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# SOME PROPERTIES OF STRONGLY-CONVEX SUBTRELLISES OF A TRELLIS

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**Abstract** A non-empty subset $X$ of a trellis $T$ is called strongly-convex if $X$ is an intersection of an ideal and a dual ideal of $T$. We study the collection $SC(T)$ of all strongly-convex subtrellises of $T$ with respect to two different orders, viz. bi-domination and inclusion. Furthermore, using the notion of strong-convexity, the existence of the smallest congruence relation on a trellis whose homomorphic image is a lattice is proved.

## 1 Introduction

Let $L$ represent a lattice, and $CS(L)$ denote the collection of all non-empty convex sublattices of $L$. In 1996, S. Lavanya and S. P. Bhatta [4] examined $CS(L)$ under a partial order $\leq$ (referred to as bi-domination) distinct from the inclusion relation. This order can be described as follows: for any $A, B \in CS(L)$, $A \leq B$ if and only if $(A] \subseteq (B]$ and $[A) \supseteq [B)$, where $(A]$ and $[A)$ respectively denote the ideal and the filter (dual ideal) generated by $A$ on $L$. They demonstrated that $(CS(L), \leq)$ forms a lattice wherein both $L$ and $CS(L)$ belong to the same equational class. Later, in 2011, Bhatta and Ramananda [5] established a congruence relation $\overline{\Theta}$ on $CS(L)$ corresponding to every congruence relation $\Theta$ on $L$, such that $CS(L/\Theta) \cong CS(L)/\overline{\Theta}$. In an attempt to generalize these ideas to trellises, K. Bhargava et al. [1] introduced the concept of strongly-convex subtrellises in trellises. They have successfully generalized many results of [4, 5] to trellises.

The concepts of a pseudo-ordered set and a trellis (also known as a weakly associative lattice) were first introduced independently by E. Fried [2] and H. L. Skala [6]. A pseudo-order defined on a non-empty set $P$ constitutes a reflexive and anti-symmetric relation $\trianglelefteq$ on it, making $(P, \trianglelefteq)$ a pseudo-ordered set. If a pseudo-order is transitive, it qualifies as a partial order. A trellis $T$ is essentially a pseudo-ordered set where each pair of elements possesses both a least upper bound and a greatest lower bound. The concepts of subtrellises, ideals, dual ideals (filters), order-preserving (isotone) maps, homomorphisms, congruence relations, etc., are defined for trellises similar to those in lattices.

Given a non-empty subset $X$ of a trellis $T$, and for $a, b \in T$, the notation $a \sqsubseteq_X b$ signifies the existence of $x_1, \ldots, x_n \in X$ such that $a \trianglelefteq x_1 \trianglelefteq \cdots \trianglelefteq x_n \trianglelefteq b$. When $X = T$, the symbol $\sqsubseteq$ is used instead of $\sqsubseteq_T$. The relation $\sqsubseteq$ represents the transitive closure of the pseudo-order $\trianglelefteq$ on $T$.

A fundamental property of lattices [3] is as follows: "The intersection of an ideal and a dual ideal of a lattice $L$ forms a convex sublattice, provided it is non-empty. Conversely, if $C$ is a convex sublattice of $L$, then $C$ can be uniquely expressed as the intersection of an ideal and a dual ideal." Motivated by this, K. Bhargava et al. define the following:

**Definition 1.1.** [1] A non-empty subset $X$ of a trellis $T$ is said to be *strongly-convex* if $X = I \cap D$ for some ideal $I$ of $T$ and for some dual ideal $D$ of $T$.

It has been demonstrated in [1], that the collection $SC(T)$ comprising of all strongly-convex subtrellises of $T$ forms a lattice under the bi-domination relation (cf. Theorem 2.2). Moreover, they have successfully extended numerous findings from [4, 5] to the lattice $(SC(T), \leq)$.

In Section 3.1 of this article, we initially provide a representation of the lattice $(SC(T), \leq)$ by means of a characterization theorem (Theorem 3.1). Following that, we identify all the distributive, standard, and neutral elements within $(SC(T), \leq)$. In Section 3.2, we study the properties of $SC(T)$ with respect to the inclusion order. We prove that the poset $(SC(T) \cup \{\emptyset\}, \subseteq)$ forms an algebraic lattice. In Section 4, as an application of the newly defined notion of strong-convexity, it is proved the existence of the smallest congruence relation on a trellis whose homomorphic image is a lattice. Additionally, in Section 2, we outline some fundamental findings from [1] for quick reference.

For terminologies and notations not mentioned here, the reader may refer to [3, 6, 7].

