christian Lomp

MSC 20210 Classifications: Primary 16D10,16D20,16S50.

Keywords and phrases: Reduced module, Semicommutative module, Weakly Semicommutative module, Symmetric module, Nil-symmetric module.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Koijam Praminda Singh was very thankful to the Manipur University for their support and facility.

Corresponding Author: K. Praminda

Abstract In this paper, we have introduced the notion of nil-symmetric modules as a generalisation of symmetric modules and reduced modules by working on the context of nilpotent elements of a module and have also investigated some of its properties. We have also extended various results on symmetric and other classes of modules to that of nil-symmetric modules and have also shown that there is a module which is nil-symmetric but not symmetric. We prove that localizations of nil-symmetric modules are nil-symmetric. It has also been shown that ${}_RM$ is nil-symmetric if and only if ${}_{T_n(R)}T_n(M)$ is nil-symmetric.

1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules over the ring R. $T_n(R)$ denotes the ring of all $n \times n$ upper triangular matrices over R. Let $T(M) = \{m \in M : rm = 0$ for some non-zero divisors $r \in R\}$. Torsion of M is defined as $Tor(M) = \{m \in M : rm = 0$ for some non-zero $r \in R\}$. Clearly, $T(M) \subseteq Tor(M)$. If R is an integral domain, they are same. $C(R)$ denotes the centre of a ring R and defined by $C(R) = \{r \in R : ra = ar$ for all $a \in R\}$. Here, D denotes a non-commutative domain. $Nil_R(M)$ is the set of all nilpotent elements of a left R-module M.

Recall in [2], J. Lambek introduced the notion of symmetric ring. A ring R is symmetric if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $bac = 0$; it is easily seen that this is left-right symmetric concept. U.S. Chakraborty and K. Das introduced the concept of nil-symmetric rings as a generalisation of symmetric rings and a particular case of nil-semicommutative rings in [12]. A ring R is called right (left) nil-symmetric if whenever, for every $a, b \in nil(R)$ and for every $c \in R$ satisfy $abc = 0(cab = 0)$, we have $acb = 0$. A ring R is nil-symmetric if it is both right and left nil-symmetric. Thus, every symmetric ring is nil-symmetric but the converse need not be true in general as in [[12], Example 3], if R is a reduced ring, then $T_2(R)$ is a nil-symmetric ring but not symmetric.

In [2] and [9], a module ${}_RM$ is symmetric if whenever $a, b \in R, m \in M$ satisfy $abm = 0$, we have $bam = 0$. M. B. Rege and A. M. Buhphang studied various properties of symmetric modules. The relationship of symmetric modules with reduced modules were also studied in [8]. Symmetric modules were generalised to α-symmetric modules by Agayev, Halicioglu and Harmanci in [6].

A ring R is reduced if it has no non-zero nilpotent elements. The reduced ring concept was extended to modules by Lee and Zhou in [11]. In [5], the relationship of reduced modules with ZI-modules was studied by Agayev and Harmanci. A left R-module M is reduced if it satisfies any of the following conditions:

(i) whenever $a \in R, m \in M$ satisfy $a^2m = 0$, we have $aRm = 0$.

(ii) whenever $a \in R, m \in M$ satisfy $am = 0$, we have $aM \cap Rm = 0$. In [4], M. Dutta and

Singh introduced the idea of weak reduced and weak rigid module as a generalisation of reduced and rigid module. They stated that a left R-module M is weak reduced if whenever $a^2m = 0$ $\forall a \in R$ and $m \in M$ implies $aRm \subseteq Nil_R(M)$ and a left R-module M is weak rigid if whenever $a^2m = 0$ $\forall a \in R$ and $m \in M$ implies $am \in Nil_R(M)$.

In [1], Ssevviiri and Groenewald introduced the concept of nilpotent elements of a module. A non-zero element $m \in M$ is said to be a nilpotent element of M if there exist $0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0$ but $rm \neq 0$. We take the zero element of M as a nilpotent element. In this paper the term "nil" is used to generalize symmetric module by using the definition of nilpotent elements of a module.

