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On a bi-nonlocal $p(x)-$Kirchhoff equation involving $p(x)-$Biharmonic operator with dependence on the Gradient via Mountain Pass Techniques.

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Abstract
In this paper, we study a bi-nonlocal elliptic problem involving the operator $p(x)-$Biharmonic. Applying variational methods (Mountain Pass Theorem-Krasnoselskii's Genus) and under suitable conditions, we prove several results on the existence of nontrivial solutions. To the best of our knowledge, this paper is one of the first contributions to the study of bi-nonlocal elliptic problem with dependence on the Gradient.

1 Introduction

The main objective of this paper is to study questions of existence of nontrivial solutions for the bi-non-local elliptic problem

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta^2_{p(x)}u &= f\left(x,u,|\nabla u|^{p(x)-2}\nabla u\right)\left[\int_\Omega F(x,u,|\nabla u|^{p(x)-2}\nabla u)dx\right]^r \quad \text{in } \Omega \\ u &= \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a bounded domain, $\Delta^2_{p(x)}u := \Delta(|\Delta|^{p(x)-2}\Delta u)$ which is the operator of fourth order called the $p(x)$-biharmonic operator, $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ and $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and f are continuous functions satisfying conditions which will be stated later. $F(x,u,|u|^{p(x)-2}u) = \int_0^u f(x,s,|\nabla s|^{p(x)-2}\nabla s)ds$, $r > 0$ is real parameter.
The problem (1.1) is called bi-nonlocal because of the presence of two terms

$$\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) \quad \text{and} \quad \Big(\int_\Omega F(x,u,|\xi|^{p(x)-2}\xi)dx\Big).$$

which imply that the problem is no longer a pointwise identity. This implies some mathematical difficulties which give the study of this type of problem an interesting perspective.
The (1.1) problem is related to the stationary version of a model, called the Kirchhoff equation. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial t}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

with ρ, P_0, h, E, L are constants, which extends the classical wave equation of D'Alembert.

The problem of differential equations and variational problems involving non-local operators have received more and more attention in recent years, particularly in the fields of optimization, continuum mechanics, phase transition phenomena, finance, population dynamics and game theory (see [2], [3], [5], [6], [7], [10], [11]).
Moreover, problems involving the $p(x)-$Laplacian operator were first motivated by Corréa and Augusto Cézar [2], [3]. In [1], the authors consider the following problem

$$\begin{cases} -M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta^2_{p(x)}u &= f\,(x,u)\left[\displaystyle\int_\Omega F(x,u)dx\right]^r \qquad \text{in } \Omega \\ u &= \Delta u = 0, \qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

Using the variational method and under appropriate conditions, they prove the existence of non-trivial weak solutions to their problem. This article is a generalization of the paper [1] mentioned above. Specifically, we treat our problem (1.1) with the gradient term.

This work is treated for the first time with the gradient term, which adds several problems to assure the existence and regularity of the solution.

To state our results, we assume that the Kirchhoff function $M : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and f are continuous and satisfy the following conditions:
(M_1) There exist real numbers $m_0, m_1 > 0$ and $\theta \geq 1$ such that

$$m_0 \leq M_1(t) \leq m_1 \qquad \forall t \geq 0.$$

(f_1)

$$A_1 t^{\alpha(x)-1} \leq f(x,t,|\nabla\xi|^{p(x)-2}\nabla\xi) \leq A_2 t^{q(x)-1}$$

with $\alpha(x) \leq q(x) < p^*(x)$ pour tout $x \in \overline{\Omega}$.
(f_2) Ambrosetti-Rabinowitz's condition holds, i.e., there exists $\theta > \frac{m_1}{m_0}(> p^+)$ such that

$$0 < \theta F(x,t,|\xi|^{p(x)-2}\xi) < (r+1)tf(x,t,|\xi|^{p(x)-2}\xi),$$

$for\ all\ |t| \geq t_0\ and\ x \in \overline{\Omega}$,
(f_3) There exist positive constants $R_1,\ R_2,\ L_2, L_3$ such that

$$|f(x,t_1,|\xi|^{p(x)-2}\xi)\left[\int_\Omega F(x,t_1,|\xi|^{p(x)-2}\xi)dx\right]^r - f(x,t_2,|\xi|^{p(x)-2}\xi)\left[\int_\Omega F(x,t_2,|\xi|^{p(x)-2}\xi)dx\right]^r| \leq L_2|t_1-t_2|^{p(x)-1}$$