## 2 Preliminaries

Let $T$ denote a trellis. The ideal (dual ideal) generated by a non-empty subset $X$ of $T$ is defined as the smallest ideal (dual ideal) containing $X$, denoted respectively by $(X]$ and $[X)$. When $X = \{x\}$, the symbols $(x]$ and $[x)$ are used instead of $\big(\{x\}\big]$ and $\big[\{x\}\big)$. Note that we do not consider the empty set $\emptyset$ as an ideal (dual ideal). The ideal lattice of $T$ with respect to the inclusion order is denoted by $I(T)$, and $D(T)$ denotes the lattice of dual ideals of $T$ with respect to the reverse inclusion.

**Theorem 2.1.** *[1] Let $T$ be a trellis, and $X \in SC(T)$. Then $X = (X] \cap [X)$.*

**Theorem 2.2.** *[1] Let $T$ be a trellis. For $X, Y \in SC(T)$, define $X \leq Y$ if and only if $(X] \subseteq (Y]$ and $[X) \supseteq [Y)$. Then $\leq$ is a partial order on $SC(T)$, and the poset $(SC(T), \leq)$ forms a lattice with respect to the following meet and join operations:*

$$X \wedge Y = \big((X] \wedge (Y]\big) \cap \big([X) \wedge [Y)\big) \tag{2.1}$$

$$\text{and} \quad X \vee Y = \big((X] \vee (Y]\big) \cap \big([X) \vee [Y)\big), \tag{2.2}$$

*where $(X] \wedge (Y] = (X] \cap (Y]$, $(X] \vee (Y] = \big((X] \cup (Y]\big]$ are the meet and join operations in $I(T)$, and $[X) \wedge [Y) = \big[[X) \cup [Y)\big)$, $[X) \vee [Y) = [X) \cap [Y)$ are the meet and join operations in $D(T)$.*

**Definition 2.3.** [1] Let $X$ be a non-empty subset of a trellis $T$.

(i) The *initial segment of $X$ with respect to* $\sqsubseteq$ is defined by

$$\downarrow_{\sqsubseteq}(X) = \{\, y \in T : y \sqsubseteq x \text{ for some } x \in X \,\}.$$

If $X = \{x\}$, then the symbol $\downarrow_{\sqsubseteq}(x)$ is used in place of $\downarrow_{\sqsubseteq}\big(\{x\}\big)$.

(ii) The *final segment of $X$ with respect to* $\sqsubseteq$ is defined by

$$\uparrow_{\sqsubseteq}(X) = \{\, y \in T : x \sqsubseteq y \text{ for some } x \in X \,\}.$$

If $X = \{x\}$, then the symbol $\uparrow_{\sqsubseteq}(x)$ is used in place of $\uparrow_{\sqsubseteq}\big(\{x\}\big)$.

Note that $X \subseteq \downarrow_{\sqsubseteq}(X) \subseteq (X]$ and $X \subseteq \uparrow_{\sqsubseteq}(X) \subseteq [X)$ for any non-empty subset $X$ of a trellis.

**Definition 2.4.** [1] A subset $X$ of a trellis $T$ is said to be p-convex if whenever $x, y \in X$ and $a \in T$ with $x \sqsubseteq a \sqsubseteq y$, then $a \in X$.

A characterization of strongly-convex subtrellises is given in the following theorem.

**Theorem 2.5.** *[1] A non-empty subset $X$ of a trellis $T$ lies in $SC(T)$ if and only if $(X] = \downarrow_{\sqsubseteq}(X)$ (that is $\downarrow_{\sqsubseteq}(X)$ is an ideal of $T$), $[X) = \uparrow_{\sqsubseteq}(X)$ (that is $\uparrow_{\sqsubseteq}(X)$ is a dual ideal of $T$), and $X$ is p-convex.*

**Remark 2.6.** [1] The partial order $\leq$ in $SC(T)$ has the following alternate description, using Theorem 2.5: for $X, Y \in SC(T)$, $X \leq Y$ if and only if "for each $x \in X$, there is a $y \in Y$ such that $x \sqsubseteq y$" and "for each $y \in Y$, there is a $x \in X$ such that $x \sqsubseteq y$".

**Definition 2.7.** [1] Let $T$ be a trellis, and $A$ be a non-empty subset of $T$. The *strongly-convex subtrellis generated by* $A$ is defined by $\langle A \rangle = (A] \cap [A)$.

If $A = \{a\}$, then the symbol $\langle a \rangle$ is used in place of $\big\langle \{a\} \big\rangle$.

**Remark 2.8.** [1]

(i) Note that $X = (A] \cap [A)$ is the smallest strongly-convex subtrellis of $T$ containing $A$.

(ii) Clearly, $\big(\langle A \rangle\big] = (A]$ and $\big[\langle A \rangle\big) = [A)$ for any non-empty subset $A$ of a trellis $T$. Also, $A$ is strongly-convex if and only if $\langle A \rangle = A$. Furthermore, for any two non-empty subsets $A$ and $B$ of $T$, if $A \subseteq B$, then $\langle A \rangle \subseteq \langle B \rangle$.