Recall, a left R-module M is called semicommutative (a ZI-module) if whenever $am = 0$ implies $aRm = 0$ for all $a \in R$ and $m \in M$. In [7], Ansari and Singh introduced weakly semicommutative module as a generalisation of semicommutative module. A left R-module M is said to be weakly semicommutative if whenever $am = 0$ implies $aRm \subseteq Nil_R(M)$ for all $a \in R$ and $m \in M$.

2 Nilpotent elements of modules

In [1], nilpotent elements of a module can be defined as:

Definition 2.1. An element $m \in M$ is said to be a nilpotent element if either $m = 0$ or there exist $0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0$ but $rm \neq 0$, i.e., $Nil_R(M) = \{m \in M | \exists 0 \neq r \in R$ and $k \in \mathbb{N}$ such that $r^k m = 0, rm \neq 0\} \cup \{0\}$.

In [7], it is stated that if m is an element of a left R-module M, then the following conditions are equivalent:

(i) There exist $r \in R$ and $n \geq 2$ such that $r^n m = 0$ but $r^{n-1} m \neq 0$.

(ii) There exists $t \in R$ such that $t^2 m = 0$ but $tm \neq 0$.

In [7], we have, if $m \in M$ satisfies any of the above equivalent conditions, then m is a nilpotent element of the left R-module M.

Example 2.2. Some examples of nilpotent elements of modules are given below:

(i) Let $M = \mathbb{Z}_8$ and $R = \mathbb{Z}_8$.

Here, $2^3.\bar{1} = 0$ but $2.\bar{1} \neq 0$

$2^2.\bar{2} = 0$ but $2.\bar{2} \neq 0$

$2^3.\bar{3} = 0$ but $2.\bar{3} \neq 0$

$2^3.\bar{5} = 0$ but $2.\bar{5} \neq 0$

$2^2.\bar{6} = 0$ but $2.\bar{6} \neq 0$

$2^3.\bar{7} = 0$ but $2.\bar{7} \neq 0$

Clearly, $Nil_{\mathbb{Z}_8}(\mathbb{Z}_8) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{5}, \bar{6}, \bar{7}\}$.

(ii) If $a \in R$ is nilpotent (with degree $n \geq 3$) in the ring R, then we have $a^{n-1}.a = a^n = 0$ and $a.a = a^2 \neq 0$. Thus, a is nilpotent in the left R-module R.

3 Nil-symmetric modules

In this section, we introduced the class of nil-symmetric modules as a generalisation of symmetric modules and reduced modules. We also show that there are nil-symmetric modules which are not symmetric.

Definition 3.1. [2] A left R-module M is said to be symmetric if whenever $a, b \in R, m \in M$ satisfy $abm = 0$ implies $bam = 0$.

Definition 3.2. A left R-module M is said to be nil-symmetric if whenever $a, b \in R, m \in M$ satisfy $abm = 0$ implies $bam \in Nil_R(M)$.

Remark 3.3. From the definition, the following remarks can be obtained.

(1) All modules over commutative rings are nil-symmetric modules.

(2) Submodules of nil-symmetric modules are nil-symmetric.

Recall, in [3] the concept of generalized weakly symmetric rings were studied. A ring R is called generalized weakly symmetric if $abc = 0$ implies that bac is nilpotent for all $a, b, b \in R$.

Theorem 3.4. *If R is a generalized weakly symmetric ring with nilpotency index greater than 2, then the left R-module R is nil-symmetric.*

Proof: Let $a, b, m \in R$ with $abm = 0$. Since R is a nil-symmetric ring $\implies$ $bam = 0 \in Nil(R)$ $\implies$ $(bam)^k = 0, k \in \mathbb{N}$ $\implies$ $(bam)^{k-1}(bam) = 0$ $\implies$ $s^{k_0}bam = 0, sbam \neq 0$ where $s = bam, k_0 = k - 1$ $\implies$ $bam \in Nil_R(R)$. Hence, ${}_RR$ is nil-symmetric.

Lemma 3.5. *[8] All reduced modules are symmetric modules.*

Theorem 3.6. *All symmetric modules are nil-symmetric modules.*

Proof: Let M be a symmetric module. Let $a, b \in R$ and $m \in M$ with $abm = 0$. Then, $bam = 0 \in Nil_R(M)$ $\implies$ $bam \in Nil_R(M)$.