$\forall x \in \overline{\Omega}, |\xi| \leq R_1, |t_1|, |t_2| \leq R_2$

$$|f(x,t,|\xi_1|^{p(x)-2}\xi_1)\left[\int_\Omega F(x,t,|\xi_1|^{p(x)-2}\xi_1)dx\right]^r - f(x,t,|\xi_2|^{p(x)-2}\xi_2)\left[\int_\Omega F(x,t,|\xi_2|^{p(x)-2}\xi_2)dx\right]^r| \leq L_3|\xi_1-\xi_2|^{p(x)-1}$$

$\forall x \in \overline{\Omega}, |\xi_1|, |\xi_2| \leq R_1, |t| \leq R_2$
(f_4)

$$f(x,t,|\xi|^{p(x)-2}\xi) = -f(x,-t,|\xi|^{p(x)-2}\xi).$$

Moreover, in the proof of the main result related to problem (1.1), we use the well-known vector inequalities; (see [16], [18], [22]).

$$\left(|\psi|^{p(x)-2}\psi - |\xi|^{p(x)-2}\xi\right)(\psi-\xi) \geq 2^{2-p^+}|\psi-\xi|^{p(x)}, \quad p(x) \geq 2, \tag{1.3}$$

for all $x \in \overline{\Omega}$ and $\psi, \xi \in \mathbb{R}^N$.

Remark 1.1.

(i) Condition (f_1) imply $f(x,0,|\xi_1|^{p(x)-2}\xi_1)=0$. Throughout this paper, we put $f(x,t,|\xi_1|^{p(x)-2}\xi_1)=0$ for $t<0$ and hence, $f(x,t,|\xi_1|^{p(x)-2}\xi_1)$ is defined on $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$and continuous.

(ii) Note that

$$\overline{\alpha}=\begin{cases}\alpha^- & \text{if } \|u\|<1\\ \alpha^+ & \text{if } \|u\|>1.\end{cases} \tag{1.4}$$

(iii) A typical example of the function f given by $f(x,t,\zeta)=t\zeta$, this function satisfies the conditions $(f_1)-(f_4)$.

Now we are in a position to give the first main result.

Theorem 1.2. *In case $q^-(r+1)>\alpha^-(r+1)>p^+$, we assume that the conditions (M_1) and $(f_1)-(f_3)$ are verified. If $2^{2-p^+}m_0>L_2$ and $\gamma=\dfrac{SL_3}{2^{2-p^+}m_0-S^{\overline{p}_{n+1}}L_2}<1$. Then, problem (1.1) has a nontrivial solution.*

Now we can give the second main result.

Theorem 1.3. *In case $q^+(r+1)<p^-$, we assume that the conditions (M_1), $(f_1)-(f_3)$ and (f_4) are verified. Then, problem (1.1) has infinitely many solutions.*

The paper is organized in the following way :
Section 2 is concerned the preliminary results, while Section 3 presents the first main result. Finally, in the 4th section, we give the second main result.

2 Preliminary

For $p(.)\in L^\infty_+(\Omega)$. We need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, which we will use later (for details, see [4], [17], [21]).
Define the generalized Lebesgue space by:

$$L^{p(x)}(\Omega)=\left\{u:\quad \Omega\longrightarrow\mathbb{R},\quad \text{measurable and}\quad \int_\Omega |u(x)|^{p(x)}dx<\infty\right\},$$

where $p\in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega}):\ \ h(x)>1,\ \ \forall x\in\overline{\Omega}\right\}.$$

Denote

$$p^+=\max_{x\in\overline{\Omega}}p(x),\quad p^-=\min_{x\in\overline{\Omega}}p(x),$$

and for all $x\in\overline{\Omega}$ and $k\geq 1$

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N,\end{cases}$$

and

$$p_k^*(x)=\begin{cases}\frac{Np(x)}{N-kp(x)} & \text{if } kp(x)<N,\\ +\infty & \text{if } kp(x)\geq N.\end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf\left\{\mu > 0; \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

and the space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.
Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,$$

then the following relations hold

Proposition 2.1 ([17, 19, 20, 21]). *If $u, u_n \in L^{p(x)}(\Omega)$, $(n = 1, 2, ...)$ then the following statements are equivalent:*

(i) $\lim_{n\longrightarrow\infty} |u_n - u|_{p(x)} = 0$*;*

(ii) $\lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n - u) = 0$*;*

(iii) $u_n \longrightarrow u$ *in measure in* Ω *and* $\lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

Proposition 2.2. *([17, 19, 20, 21]) If $u, u_n \in L^{p(x)}(\Omega)$, $(n = 1, 2, ...)$, we have*

(i) $|u|_{p(x)} < 1$ *(respectively=1; > 1)* $\Longleftrightarrow \rho_{p(x)}(u) < 1$ *(respectively $= 1$; > 1);*

(ii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}$*;*

(iii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}$*;*

(iv) $|u_n|_{p(x)} \to 0$ *(respectively $\to \infty$)* $\Longleftrightarrow \rho_{p(x)}(u_n) \to 0$ *(respectively $\to \infty$).*

If we consider $I(u) = \int_\Omega |\Delta u(x)|^{p(x)} dx$ instead of $\rho_{p(x)}(u)$, then the statements of Proposition 2.1 and Proposition 2.2 also hold for $u \in X$.