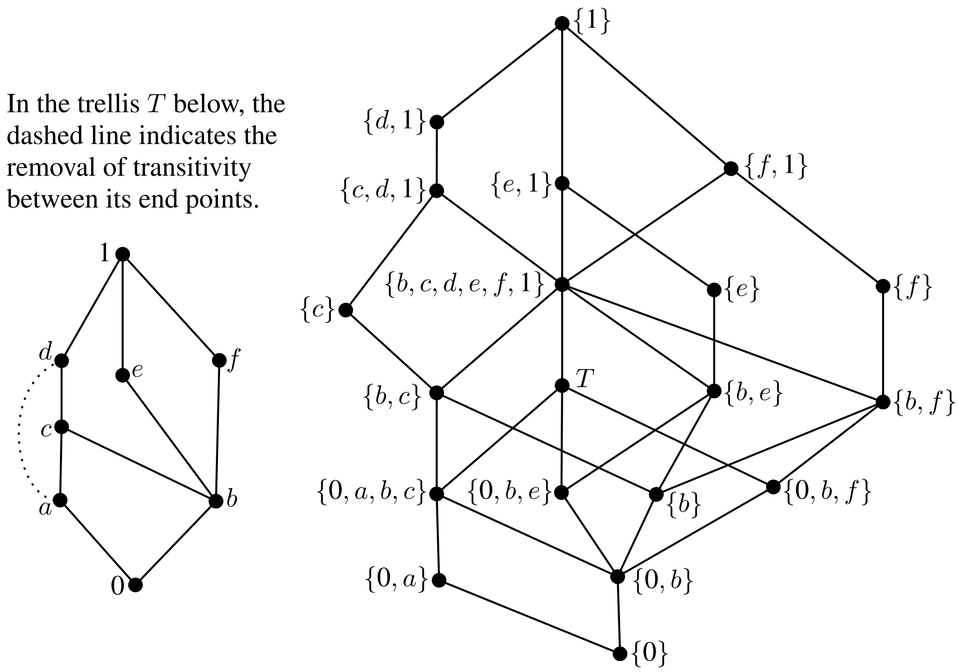


**Figure 1.** A trellis $T$ and the corresponding lattice $(SC(T), \leq)$.

## 3 Main results

### 3.1 Some results on $SC(T)$ with respect to the bi-domination order

Let $T$ be a trellis and let $\mathcal{P}^*(T)$ denote the collection of all non-empty subsets of $T$. For $A, B \in \mathcal{P}^*(T)$, define $A \leq B$ if and only if $(A] \subseteq (B]$ and $[A) \supseteq [B)$. Clearly $\leq$ is a preorder on $\mathcal{P}^*(T)$, i.e. $\leq$ is reflexive and transitive. Now define a new relation $\theta$ on $\mathcal{P}^*(T)$ as follows: for $A, B \in \mathcal{P}^*(T)$, define $A \equiv B \ (\theta)$ if and only if $A \leq B$ and $B \leq A$. Clearly, $\theta$ is an equivalence relation on $\mathcal{P}^*(T)$. Note that $A \equiv \langle A \rangle \ (\theta)$, and $A \equiv B \ (\theta)$ if and only if $\langle A \rangle = \langle B \rangle$. Hence the relation $\leq$ defined on $\mathcal{P}^*(T)/\theta$ by, for $[A], [B] \in \mathcal{P}^*(T)/\theta$, $[A] \leq [B]$ if and only if $\langle A \rangle \leq \langle B \rangle$, is a partial order on $\mathcal{P}^*(T)/\theta$. It follows that the posets $(\mathcal{P}^*(T)/\theta, \leq)$ and $(SC(T), \leq)$ are order isomorphic. In fact $[A] \mapsto \langle A \rangle$ is an order isomorphism between the two posets. But since the poset $(SC(T), \leq)$ is also a lattice, the poset $(\mathcal{P}^*(T)/\theta, \leq)$ must also be a lattice, and hence we have the following characterization theorem:

**Theorem 3.1.** *For any trellis $T$,*

$$(SC(T), \leq) \cong (\mathcal{P}^*(T)/\theta, \leq).$$

Next we determine the distributive, standard, and neutral elements of the lattice $(SC(T), \leq)$. An element $a$ of a lattice $L$ is called [3]:

(i) distributive if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ for all $x, y \in L$.

(ii) standard if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ for all $x, y \in L$.

(iii) neutral if $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$ for all $x, y \in L$.