Remark 3.7. The converse of Theorem 3.6 is not true in general which is shown in Example 3.30. The above Lemma 3.5 and Theorem 3.6 give Corollary 3.8.

Corollary 3.8. *All reduced modules are nil-symmetric modules.*

Lemma 3.9. *[8] Symmetric modules are semicommutative.*

Remark 3.10. Nil-symmetric modules are not semicommutative.

Example 3.11. Let $M = \mathbb{Z}$. Then, M is nil-symmetric. Hence, ${}_{T_n(\mathbb{Z})}T_n(\mathbb{Z})$ is nil-symmetric by Theorem 3.28. Let $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $e_{11}e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But $e_{11}e_{12}e_{22} = e_{12}e_{22} = e_{12} \neq 0$. So, M is not semicommutative.

Theorem 3.12. *All nil-symmetric modules are weakly semicommutative.*

Proof: Let M be a nil-symmetric module. Let $a \in R, m \in M$ with $am = 0 \Longrightarrow bam = 0$ for all $b \in R$. Since M is nil-symmetric $\implies$ $abm \in Nil_R(M)$ $\implies$ $aRm \subseteq Nil_R(M)$. Hence, M is weakly semicommutative.

Next, we recall a torsion free module. A module having no non-zero torsion elements is called a torsion free-module,i.e., $0 \neq m$ is torsion free if $rm = 0, r \in R$ $\implies$ $r = 0$. We recall a result in [7].

Theorem 3.13. *If M is a torsion free left R-module, then $Nil_R(M) = \{0\}$.*

In [7], the converse of the above Theorem 3.13 need not be true in general,i.e., there exists a left R-module M such that $Nil_R(M) = 0$ but M is not torsion free by the following example.

Example 3.14. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p$, where p is a prime number. Then, $\bar{1} \in Tor({}_{\mathbb{Z}}\mathbb{Z}_p)$ as $p.\bar{1} = \bar{0}$. Thus, $Tor({}_{\mathbb{Z}}\mathbb{Z}_p) \neq 0$. Let $\bar{0} \neq \bar{a} \in Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. Then, by definition there exist $r \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $r^k\bar{a} = \bar{0}$ and $r\bar{a} \neq \bar{0}$ implies $p|r^ka$ which again implies $p|r^k$ or $p|a$. If $p|a$, then $r\bar{a} = 0$ and thus $\bar{a} \notin Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. Suppose $p|r^k$ which implies $p|r.r^{k-1}$. Again, $p|r$ or $p|r^{k-1}$. If $p|r$, then $r\bar{a} = \bar{0}$ and hence $\bar{a} \notin Nil_{\mathbb{Z}}(\mathbb{Z}_p)$. On the other suppose $p|r^{k-1}$, then by continuing we get $p|r$ and hence $r\bar{a} = 0$. Thus, $Nil_{\mathbb{Z}}(\mathbb{Z}_p) = 0$.

Here, we have found some conditions for which symmetric and nil-symmetric modules are equivalent which is given below.

Theorem 3.15. *Let M be a torsion free left R-module. Then, M is symmetric if and only if M is nil-symmetric.*

Proof: Let M be nil-symmetric. Also, let $a, b \in R, m \in M$ with $abm = 0$. Since M is nil-symmetric, $bam \in Nil_R(M)$. Now, since M is torsion free, $Nil_R(M) = 0$. Therefore, $bam \in \{0\}$ $\implies$ $bam = 0$. Hence, M is symmetric.

The converse part follows from Theorem 3.6.