Proposition 2.3. *([19, 21]) Let $p(x) > 1$ for all $x \in \Omega$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then, for all $a, b \ge 0$*

$$ab \le \frac{a^{p(x)}}{p(x)} + \frac{b^{q(x)}}{q(x)}.$$

Proposition 2.4. *([19, 21]) The space $(L^{p(x))}(\Omega), |\cdot|_{p(x))})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$), i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x))}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

In order to prove the existence of a weak solution for problem (1.1), we introduce the space

$$X = \left\{W^{2,p(x)}(\Omega) \cap W^{1,p(x)}(\Omega); \quad u = 0 \quad and \quad \Delta u = 0 \quad on \quad \partial\Omega\right\},$$

equipped with the norm

$$\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} := |\nabla u|_{p(x)} + |u|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|L|=2} |D^L u|_{p(x)}. \quad (2.1)$$

We know that $(X, \|.\|)$ is also a separable and reflexive Banach space (cf. [8] and [1]).

Remark 2.5. From [9], the norm $\|\cdot\|_{2,p(\cdot)}$ is equivalent to the norm $\|\cdot\|_{p(\cdot)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(\cdot)}$, $\|\cdot\|_{1,p(\cdot)}$ and $\|\cdot\|$ are equivalent.

Theorem 2.6 ([8]). *Let $p, q \in C_+(\Omega)$ such that $q(x) \leq p_2^*$. Then, there is a continuous and compact imbedding X into $L^{q(x)}(\Omega)$.*

Definition 2.7. Let X be a real Banach space and let be a functional $I \in \mathcal{C}^1(X, \mathbb{R})$. We say that I satisfies the *Palais-Smale condition* on X ((PS)-condition, for short) if any sequence $(u_n) \subset X$ with $(I(u_n))$ bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. By (PS)-*sequence* for I we understand a sequence $(u_n) \subset X$ which satisfies $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$.

Define the mappings $L_{p(x)}, N : X \longrightarrow X'$ by

$$< L_{p(x)}(u), v >= \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx \quad \forall u, v \in X,$$

and

$$< N(u), v >= \int_\Omega |u|^{q(x)-2} uv dx \quad \forall u, v \in X.$$

Proposition 2.8 ([8] Proposition 4.2).

1. *$N(u)$ is sequentially weakly-strongly continuous.*
2. *$L_{p(x)}(u)$ satisfies condition $(S+)$.*

3 The First Result : (The case: $q^-(r+1) > \alpha^-(r+1) > p^+$)

In this section, we'll give the first main result, but before that we'll prove the following lemmas. To establish the variational frame, we first use a "freezing" technique whose formulation appears initially in [15]. This technique will help us transform the problem (1.1) into a family of problems without dependence of ∇u. That is, for every $w \in X$ fixed, we consider the " freezed " problem as given by

$$(P_w) \begin{cases} -M\Big(\displaystyle\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\Big) \Delta^2_{p(x)} u & = f\left(x, u, |\nabla w|^{p(x)-2} \nabla w\right) \Big[\displaystyle\int_\Omega F(x, u, |\nabla w|^{p(x)-2} \nabla w) dx\Big]^r \quad \text{in} \\ u & = \Delta u = 0, \end{cases}$$

The energy functional associated to problem (P_w) is defined as $J_w : X \to \mathbb{R}$,

$$J_w(u) = \tilde{M}\Big(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\Big)\Big) - \frac{1}{r+1} \Big[\int_\Omega F(x, u, |\nabla w|^{p(x)-2} \nabla w) dx\Big]^{r+1},$$

where

$$\tilde{M}(x, t) = \int_0^t M(x, s) ds.$$

In a standard way, it can be proved that $J \in C^1(X, \mathbb{R})$. Moreover, we have, for all $(u, v) \in X$,

$$\begin{aligned} J'_w(u)(v) \quad &= \quad M\Big(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\Big) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx \\ &\quad - \Big[\int_\Omega F(x, u, |\nabla w|^{p(x)-2} \nabla w) dx\Big]^r \int_\Omega f(x, u, |\nabla w|^{p(x)-2} \nabla w) v dx. \end{aligned}$$

Thus, weak solutions of problem (P_w) are exactly the critical points of the functional J_w.