First, note that for any $X, Y \in SC(T)$, using the Theorem 2.2, we have

$$(X \wedge Y] = (X] \wedge (Y] = (X] \cap (Y] \tag{3.1}$$

$$\text{and } [X \wedge Y) = [X) \wedge [Y) = \big[[X) \cup [Y)\big). \tag{3.2}$$

Also,

$$(X \vee Y] = (X] \vee (Y] = \big((X] \cup (Y]\big] \tag{3.3}$$

$$\text{and } [X \vee Y) = [X) \vee [Y) = [X) \cap [Y). \tag{3.4}$$

Hence if $p$ is any $n$-ary lattice polynomial, and $X_1, \ldots, X_n \in SC(T)$, then

$$\big(p(X_1, \ldots, X_n)\big] = p\big((X_1], \ldots, (X_n]\big) \tag{3.5}$$

$$\text{and } \big[p(X_1, \ldots, X_n)\big) = p\big([X_1), \ldots, [X_n)\big). \tag{3.6}$$

Consequently, the result below follows immediately:

**Lemma 3.2.** *Let $T$ be a trellis and let $A, B \in SC(T)$. Suppose $p, q$ are any two $(n+1)$-ary lattice polynomials. Then*

$$p(A, X_1, \ldots, X_n) = q(B, X_1, \ldots, X_n) \ \ \forall\, X_1, \ldots, X_n \in SC(T) \tag{3.7}$$

*if and only if*

$$p\big((A], I_1, \ldots, I_n\big) = q\big((B], I_1, \ldots, I_n\big) \ \ \forall\, I_1, \ldots, I_n \in I(T) \tag{3.8}$$

*and*

$$p\big([A), D_1, \ldots, D_n\big) = q\big([B), D_1, \ldots, D_n\big) \ \ \forall\, D_1, \ldots, D_n \in D(T). \tag{3.9}$$

Taking $A = B$ in the above lemma, we have the following theorem:

**Theorem 3.3.** *Let $T$ be a trellis and $A \in SC(T)$. Suppose $p, q$ are any two $n$-ary lattice polynomials. Then $A$ satisfies the polynomial identity $p = q$ in the lattice $(SC(T), \leq)$ if and only if $(A]$ satisfies the polynomial identity $p = q$ in $I(T)$, and $[A)$ satisfies the polynomial identity $p = q$ in $D(T)$.*

As a special case, we have the following:

**Corollary 3.4.** *Let $T$ be a trellis and $A \in SC(T)$. Then*

*(i) $A$ is a distributive element of $(SC(T), \leq)$ if and only if $(A]$ is a distributive element of $(I(T), \subseteq)$ and $[A)$ is a distributive element of $(D(T), \supseteq)$.*

*(ii) $A$ is a standard element of $(SC(T), \leq)$ if and only if $(A]$ is a standard element of $(I(T), \subseteq)$ and $[A)$ is a standard element of $(D(T), \supseteq)$.*

*(iii) $A$ is a neutral element of $(SC(T), \leq)$ if and only if $(A]$ is a neutral element of $(I(T), \subseteq)$ and $[A)$ is a neutral element of $(D(T), \supseteq)$.*

### 3.2 A study of $SC(T)$ with respect to the inclusion order

**Theorem 3.5.** *For any trellis $T$, the poset $(SC(T) \cup \{\emptyset\}, \subseteq)$ forms a complete lattice, where for $X, Y \in SC(T)$,*

$$X \wedge Y = X \cap Y, \tag{3.10}$$

$$X \vee Y = \langle X \cup Y \rangle. \tag{3.11}$$

*Proof.* Follows by noting that the intersection of any collection of strongly-convex subtrellises of $T$ is either empty or strongly-convex. □

**Lemma 3.6.** *Let $T$ be a trellis. If $\{X_\alpha\}$ is a chain in $(SC(T) \cup \{\emptyset\}, \subseteq)$, then $X = \bigcup_\alpha X_\alpha \in SC(T)$.*

*Proof.* To prove that $X$ is p-convex, let $x, y \in X$ and $a \in T$ be such that $x \sqsubseteq a \sqsubseteq y$. Since $\{X_\alpha\}$ is a chain with respect to $\subseteq$, it follows that $x, y \in X_\alpha$ for some $\alpha$. Then $a \in X_\alpha$, as $X_\alpha$ is p-convex. Thus $a \in X$.

To prove that $\downarrow_{\sqsubseteq}(X)$ is an ideal, let $a, b \in \downarrow_{\sqsubseteq}(X)$. Then $a \sqsubseteq x$ and $b \sqsubseteq y$ for some $x, y \in X$. Again, since $\{X_\alpha\}$ is a chain with respect to $\subseteq$, it follows that $x, y \in X_\alpha$ for some $\alpha$. Then $a, b \in (X_\alpha]$, and hence $a \vee b \in (X_\alpha] = \downarrow_{\sqsubseteq}(X_\alpha)$, as $X_\alpha$ is strongly convex. Thus $a \vee b \in \downarrow_{\sqsubseteq}(X)$. Hence $\downarrow_{\sqsubseteq}(X)$ is an ideal of $T$.