Theorem 3.16. *Let M be a nil-symmetric module over a domain D. Then, $T(M)$ is a submodule of M.*

Proof: Let $m_1, m_2 \in T(M)$. Then, there exist $0 \neq r_1, 0 \neq r_2 \in R$ such that $r_1m_1 = 0, r_2m_2 = 0 \implies r_2r_1m_1 = 0, r_1r_2m_2 = 0 \implies r_1r_2m_1 \in Nil_D(M), r_2r_1m_2 \in Nil_D(M)$. Then, there exist $0 \neq t \in D$ and $n \in \mathbb{N}$ such that $t^nr_1r_2m_1 = 0, tr_1r_2m_1 \neq 0$. Now, $t^nr_1r_2(m_1 - m_2) = t^nr_1r_2m_1 - t^nr_1r_2m_2 = 0$ which implies $m_1 - m_2 \in T(M)$. Also, let $m \in T(M) \implies rm = 0$ for some $0 \neq r \in D \implies arm = 0\ \forall a \in D$. Since M is a nil-symmetric module and $arm = 0 \implies ram \in Nil_D(M)$. Then, there exist $0 \neq t \in D$ and $n \in \mathbb{N}$ such that $t^nram = 0, tram \neq 0$. Since D is domain, we have $t^nr \neq 0$. Thus, $am \in T(M)$. Hence proved.

Lemma 3.17. *[4] If ${}_RN$ is a submodule of ${}_RM$, then $Nil_R(N) \subseteq Nil_R(M)$.*

Theorem 3.18. *A left R-module M is nil-symmetric if and only if every cyclic submodule of M is nil-symmetric.*

Proof: Let M be nil-symmetric. Since submodules of nil-symmetric modules are nil-symmetric, every cyclic submodule of M is nil-symmetric.

Conversely, let $a, b \in R, m \in M$ satisfying $abm = 0$. Since $m \in M, m = 1.m \in Rm$ which is cyclic $\implies m \in Rm \subseteq M \implies abm = 0$. Since Rm is a nil-symmetric module $\implies bam \in Nil_R(Rm) \implies bam \in Nil_R(M)$. Hence M is nil-symmetric.

Theorem 3.19. *A left R-module M is nil-symmetric if and only if every finitely generated submodule of M is nil-symmetric.*

Proof: Let M be nil-symmetric. Since submodules of nil-symmetric modules are nil-symmetric, every finitely generated submodule of M is nil-symmetric.

The converse is clear by Theorem 3.18.

In the next theorem, we give a condition on a submodule N of a left R-module M which is sufficient for the nil-symmetricity of $\frac{M}{N}$ to imply nil-symmetricity of M.

Theorem 3.20. *Let M be a left-R module over a commutatative ring R and N be a submodule of M such that $N \subseteq Nil_R(M)$. If $\frac{M}{N}$ is nil-symmetric, then M is nil-symmetric.*

Proof: Let $a, b \in R$ and $m \in M$ with $abm = 0$. Then, we have, $ab\bar{m} = 0$. Since $\frac{M}{N}$ is nil-symmetric, $ba\bar{m} \in Nil_R(\frac{M}{N})$. Then, there exist $r \in R, k \in \mathbb{N}$ such that $r^kba\bar{m} = \bar{0}, rba\bar{m} \neq \bar{0} \implies r^kba(m+N) = \bar{0}, rba(m+N) \neq \bar{0} \implies r^kbam + N = 0 + N, rbam + N \neq 0 + N \implies r^kbam \in N$. Since $N \subseteq Nil_R(M)$, we have, $r^kbam \in Nil_R(M)$. Then, there exist $p \in R, s \in \mathbb{N}$ such that $p^sr^kbam = 0, pr^kbam \neq 0$. Since R is commutative, we have, $(pr)^{max(s,k)}bam = 0, prbam \neq 0$ as $pr^kbam \neq 0 \implies bam \in Nil_R(M)$. Hence, M is nil-symmetric.

Theorem 3.21. *Let M be a left R-module over an integral domain R. If M is nil-symmetric, then $\frac{M}{T(M)}$ is symmetric.*

Proof: The proof is obvious as R is commutative.

Corollary 3.22. *Let M be a left R-module over an integral domain R. If M is nil-symmetric, then $\frac{M}{T(M)}$ is nil-symmetric.*

Theorem 3.23. *Let $\theta : R \to R'$ be a ring homomorphism and let M be an R'-module. Then, M can be made as an R-module by defining $am = \theta(a)m$. If θ is onto, the following are equivalent:*

(1) M is a nil-symmetric R'-module.

(2) M is a nil-symmetric R-module.