Definition 3.1. We recall that $u \in X$ is a weak solution to the problem (P_w) if it verifes

$$M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx =$$
$$\Big[\int_\Omega F(x,u,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r \int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)v dx.$$

We start with the following auxiliary result

Lemma 3.2. *There exist $\rho, k > 0$ which are independant of w such that*

$$J_w(u) \geq k \qquad \forall u \in X, \qquad \textit{with} \qquad \|u\| = \rho$$

Proof. By using the condition (M_1) and (f_1), the embedding from X to $L^{q(x)}(\Omega)$ is is continuously, (see [17, 19, 20, 21]).

In other hand, we have

$$\begin{aligned} J_w(u) &= \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) - \frac{1}{r+1}\Big[\int_\Omega F(x,u,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^{r+1} \\ &\geq \frac{m_0}{p^+}\int_\Omega |\Delta u|^{p(x)}dx - \frac{1}{r+1}(\frac{A_2}{q^-})^{r+1}\Big[\int_\Omega |u|^{q(x)}dx\Big]^{r+1} \\ &\geq \frac{m_0}{p^+}I(u) - \frac{1}{r+1}(\frac{A_2}{q^-})^{r+1}\Big[\int_\Omega |u|^{q(x)}dx\Big]^{r+1} \\ &\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{C^{q^-(r+1)}}{r+1}(\frac{A_2}{q^-})^{r+1}\|u\|^{q^-(r+1)} \end{aligned}$$

Since $q^-(r+1) > \alpha^-(r+1) > p^+$, we find positive $\rho, k > 0$ such that

$$J_w(u) \geq k \quad \forall u \in X \quad \|u\| = \rho.$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *There exist $\widehat{\varphi} \in X$, $\widehat{\varphi} \geq 0$ such that*

$$\lim_{t\longrightarrow\infty} J(t\widehat{\varphi}) = -\infty.$$

Proof. Let choose $\widehat{\varphi} \in C_0^\infty(\Omega)$ such that $\widehat{\varphi} > 0$. Then, for any $t > 1$, by (M_1) and (f_1) it follows

$$\begin{aligned} J_w(t\widehat{\varphi}) &= \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta t\widehat{\varphi}|^{p(x)}dx\Big) - \frac{1}{r+1}\Big[\int_\Omega F(x,t\widehat{\varphi},|\nabla w|^{p(x)-2}\nabla w)dx\Big]^{r+1} \\ &\leq \frac{m_1 t^{p^+}}{p^-}I(\widehat{\varphi}) - \frac{1}{r+1}(\frac{A_1}{q^+})^{r+1}t^{\alpha^+(r+1)}\Big[\int_\Omega |\widehat{\varphi}|^{q(x)}dx\Big]^{r+1} \end{aligned}$$

Since $\alpha^+(r+1) > p^+$, we get $J_w(t\widehat{\varphi}) \longrightarrow -\infty$ as $t \longrightarrow \infty$.

□

Remark 3.4. It follows J_w satisfies the geometry of the Montain Pass Theorem.

Lemma 3.5. *The functional J_w satisfies the Palais-Smale condition in X.*

Proof. Let $(u_n)_n \subset X$ be a sequence such that

$$J_w(u_n) \longrightarrow c > 0, \qquad J'_w(u_n) \longrightarrow 0 \qquad \text{in} \quad X^*. \tag{3.1}$$

where X^* is the dual space of X.

First, we show that $(u_n)_n$ is bounded in X. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $(u_n)_n$, we may assume that $\|u_n\| \longrightarrow \infty$. Thus, we may consider that

$$\|u_n\| > 1 \quad \text{for any integer } n,$$

thus

$$\begin{aligned}
c+1 &+ \|u_n\| \geq J_w(u_n) - \frac{1}{\theta} < J'_w(u_n), u_n > \\
&\geq \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big) - \frac{1}{r+1}\Big[\int_\Omega F(x,u_n,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^{r+1} \\
&- \frac{1}{\theta}M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta v dx \\
&+ \frac{1}{\theta}\frac{1}{r+1}\Big[\int_\Omega F(x,u_n,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r\int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)vdx \\
&\geq \frac{m_0}{p^+}\int_\Omega |\Delta u_n|^{p(x)}dx - \frac{1}{r+1}\Big[\int_\Omega F(x,u_n,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^{r+1} \\
&- \frac{m_1}{\theta}\int_\Omega |\Delta u_n|^{p(x)}dx + \frac{1}{\theta}\frac{\theta}{r+1}\Big[\int_\Omega F(x,u_n,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^{r+1} \\
&\geq \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\int_\Omega |\Delta u_n|^{p(x)}dx \\
&\geq \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)I(u_n) \\
&\geq \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\|u_n\|^{p^-}
\end{aligned}$$