Similarly $\uparrow_{\sqsubseteq}(X)$ is a dual ideal of $T$. Thus $X \in SC(T)$ by Theorem 2.5. □

**Theorem 3.7.** *Let $T$ be a trellis, and let $A \subseteq T$, $A \neq \emptyset$. Then*

$$\langle A \rangle = \bigcup \{\, \langle F \rangle : F \subseteq A,\ F \textit{ finite} \,\}. \tag{3.12}$$

*Proof.* The proof is by transfinite induction on $|A|$. If $A$ is finite, then the result holds trivially. Assume that $A$ is infinite and the result holds for all sets of cardinality smaller than $|A|$. We well-order the set $A$ in the order type of its cardinal. Then for each $a \in A$, the initial segment $S(a) = \{\, x \in A : x < a \,\}$ determined by $a$ has cardinality strictly less than $|A|$. Furthermore, as $A$ is infinite, its cardinal is a limit ordinal, and hence it follows that

$$A = \bigcup_{a \in A} S(a). \tag{3.13}$$

Let $\mathcal{C} = \{\, \langle S(a) \rangle : a \in A \,\}$. Since $A$ is a chain with respect to $\leq$, it follows that $\mathcal{C}$ is a chain with respect to $\subseteq$. For, if $a, b \in A$ with $a \leq b$, then $S(a) \subseteq S(b)$. Now the last part of Remark 2.8(ii) implies that $\langle S(a) \rangle \subseteq \langle S(b) \rangle$. Let

$$C = \bigcup \mathcal{C} = \bigcup_{a \in A} \langle S(a) \rangle.$$

By Lemma 3.6, it follows that $C$ is strongly-convex. For each $a \in A$, since $S(a) \subseteq \langle S(a) \rangle \subseteq C$, it follows from (3.13) that $A \subseteq C$. Hence $\langle A \rangle \subseteq \langle C \rangle = C$ as $C \in SC(T)$. Also for any $a \in A$, the fact that $S(a) \subseteq A$ implies that $\langle S(a) \rangle \subseteq \langle A \rangle$, and hence $C \subseteq \langle A \rangle$. Thus we get

$$\langle A \rangle = C = \bigcup_{a \in A} \langle S(a) \rangle.$$

Note that, by induction hypothesis, for each $a \in A$, we have

$$\langle S(a) \rangle = \bigcup \{\, \langle F \rangle : F \subseteq S(a),\ F \text{ finite} \,\},$$

as $|S(a)| < |A|$. Thus

$$\langle A \rangle = \bigcup_{a \in A} \langle S(a) \rangle = \bigcup \{\, \langle F \rangle : F \subseteq A,\ F \text{ finite} \,\},$$

as any finite subset $F$ of $A$ is contained in $S(a)$ for some $a \in A$. □

An element $a$ of a complete lattice $L$ is compact if whenever $a \leq \vee S$ for a subset $S$ of $L$, then $a \leq \vee F$ for a finite subset $F$ of $S$. A complete lattice $L$ is said to be algebraic (compactly generated) if every element of $L$ is a join of some compact elements of $L$.

**Lemma 3.8.** *Let $T$ be a trellis and $F = \{x_1, \ldots, x_n\}$ be a non-empty finite subset of $T$. Then*

$$\langle F \rangle = \bigvee_{i=1}^{n} \langle x_i \rangle. \tag{3.14}$$

*Proof.* As $x_i \in \langle x_i \rangle$ for each $i$, clearly

$$F \subseteq \bigcup_{i=1}^{n} \langle x_i \rangle \text{ and hence } \langle F \rangle \subseteq \left\langle \bigcup_{i=1}^{n} \langle x_i \rangle \right\rangle = \bigvee_{i=1}^{n} \langle x_i \rangle.$$

For the reverse inclusion, observe that $x_i \in F$ implies that $\langle x_i \rangle \subseteq \langle F \rangle$ for each $i$. Thus

$$\bigcup_{i=1}^{n} \langle x_i \rangle \subseteq \langle F \rangle, \text{ and hence } \bigvee_{i=1}^{n} \langle x_i \rangle = \left\langle \bigcup_{i=1}^{n} \langle x_i \rangle \right\rangle \subseteq \langle F \rangle.$$