Proof: (1) $\implies$ (2) Let $abm = 0\ \forall a, b \in R, m \in M \implies \theta(ab)m = 0 \implies \theta(a)\theta(b)m = 0$ in ${}_{R'}M$. Since ${}_{R'}M$ is nil-symmetric, $\theta(b)\theta(a)m \in Nil_{R'}(M) \implies \exists t \in R'$ and $k \in \mathbb{N}$ such that $t^k\theta(b)\theta(a)m = 0, t\theta(b)\theta(a)m \neq 0$. Since θ is onto, there exists $l \in R$ such that $\theta(l) = t$. Now, $l^kbam = \theta(l)^k\theta(b)\theta(a)m = t^k\theta(b)\theta(a)m = 0$ and $t\theta(b)\theta(a)m \neq 0$ implies $\theta(l)\theta(b)\theta(a)m \neq 0$, and so $lbam \neq 0$. Therefore, $bam \in Nil_R(M)$. Hence, M is a nil-symmetric R-module.

(2) $\implies$ (1) Let $a'b'm = 0\ \forall a', b' \in R', m \in M$. Since θ is onto, there exist $r \in R, l \in R$ such that $\theta(r) = a', \theta(l) = b'$. Now, $\theta(r)\theta(l)m = 0 \implies \theta(rl)m = 0 \implies rlm = 0 \implies lrm \in Nil_R(M)$. Then, there exist $t \in R$ and $n \in \mathbb{N}$ such that $t^nlrm = 0$ and $tlrm \neq 0$. Then, $b'a'm \in Nil_{R'}(M)$. Hence, M is a nil-symmetric R'-module.

Next, we study localisations. Recall that if R is a commutative ring and S is a multiplicatively closed subset of R consisting of $C(R)-\{0\}$ and without zero divisor, then $S^{-1}R$ has a ring structure with unity known as ring of fractions. If R is an integral domain and $S = R - \{0\}$, then the ring of fractions $S^{-1}R$ is called field of fractions. If M is a left R-module, then $S^{-1}M$ can be made as an $S^{-1}R$-module. By applying standard localisations techniques, we can prove Theorem 3.24 and Corollary 3.25.

Theorem 3.24. *Let R be a ring and S be a multiplicatively closed subset of $C(R)-\{0\}$. Then, M is a nil-symmetric R-module if and only if $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module.*

Proof: Consider M to be a nil-symmetric R-module. Let $\frac{a}{r}\frac{b}{s}\frac{m}{t} = 0$ in $S^{-1}M$ where $\frac{m}{t} \in S^{-1}M, \frac{a}{r}, \frac{b}{s} \in S^{-1}R \implies u_1abm = 0$ for some $u_1 \in R \implies abm = 0$. Since M is a nil-symmetric R-module, we have $bam \in Nil_R(M)$. Then, there exist $0 \neq t \in R$ and $n \in \mathbb{N}$ such that $t^n bam = 0$ and $tbam \neq 0$. Now, $t^n \frac{b}{s}\frac{a}{r}\frac{m}{t} = \frac{t^n bam}{srt} = 0$ and $t\frac{b}{s}\frac{a}{r}\frac{m}{t} = \frac{tbam}{srt} \neq 0$ as $tbam \neq 0$. Therefore, $\frac{b}{s}\frac{a}{r}\frac{m}{t} \in Nil_{S^{-1}R}(S^{-1}M)$. Hence, $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module.

Conversely, let $a, b \in R$ and $m \in M$ with $abm = 0 \implies \frac{a}{1}\frac{b}{1}\frac{m}{1} = 0$. Since $S^{-1}M$ is a nil-symmetric $S^{-1}R$-module, we have $\frac{b}{1}\frac{a}{1}\frac{m}{1} \in Nil_{S^{-1}R}(S^{-1}M)$. Then, there exist $\frac{t}{s} \in S^{-1}R$ and $n \in \mathbb{N}$ such that $(\frac{t}{s})^n \frac{b}{1}\frac{a}{1}\frac{m}{1} = 0 \implies t^n bam = 0 \implies u_1(t^n bam - 0) = 0$ for some $u_1 \in S \implies u_1 t^n bam = 0 \implies t^n bam = 0$ and $\frac{t}{s}bam \neq 0 \implies u(tbam - 0.s) \neq 0$ for all $u \in S \implies utbam \neq 0$ for all $u \in S \implies tbam \neq 0$ for $u = 1$. Therefore, $bam \in Nil_R(M)$. Hence, M is a nil-symmetric R-module.