Since $1 < p^-$, dividing the above inequality by $\|u_n\|$ and passing to the limit as $n \longrightarrow \infty$ we obtain a contradiction. It follows that the sequence $(u_n)_n$ is bounded in X.
Thus, there exists $u \in X$ such that passing to a subsequence, still denoted by $(u_n)_n$, it converges weakly to u in X.
We have

$$\begin{aligned}
J'_w(u_n)(u_n-u) &= M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)dx \\
&-\Big[\int_\Omega F(x,u,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r\int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)(u_n-u)dx
\end{aligned}$$

The embedding from X to the weighted spaces $L^{q(x)}(\Omega)$ is compact. Then, using the Hölder inequalities, we obtain

$$\begin{aligned}
\left|\int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)(u_n-u)dx\right| &\leq A_2\int_\Omega |u_n|^{q(x)-1}|u_n-u|\mathrm{dx} \\
&\leq K_1\Big||u_n|^{q(x)-1}\Big|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{q(x)}
\end{aligned}$$

where $q(x) < p^*(x)$ for a.e $x \in \Omega$. As $n \longrightarrow \infty$, we deduce

$$\int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)(u_n-u)dx \longrightarrow 0 \tag{3.2}$$

From (f_1) and when $(u_n)_n$ is bounded, we get

$$\Big[\int_\Omega F(x,u,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r\int_\Omega f(x,u,|\nabla w|^{p(x)-2}\nabla w)(u_n-u)dx \longrightarrow 0. \tag{3.3}$$

From (M_1), we have also

$$< L_{p(x)}(u_n), u_n-u >= \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)dx \longrightarrow 0 \tag{3.4}$$

By the Proposition 2.8, $L_{p(x)}$ satisfies condition $(S+)$, we have $u_n \longrightarrow u$ in X. So we conclude that functional J_w satisfies the Palais-Smale condition. □

Remark 3.6. By Lemmas 3.2,3.3 and 3.5, all assumptions of the mountain pass theorem in [14] are satisfied. Then we deduce u_w as a nontrivial weak solution of problem (P_w) such that

$$|u_w|_{C^{0,\alpha}(\Omega)} \leq \rho_1 \quad \text{and} \quad |\nabla u_w|_{C^{0,\alpha}(\Omega)} \leq \rho_2. \tag{3.5}$$

.

We now show that this solution is positive. First of all, the problem (P_w) is equivalent to the following problem

$$\begin{cases} -\Delta^2_{p(x)} u &= \dfrac{f\big(x,u,|\nabla w|^{p(x)-2}\nabla w\big)\Big[\displaystyle\int_\Omega F(x,u,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)} \qquad \text{in } \Omega \\ \quad u &= \Delta u = 0, \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

Lemma 3.7. *Let $w \in X$. There exists a positive constant K_* dependent on m_0 and A_2 but independent of w, such that*

$$\|u_w\| \geq K_*.$$

Proof. Since $u_w \neq 0$ is a solution of problem (P_w), we have

$$< -\Delta^2_{p(x)}u_w, u_w >= \Big[\int_\Omega \frac{f\Big(x,u_w,|\nabla w|^{p(x)-2}\nabla w\Big)u_w}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_w|^{p(x)}dx\Big)}dx\Big]\Big[\int_\Omega F(x,u_w,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r$$

wish is equivalent

$$\int_\Omega \frac{1}{p(x)}|\Delta u_w|^{p(x)}dx = \Big[\int_\Omega \frac{f\Big(x,u_w,|\nabla w|^{p(x)-2}\nabla w\Big)u_w}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_w|^{p(x)}dx\Big)}dx\Big]\Big[\int_\Omega F(x,u_w,|\nabla w|^{p(x)-2}\nabla w)dx\Big]^r$$

Then,

$$\frac{f\Big(x,u_w,|\nabla w|^{p(x)-2}\nabla w\Big)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_w|^{p(x)}dx\Big)} \leq \frac{A_2}{m_0}u_w^{q(x)-1}$$

implique que

$$I(u_w) \leq \frac{A_2^{r+1}}{m_0}\rho^{r+1}_{q(x)}(u_w)$$

Then, from (M_1) and (f_1), we have

$$\frac{m_0}{A_2^{r+1}} \leq \|u_w\|^{q^-(r+1)-p^+}$$

Then

$$\|u_w\| \geq K_* = \left(\frac{m_0}{A_2^{r+1}}\right)^{\frac{1}{q^-(r+1)-p^+}} \tag{3.6}$$