This completes the proof. □

**Corollary 3.9.** *For any trellis $T$, the complete lattice $(SC(T) \cup \{\emptyset\}, \subseteq)$ is algebraic. The set of all compact elements of this lattice is*

$$\{\, \langle F \rangle : \ F \text{ finite subset of } T \,\}. \tag{3.15}$$

*Proof.* Let $F = \{f_1, \ldots, f_n\}$ be a finite subset of $T$, and suppose

$$\langle F \rangle \subseteq \bigvee_{\alpha \in \Lambda} X_\alpha = \left\langle \bigcup_{\alpha} X_\alpha \right\rangle$$

for some sub-collection $\{\, X_\alpha : \alpha \in \Lambda \,\}$ in $SC(T)$. By Theorem 3.7, we have

$$\left\langle \bigcup_{\alpha} X_\alpha \right\rangle = \bigcup \{\, \langle S \rangle : S \subseteq \bigcup_{\alpha} X_\alpha, \ S \text{ finite} \,\}.$$

Since $f_i \in F \subseteq \langle F \rangle$, for each $i$, there exists a finite subset $S_i$ of $\cup_\alpha X_\alpha$ such that $f_i \in \langle S_i \rangle$. Since $S_i$ is a finite subset of $\cup_\alpha X_\alpha$, for each $i$, there is a finite subset $\Lambda_i$ of $\Lambda$ such that

$$S_i \subseteq \bigcup_{\alpha \in \Lambda_i} X_\alpha.$$

Put $\Lambda_F = \cup_{i=1}^{n} \Lambda_i$. Then $\Lambda_F$ is a finite subset of $\Lambda$ and

$$\begin{aligned} F = \{\, f_1, \ldots, f_n \,\} &\subseteq \bigcup_{i=1}^{n} \langle S_i \rangle \\ &\subseteq \left\langle \bigcup_{i=1}^{n} S_i \right\rangle \\ &\subseteq \left\langle \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_i} X_\alpha \right\rangle = \left\langle \bigcup_{\alpha \in \Lambda_F} X_\alpha \right\rangle. \end{aligned}$$

Hence we conclude that

$$\langle F \rangle \subseteq \left\langle \bigcup_{\alpha \in \Lambda_F} X_\alpha \right\rangle = \bigvee_{\alpha \in \Lambda_F} X_\alpha.$$

Thus $\langle F \rangle$ is a compact element in $(SC(T) \cup \{\emptyset\}, \subseteq)$.

Next let $X \in SC(T)$ be any compact element. We prove that $X = \langle F \rangle$ for some finite subset $F$ of $T$. Clearly

$$\begin{aligned} X &\subseteq \bigcup_{x\in X} \langle x \rangle \\ &\subseteq \left\langle \bigcup_{x\in X} \langle x \rangle \right\rangle = \bigvee_{x\in X} \langle x \rangle. \end{aligned}$$

As $X$ is compact, there exist finitely many elements $x_1, \dots, x_n \in X$ such that

$$X \subseteq \bigvee_{i=1}^{n} \langle x_i \rangle.$$

Put $F = \{x_1, \dots, x_n\}$. Then

$$\langle F \rangle = \bigvee_{i=1}^{n} \langle x_i \rangle, \text{ by Lemma 3.8.}$$

Also since $F \subseteq X$, we have $\langle F \rangle \subseteq \langle X \rangle = X$. Thus we conclude that $X = \langle F \rangle$, where $F$ is a finite subset of $T$.

Thus the set of all compact elements of the complete lattice $(SC(T) \cup \{\emptyset\}, \subseteq)$ is

$$\{\, \langle F \rangle : \ F \text{ finite subset of } T \,\}.$$

We now show that this lattice is algebraic by proving that every element of $SC(T)$ is a join of some compact elements in $SC(T)$. Let $X \in SC(T)$ be arbitrary. Then by Theorem 3.7, we have

$$X = \bigcup \{\, \langle F \rangle : F \subseteq X, \ F \text{ finite} \,\}.$$

But then

$$\begin{aligned} \bigvee \{\, \langle F \rangle : F \subseteq X, \ F \text{ finite} \,\} &= \left\langle \bigcup \{\, \langle F \rangle : F \subseteq X, \ F \text{ finite} \,\} \right\rangle \\ &= \langle X \rangle \\ &= X, \text{ as } X \text{ is strongly convex.} \end{aligned}$$

Thus $X$ is a join of some compact elements in $SC(T)$. Hence $(SC(T) \cup \{\emptyset\}, \subseteq)$ is algebraic. □

For the trellis $T$ of Figure 1, the lattice $(SC(T) \cup \{\emptyset\}, \subseteq)$ is shown in Figure 2.