Corollary 3.25. *For a left R-module M, ${}_{R[x]}M[x]$ is nil-symmetric if and only if ${}_{R[x,x^{-1}]}M[x, x^{-1}]$ is nil-symmetric.*

Proof: Let $S = \{1, x, x^2, ...\}$. Then, S is a multiplicatively closed subset of $R[x]$ consisting of central elements of $R[x]$. Since $S^{-1}M[x] = M[x, x^{-1}]$ and $S^{-1}R[x] = R[x, x^{-1}]$, the result is clear from Theorem 3.24.

Lemma 3.26. *[4] Let M be a left R-module. Then, $Nil_{M_n(R)}M_n(M) = M_n(M)$ for $n \geq 2$.*

Theorem 3.27. *For a left R-module M, ${}_{M_n(R)}M_n(M)$ is nil-symmetric for $n \geq 2$.*

Proof: Let $ABL = 0\ \forall A, B \in M_n(R)$ and $L \in M_n(M)$. Then, $BAL \in M_n(M) = Nil_{M_n(R)}M_n(M) \implies BAL \in Nil_{M_n(R)}M_n(M)$. Hence, ${}_{M_n(R)}M_n(M)$ is a nil-symmetric module.

Theorem 3.28. *A left R-module M is a nil-symmetric module if and only if for any $n \in \mathbb{N}$, ${}_{T_n(R)}T_n(M)$ is a nil-symmetric module.*

Proof: Consider M to be a nil-symmetric module. Let $A = (a_{ij}), B = (b_{ij}) \in T_n(R)$ and $L = (m_{ij}) \in T_n(M)$ with $ABL = 0$. Then, $a_{ii}b_{ii}m_{ii} = 0\ \forall\ 0 < i \leq n$. Since ${}_RM$ is nil-symmetric, we have $b_{ii}a_{ii}m_{ii} \in Nil_R(M)\ \forall\ 0 < i \leq n$.

$$\text{Now, } BAL = \begin{bmatrix} b_{11}a_{11}m_{11} & * & * & \cdots & * \\ 0 & b_{22}a_{22}m_{22} & * & \cdots & * \\ 0 & 0 & b_{33}a_{33}m_{33} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn}a_{nn}m_{nn} \end{bmatrix}.$$

Since $b_{nn}a_{nn}m_{nn} \in Nil_R(M)$, there exist $t_n \in R$ and $n \in \mathbb{N}$ such that $t_n^k b_{nn}a_{nn}m_{nn} = 0$ and $t_n b_{nn}a_{nn}m_{nn} \neq 0$. Choose $T = diag(0, 0, ..., t_n)$, we have $T^k BAL = 0$ and $TBAL \neq 0$.

The converse part is easily seen that submodules of nil-symmetric modules are nil-symmetric, then so is ${}_RM$.

Corollary 3.29. *Let ${}_RM$ be a symmetric module. Then, for any $n \in \mathbb{N}$, ${}_{T_n(R)}T_n(M)$ is a nil-symmetric module.*

Here, we have given an example of a module which is nil-symmetric but not symmetric.