The proof is complete. □

Lemma 3.8. *Let $w \in X$. There exists a positive constant K^* dependent on m_1, θ and A_1 but independent of w such that*

$$\|u_w\| \leq K^*.$$

Proof. from (M_1) and (f_2), we have

$$\frac{f\Big(x, u_w, |\nabla w|^{p(x)-2}\nabla w\Big)}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_w|^{p(x)}dx\Big)} \geq \frac{\theta A_1}{(r+1)m_1}\rho_{\alpha(x)}(u_w)$$

implique que

$$I(u_w) \geq \frac{\theta A_1^{r+1}}{(r+1)m_1}\rho_{\alpha(x)}^{r+1}(u_w)$$

Then

$$\frac{(r+1)m_1}{\theta A_1^{r+1}} \geq \|u_w\|^{\alpha^-(r+1)-p^+}$$

Then

$$\|u_w\| \leq K^* = \left(\frac{(r+1)m_1}{\theta A_1^{r+1}}\right)^{\frac{1}{\alpha^-(r+1)-p^+}} \tag{3.7}$$

The proof is complete. □

Now we can prove the first result, we can construct a sequence $(u_n)_n \subset X$ as solutions of the following:

$$(P_n)\begin{cases} -\Delta^2_{p(x)}u_n &= \dfrac{f\big(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\big)\Big[\int_\Omega F(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_{n-1}|^{p(x)}dx\Big)} \quad \text{in} \quad \Omega \\ u_n &= \Delta u_{n-1} = 0, \qquad \text{on } \partial\Omega, \end{cases}$$

obtained by the mountain pass theorem in Lemmas 3.2,3.3 and 3.5 , starting with an arbitrary $u_0 \in X \cap C^{1,\beta}(\overline{\Omega})$. By Remark 3.6 we see that

$$\|u_n\|_{C^{0,\beta}(\overline{\Omega})} \leq K_1 \quad \text{and} \quad \|\nabla u_n\|_{C^{1,\beta}(\overline{\Omega})} \leq K_2,$$

for some constants K_1, $K_2 > 0$.

Proposition 3.9. *The sequence $(u_n)_n$ is γ-lipschitzian, where $\gamma = \dfrac{SL_3}{2^{2-p^+}m_0 - S^{\overline{p}_{n+1}}L_2}$.*

Proof. Using (P_n) and (P_{n+1}), we obtain

$$\int_\Omega |\Delta u_{n+1}|^{p(x)-2}\Delta u_{n+1}(\Delta u_{n+1}-\Delta u_n)dx$$
$$= \Big[\int_\Omega \frac{f\Big(x,u_{n+1},|\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x,u_{n+1},|\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r$$

and

$$\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_{n+1}-\Delta u_n)dx$$
$$= \Big[\int_\Omega \frac{f\Big(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\Big)(u_{n+1}-u_n)}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_{n-1}|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r$$

Then

$$
\begin{aligned}
&\int_\Omega \Big(|\Delta u_{n+1}|^{p(x)-2}\Delta u_{n+1} - |\Delta u_n|^{p(x)-2}\Delta u_n\Big)(\Delta u_{n+1} - \Delta u_n)\\
&\quad = \Big[\int_\Omega \frac{f\Big(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\quad -\Big[\int_\Omega \frac{f\Big(x, u_n, |\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_{n-1}|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_n, |\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r\\
&\quad = \Big[\int_\Omega \frac{f\Big(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\quad -\Big[\int_\Omega \frac{f\Big(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\quad +\Big[\int_\Omega \frac{f\Big(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\quad -\Big[\int_\Omega \frac{f\Big(x, u_n, |\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_{n-1}|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_n, |\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r
\end{aligned}
$$

Using Hölder inequality, Sobolev embedding theorem, (M_1) and (f_3), we get

$$
\begin{aligned}
I_1 &= \Big[\int_\Omega \frac{f\Big(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\quad -\Big[\int_\Omega \frac{f\Big(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\le \frac{1}{m_0}\Big|\int_\Omega f\Big(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)dx.\Big[\int_\Omega F(x, u_{n+1}, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\\
&\qquad -\int_\Omega f\Big(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n).\Big[\int_\Omega F(x, u_n, |\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r\Big|
\end{aligned}
$$

thus

$$
I_1 \le \frac{L_2}{m_0}\int_\Omega |u_{n+1}-u_n|^{p(x)} \tag{3.8}
$$

And

$$I_2=\Big[\int_\Omega \frac{f\Big(x,u_n,|\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x,u_n,|\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r$$