## 4 An application

**Lemma 4.1.** *Let $\Theta$ be a congruence relation on a trellis $T$ and $a, b \in T$. Suppose that $[a]\Theta, [b]\Theta \in SC(T)$. Then $[a] \le [b]$ in $SC(T)$ if and only if $[a] \trianglelefteq [b]$ in the quotient trellis $T/\Theta$.*

*Proof.* Assume that $[a] \le [b]$ in $SC(T)$. To prove that $[a] \trianglelefteq [b]$ in $T/\Theta$, it suffices to show that $a \vee b \equiv b \ (\Theta)$. As $[a] \le [b]$ in $SC(T)$, there is a $x \in [b]$ such that $a \sqsubseteq x$. Since $\downarrow_{\sqsubseteq}([b])$ is an ideal of $T$, we have $a \vee b \in \downarrow_{\sqsubseteq}([b])$. Thus $a \vee b \sqsubseteq y$ for some $y \in [b]$. As $b, y \in [b]$ and $b \trianglelefteq a \vee b \sqsubseteq y$, we have $a \vee b \in [b]$, by $p$-convexity of $[b]$.

Conversely, assume that $[a] \trianglelefteq [b]$ in $T/\Theta$. To show that $[a] \le [b]$ in $SC(T)$, consider an element $x \in [a]$. As $[a] = [a \wedge b]$, $x \equiv a \wedge b \ (\Theta)$. Hence $x \vee b \equiv b \ (\Theta)$. Take $y = x \vee b$. Then $y \in [b]$ and $x \trianglelefteq y$. Similarly for each $y \in [b]$, there is a $x \in [a]$ such that $x \trianglelefteq y$. Thus $[a] \le [b]$ holds in $SC(T)$. □

**Theorem 4.2.** *Let $\Theta$ be a congruence relation on a trellis $T$. Then $[a]\Theta \in SC(T)$ for all $a \in T$ if and only if the quotient trellis $T/\Theta$ is a lattice.*

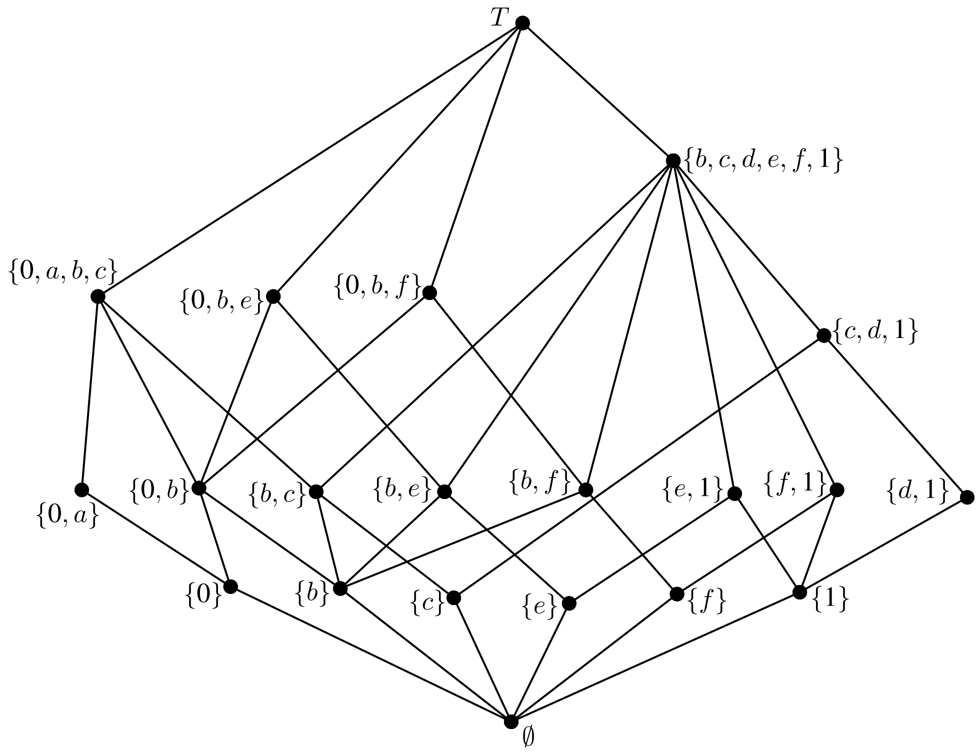


**Figure 2.** The lattice $(SC(T) \cup \{\emptyset\}, \subseteq)$ corresponding to the trellis $T$ given in Figure 1.

*Proof.* Assume that $[a] \in SC(T)$ for all $a \in T$. Let $[a] \trianglelefteq [b] \trianglelefteq [c]$ in $T/\Theta$. Then by Lemma 4.1, $[a] \leq [b] \leq [c]$ holds in $SC(T)$. Therefore $[a] \leq [c]$, as $SC(T)$ is a poset. Hence $[a] \trianglelefteq [c]$ in $T/\Theta$ by Lemma 4.1 again.