Example 3.30. Let $M = \mathbb{Z}, R = \mathbb{Z}$. Then, ${}_{\mathbb{Z}}\mathbb{Z}$ is a nil-symmetric module by Remark 3.3(1). So, ${}_{T_2(\mathbb{Z})}T_2(\mathbb{Z})$ is a nil-symmetric module but it is not symmetric module as let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then, $ABC = 0$. But $BAC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. For a left R-module M and $B = (a_{ij}) \in M_n(R)$, let $MB = \{(a_{ij}m) : m \in M\}$. For unit matrices $\{E_{ij} : 1 \leq i, j \leq n\}$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ for $n \geq 2$. Let $V_n(R) = RI_n + RV + RV^2 + ... + RV^{n-1}$ and $V_n(M) = MI_n + MV + MV^2 + ... + MV^{n-1}$. Then, $V_n(R)$ forms a ring and $V_n(M)$ forms a left R-module over $V_n(R)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\theta : V_n(R) \to \frac{R[x]}{(x^n)}$ given by $\theta(r_o I_n + r_1 V + ... + r_{n-1}V^{n-1}) = r_o + r_1 x + ... + r_{n-1}V^{n-1} + (x^n)$ and an abelian group isomorphism $\phi : V_n(M) \to \frac{M[x]}{M[x](x^n)}$ defined by $\phi(m_o I_n + m_1 V + ... + m_{n-1}V^{n-1}) = m_o + m_1 x + ... + m_{n-1}V^{n-1} + M[x](x^n)$ such that $\phi(AW) = \theta(A)\phi(W)$ for all $A \in V_n(R)$ and $W \in V_n(M)$.

Theorem 3.31. *Let M be a left R-module. If M is nil-symmetric module, then for any $n \geq 2$, $\frac{M[x]}{M[x](x^n)}$ is a nil-symmetric module over $\frac{R[x]}{(x^n)}$.*

Proof: From the above remark we can easily prove that if ${}_RM$ is nil-symmetric, then ${}_{V_n(R)}V_n(M)$ is a nil-symmetric for $n \geq 2$. Thus, the proof follows from Theorem 3.28 given above.

4 Conclusion remarks

Remark 4.1. We conclude this note with the following questions.

(1) Is a direct product of nil-symmetric modules nil-symmetric?

(2) Is there any relation between $Nil_R(M)[x]$ and $Nil_{R[x]}M[x]$?

(3) Is a direct sum of nil-symmetric modules nil-symmetric?

References

[1] D. Ssevviiri and N. Groenewald, Generalization of nilpotency of ring elements to module elements, Comm. Algebra 42, 571-777, (2014).

[2] J. Lambek, On the representation of modules by sheaves of factor modules, Canad. Math. Bull. 14, 359-368, (1971).

[3] Junchao Wei, Generalized weakly symmetric rings, J. of Pure and Applied Algebra 218, 1594-1603, (2014).

[4] M. Dutta and K. H. Singh, On some generalisations of reduced and rigid modules, IOP Conf. Ser.: Mater. Sci. Eng. 1020 012024, (2021).

[5] N. Agayev and A. Harmanci, On semicommutative modules and rings, Kyungpook Math. J. 47, 21-30, (2007).

[6] N. Agayev, S.Halicioglu and A. Harmanci, On symmetric modules, Riv. Mat. Univ. Parma (8) 2 , 91-99, (2009).

[7] N. Ansari and Kh.H. Singh, On weakly semicommutative modules, Indian J. of Mathematics, 62, 321-334, (2020).

[8] M. B. Rege and A. M. Buhphang, On reduced modules and rings, Int. Electron. J. of Algebra 3, 58-74, (2008).

[9] R. Raphael, Some remarks on regular and strongly regular rings, Canad. Math. Bull.17, 709-712, (1974/75).

[10] M.V. Mehrabadi and S. Sahebi, Central α-rigid rings, Palestine Journal of Mathematics. 6(2), 569-572, (2017).

[11] T.K. Lee and Y. Zhou, Reduced modules, Rings, Modules, Algebras and Abelian Groups, Lect. Pure Appl. Math. 16, 236, (2004).

[12] U.S Chakraborty and K. Das, On Nil-Symmetric rings, Hindawi Publishing Corp. J. of Mathematics, volume 2014, Article ID 483784, 1-7.

[13] N. Ansari and K.H. Singh, Central elements and Armendariz bimodules, Palestine Journal of Mathematics. 13(3), 762-772, (2024).

Author information

Kh. Herachandra, Department of Mathematics, Manipur University, India.
E-mail: heramath@manipuruniv.ac.in

K. Praminda, Department of Mathematics, Manipur University, India.
E-mail: koijampraminda@gmail.com

M. Rhoades, Department of Mathematics, Manipur University, India.
E-mail: rhoades.phd.math@manipuruniv.ac.in

Received: 2024-08-06
Accepted: 2024-12-13