$$-\Big[\int_\Omega \frac{f\Big(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\Big)(u_{n+1}-u_n)}{M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_{n-1}|^{p(x)}dx\Big)}dx\Big].\Big[\int_\Omega F(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r$$

$$\leq \frac{1}{m_0}\Big|\int_\Omega f\Big(x,u_n,|\nabla u_n|^{p(x)-2}\nabla u_n\Big)(u_{n+1}-u_n)dx.\Big[\int_\Omega F(x,u_n,|\nabla u_n|^{p(x)-2}\nabla u_n)dx\Big]^r$$
$$-\int_\Omega f\Big(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1}\Big)(u_{n+1}-u_n).\Big[\int_\Omega F(x,u_n,|\nabla u_{n-1}|^{p(x)-2}\nabla u_{n-1})dx\Big]^r\Big|$$

thus

$$I_2 \leq \frac{L_3}{m_0}\int_\Omega |\nabla u_n-\nabla u_{n-1}|^{p(x)-1}|u_{n+1}-u_n|dx \tag{3.9}$$

where

$$\int_\Omega\Big(|\Delta u_{n+1}|^{p(x)-2}\Delta u_{n+1}-|\Delta u_n|^{p(x)-2}\Delta u_n\Big)(\Delta u_{n+1}-\Delta u_n)=I_1+I_2$$

Then

$$I_1+I_2\leq\Big(\frac{L_2}{m_0}\int_\Omega |u_{n+1}-u_n|^{p(x)}dx+\frac{L_3}{m_0}\int_\Omega |\nabla u_n-\nabla u_{n-1}|^{p(x)-1}|u_{n+1}-u_n|dx\Big)$$
$$\leq\Big(\frac{S^{\overline{p}_{n+1}}L_2}{m_0}\|u_{n+1}-u_n\|^{\overline{p}_{n+1}}+\frac{SL_3}{m_0}\|u_n-u_{n-1}\|^{\overline{p}_n-1}\|u_{n+1}-u_n\|\Big)$$

where S is the best Sobolev constant in the embedding X into the space $L^{p(x)}(\Omega)$, and

$$\overline{p}_{n+1}=\begin{cases} p^- & \text{if } \|u_{n+1}-u_n\|<1\\ p^+ & \text{if } \|u_{n+1}-u_n\|>1.\end{cases} \tag{3.10}$$

By using the inequality (1.3), it implies that

$$2^{2-p^+}\|u_{n+1}-u_n\|^{\overline{p}_{n+1}}\leq\frac{S^{\overline{p}_{n+1}}L_2}{m_0}\|u_{n+1}-u_n\|^{\overline{p}_{n+1}}+\frac{SL_3}{m_0}\|u_n-u_{n-1}\|^{\overline{p}_n-1}\|u_{n+1}-u_n\|$$

That is,

$$\|u_{n+1}-u_n\|^{\overline{p}_{n+1}-1}\leq\gamma\|u_n-u_{n-1}\|^{\overline{p}_n-1} \tag{3.11}$$

where $\gamma=\dfrac{SL_3}{2^{2-p^+}m_0-S^{\overline{p}_{n+1}}L_2}$.

If $\overline{p}_{n+1}=\overline{p}_n=p^+$ (or $=p^-$), we have

$$\|u_{n+1}-u_n\|\leq\gamma^{\frac{1}{p^+-1}}\|u_n-u_{n-1}\| \tag{3.12}$$

If $\overline{p}_{n+1}=p^+$ and $\overline{p}_n=p^-$ (the same for the inverse case), we have

$$\|u_{n+1}-u_n\|^{p^+-1}\leq\gamma\|u_n-u_{n-1}\|^{p^--1}$$

that is equivalent to

$$\|u_{n+1}-u_n\|^{\frac{p^+-1}{p^--1}} \leq \gamma^{\frac{1}{p^--1}} \|u_n-u_{n-1}\|$$

Since $\frac{p^+-1}{p^--1} > 1$ and $\|u_{n+1}-u_n\| > 1$, then we have

$$\|u_{n+1}-u_n\| \leq \|u_{n+1}-u_n\|^{\frac{p^+-1}{p^--1}} \leq \gamma^{\frac{1}{p^--1}} \|u_n-u_{n-1}\|. \tag{3.13}$$

In all cases, the following result is confirmed

$$\|u_{n+1}-u_n\| \leq \gamma^{\frac{1}{p^--1}} \|u_n-u_{n-1}\|. \tag{3.14}$$

□

Remark 3.10. Because $\gamma < 1$, it follows that the sequence $\{u_n\}_n$ converges strongly in X to some function $u \in X$, which is a solution of problem (1.1). From the claim in the proof of Lemma 3.8, we deduce that $u > 0$ in X.