Conversely, assume that the quotient trellis $T/\Theta$ is a lattice. Let $a \in T$. To prove that $[a]$ is $p$-convex, consider any $x, y, z \in T$ with $x \sqsubseteq y \sqsubseteq z$, where $x, z \in [a]$. Then $[x] \sqsubseteq [y] \sqsubseteq [z]$ holds in $T/\Theta$, where $[x] = [z] = [a]$. But then, as the pseudo-order in $T/\Theta$ is a partial order, we have $[y] = [a]$. Hence $y \in [a]$, and thus $[a]$ is $p$-convex. Next we prove that $\downarrow_{\sqsubseteq}([a])$ is an ideal of $T$. Let $x \sqsubseteq p$ and $y \sqsubseteq q$ for some $p, q \in T$ with $p, q \in [a]$. Then $[x] \sqsubseteq [p] = [a]$ and $[y] \sqsubseteq [q] = [a]$ hold in $T/\Theta$. Thus, as $T/\Theta$ is a lattice, we have $[x] \vee [y] \trianglelefteq [a]$, and hence by Lemma 4.1, $[x \vee y] \leq [a]$ holds in $SC(T)$. Thus $x \vee y \sqsubseteq z$ for some $z \in [a]$. This proves that $\downarrow_{\sqsubseteq}([a])$ is an ideal of $T$. By the dual argument, $\uparrow_{\sqsubseteq}([a])$ is a dual ideal of $T$. □

**Theorem 4.3.** *Let $T$ be a trellis, and let $\mathcal{A}$ be the collection of all congruence relations $\Theta$ on $T$ such that $T/\Theta$ is a lattice. Then $\mathcal{A}$ forms a principal dual ideal in Con$(T)$, the congruence lattice of $T$.*

*Proof.* We have,

$$\begin{aligned}\mathcal{A} &= \{\, \Theta \in \mathrm{Con}(T) \colon T/\Theta \text{ is a lattice} \,\} \\ &= \{\, \Theta \in \mathrm{Con}(T) \colon [a]\Theta \in SC(T) \text{ for all } a \in T \,\}, \text{ using Theorem 4.2.}\end{aligned}$$

The universal relation $\iota$—that is, the relation in which every pair of elements in $T$ is related—lies in $\mathcal{A}$. Thus, $\mathcal{A} \neq \emptyset$. Let

$$\pi = \bigwedge_{\Theta \in \mathcal{A}} \Theta.$$

Clearly $\pi$ is a congruence relation on $T$. Further, for any $a \in T$, we have

$$[a]\pi = \bigcap \{\, [a]\Theta \colon \Theta \in \mathcal{A} \,\},$$

which is a non-empty intersection of a collection of strongly-convex subtrellises of $T$. Thus $[a]\pi \in SC(T)$ for all $a \in T$. Thus $T/\pi$ is a lattice, by Theorem 4.2, and hence $\pi \in \mathcal{A}$. In fact, $\mathcal{A} = [\pi)$, the dual ideal of $\mathrm{Con}(T)$ generated by $\pi$. This follows from the fact that $\mathcal{A}$ is closed under intersection, and if $\Theta$ is a congruence relation on $T$ with $\Theta \geq \pi$, then $T/\Theta$ is a homomorphic image of $T/\pi$. $\square$

The congruence lattice of the trellis $T$, shown in Figure 1, is a 3-element chain, with the only non-trivial element being the congruence relation $\Theta(0, a)$—the smallest congruence on $T$ under which 0 is congruent to $a$. The set all of congruence classes of $\Theta(0, a)$ is $\big\{\{0, a\}, \{b, c, d, e, f, 1\}\big\}$. The congruence relation $\pi$ referred to in the above theorem for $T$ is simply $\Theta(0, a)$.

## References


[1] Bhargava K., S. P. Bhatta and C. Hegde, *On strongly-convex subtrellises of a trellis*, communicated.

[2] E. Fried, *Tournaments and non-associative lattices*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **13**, 151–164, (1970).

[3] G. Grätzer, *General Lattice Theory*, 2nd edition, Birkhäuser Verlag, Basel, (2003).

[4] S. Lavanya and S. Parameshwara Bhatta, *A new approach to the lattice of convex sublattices of a lattice*, Algebra Univers., **35**(1), 63–71, (1996).

[5] S. Parameshwara Bhatta and H. S. Ramananda, *Natural extension of a congruence of a lattice to its lattice of convex sublattices*, Arch. Math. (Brno), **47**(2), 133–138, (2011).

[6] H. L. Skala, *Trellis theory*, Algebra Univers., **1**, 218–233, (1971).

[7] H. Skala, *Trellis Theory*, Mem. Am. Math. Soc. **121**, AMS, Providence, RI, (1972).


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