4 The Second Main Result : (The case: $q^+(r+1) < p^-$)

In this section, we'll give the second principal result, but before that, we'll establish the following lemmas that help us to obtain the second result.

From (f_4) we know that J is even, next we will prove the two important lemmas for our proof.

Lemma 4.1. *J is bounded from below.*

Proof. By using the condition (M_1) and (f_1), we have

$$\begin{aligned} J(u) &= \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Big) - \frac{1}{r+1}\Big[\int_\Omega F(x,u,|\nabla u|^{p(x)-2}\nabla u)dx\Big]^{r+1} \\ &\geq \frac{m_0}{p^+}\int_\Omega |\Delta u|^{p(x)}dx - \frac{1}{r+1}(\frac{A_2}{q^-})^{r+1}\Big[\int_\Omega |u|^{q(x)}dx\Big]^{r+1} \\ &\geq \frac{m_0}{p^+}I(u) - \frac{1}{r+1}(\frac{A_2}{q^-})^{r+1}\Big[\int_\Omega |u|^{q(x)}dx\Big]^{r+1} \end{aligned}$$

Taking $\|u\| \geq 1$, we have

$$J(u) \geq \frac{m_0}{p^+}\|u\|^{p^-} - \frac{1}{r+1}(\frac{A_2}{q^-})^{r+1}\|u\|^{q^+(r+1)}$$

Since $q^+(r+1) < p^-$, so J is bounded from below. The proof of Lemma 3.2 is complete. □

Lemma 4.2. *The functional J satisfies the Palais-Smale condition in X.*

Proof. Let $(u_n)_n \subset X$ be a sequence such that

$$J_w(u_n) \longrightarrow c > 0, \qquad J'_w(u_n) \longrightarrow 0 \qquad \text{in} \quad X^*. \tag{4.1}$$

where X^* is the dual space of X.
By the ceorcivity of J, the sequence $(u_n)_n$ is bounded in X.
By the reflexity of X; for a subsequence still denoted $(u_n)_n$, such that $u_n \rightharpoonup u$ in X. Similar to proof of Theorem 1.2 we deduce that $u_n \longrightarrow u$ in X.

□

In the sequel, for each $k \in \mathbb{N}$ consider $X_k = span\{e_1, e_2, ..., e_k\}$ the subspace of X (see Theorem 4.1 in [2]). Note that $X_k \hookrightarrow L^{\alpha(x)}(\Omega)$, $1 < \alpha(x) < p^*(x)$ with continuous immersions. Then, the norm X, $L^{\alpha(x)}(\Omega)$ are equivalent on X_k.
By using the condition (M_1) and (f_1), we have

$$\begin{aligned} J(u) &= \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) - \frac{1}{r+1}\Big[\int_\Omega F(x,u,|\nabla u|^{p(x)-2}\nabla u)dx\Big]^{r+1} \\ &\leq \frac{m_1}{p^-}I(u) - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}\Big[\int_\Omega |u|^{\alpha(x)}dx\Big]^{r+1} \\ &\leq \frac{m_1}{p^-}I(u) - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}\Big[\rho_{\alpha(x)}(u)\Big]^{r+1} \\ &\leq \frac{m_1}{p^-}\|u\|^{p^-} - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}C(k)\|u\|^{\alpha^+(r+1)} \end{aligned}$$

where $C(k)$ is a positive constant and $\|u\|$ is small enough. Hence

$$J(u) \leq \|u\|^{\alpha^+(r+1)}\Big[\frac{m_1}{p^-}\|u\|^{p^- - \alpha^+(r+1)} - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}C(k)\Big]$$

Let R be a positive constant such that

$$\frac{m_1}{p^-}R^{p^- - \alpha^+(r+1)} \leq \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}C(k)$$

Then, for all $r_0 \in)0, R($ and considering $K = \{u \in X : \quad \|u\| = r_0\}$, we have

$$\begin{aligned} J(u) &\leq r_0^{\alpha^+(r+1)}\Big[\frac{m_1}{p^-}r_0^{p^- - \alpha^+(r+1)} - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}C(k)\Big] \\ &< R^{\alpha^+(r+1)}\Big[\frac{m_1}{p^-}R^{p^- - \alpha^+(r+1)} - \frac{1}{r+1}(\frac{A_1}{\alpha^+})^{r+1}C(k)\Big] < 0 = J(0). \end{aligned}$$

Which implies

$$Sup_K J(u) < 0 = J(0).$$

By the Clark theorem, J has at least k different critical points. Because k is arbitrary, we obtain infinitely many critical points of J.

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