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Generalized $r-$ Minkowski formula on weighted manifolds

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Abstract In this paper we derive an integral formula for hypersurfaces embedded in weighted manifolds involving the weighted higher order mean curvatures σ_k^∞ of the hypersurface and the weighted Newton transformations T_k^∞ (see [11]). This formula generalizes the classical $r-$Minkowski integral formula on Riemannian manifolds.

1 Introduction

Integral formulas are useful tools for solving many problems in Riemannian geometry, such as Alexandrov's theorem or the characterization of certain hypersurfaces (see [4, 3, 20]) and references therein.

Let $x : M^n \longrightarrow \overline{M}^{n+1}$ be a closed oriented hypersurface immersed into a space form $\overline{M}^{n+1}$ with unit normal vector field N. Then we have for $1 \leq r \leq n$:

$$\int_{M^n} H_{r-1} dv + \int_{M^n} \langle Y, N\rangle . H_r dv = 0, \tag{1.1}$$

where Y is a conformal vector field, and H_r is the higher order mean curvature of the hypersurface.

Higher order Minkowski formulae were first obtained by Hsiung [15] in Euclidean space, and by Bivens [9] in the Euclidean sphere and hyperbolic space.

Many generalizations of the previous results have been investigated in the case where the ambient space is not a space form and in other contexts, see for instance [3, 4, 9, 16, 17, 20].

In a series of recent papers, the first author together with M. Benalili [1, 2] obtained a series of integral formulas on weighted manifolds . The idea is to compute the divergence of certain vector fields and applying the divergence theorem.

In this paper, using the weighed symmetric functions σ_k^∞ and the weighted Newton transformations T_k^∞ introduced by J. S. Case [11], we derive some new integral formulae on manifolds with density. We give also some applications and special cases.

2 Preliminaries

In this section, we will fix the notations and recall some definitions and basic results of weighted manifolds and the weighted Newton transformations. For more details, see [2, 11, 12, 13].

A weighted manifold $(\overline{M}, \langle , \rangle , dv_f)$ is a Riemannian manifold $\overline{M}$ endowed with a weighted volume form $dv_f = e^{-f} dv$, where f is a real-valued smooth function on $\overline{M}$, and dv is the Riemannian volume form associated with the metric $\langle , \rangle$.

Let $\overline{M}_f^{n+1}$ be an $(n + 1)-$dimensional oriented weighted Riemannian manifold, and ψ : $M^n \longrightarrow \overline{M}_f^{n+1}$ be an isometrically immersed hypersurface with unit normal vector field N in the normal bundle NM^n. It's Weingarten operator A is defined by

$$AX = -\left(\overline{\nabla}_X N\right)^{\mathsf{T}} ,$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}_f^{n+1}$, and X$\in \varkappa(M^n)$.

By the Codazzi equation, we can see that the normal component of the curvature tensor $\overline{R}$ of $\overline{M}_f^{n+1}$ is given in terms of A by

$$\langle \overline{R}(U,V)W,N\rangle = \left\langle \left(\overline{\nabla}_V A\right) U - \left(\overline{\nabla}_U A\right) V, W\right\rangle,$$

where $U,V,W \in \varkappa(M^n)$. In particular if the ambient space has constant sectional curvature, then we have

$$\left(\overline{\nabla}_V A\right) U = \left(\overline{\nabla}_U A\right) V.$$

It is well known that A is a linear self adjoint operator and at each point $p\in M^n$, its eigenvalues $\mu_1,...,\mu_n$ are the principal curvatures of M^n.

The weighted elementary symmetric polynomial $\sigma_k^\infty : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ are defined recursively by (see [11])

$$\begin{cases} \sigma_0^\infty(u,\mu) = 1, \\ k\sigma_k^\infty(u,\mu) = u\sigma_{k-1}^\infty(u,\mu) + \sum\limits_{j=0}^{k-1}\sum\limits_{i=1}^{n} (-1)^j \sigma_{k-1-j}^\infty(u,\mu)\mu_i^{j+1} \quad \text{for } 1\le k\le n, \\ \sigma_k^\infty(u,\mu) = 0 \quad \text{for } k>n, \end{cases} \tag{2.1}$$

where $u\in\mathbb{R}$ and $\mu = (\mu_1,...,\mu_n)\in\mathbb{R}^n$. The weighted elementary symmetric functions $\sigma_k^\infty(u,A)$ of A are defined by

$$\sigma_k^\infty(u,A) = \sigma_k^\infty(u,\mu),$$

where $\mu_1,...,\mu_n$ are the eigenvalues of A.

In particular, for $u=0$, $\sigma_k^\infty(0,A) = \sigma_k(A)$ is the classical elementary symmetric functions defined in [19].

Associate to A, we can define the weighted Newton transformations $T_k^\infty(\mu_0,A)$ by

$$\begin{cases} T_0^\infty(u,A) = I, \\ T_k^\infty(u,A) = \sigma_k^\infty(u,A)I - AT_{k-1}^\infty(u,A) \quad \text{for } k\ge 1. \end{cases}$$

This is equivalent to

$$T_k^\infty(u,A) = \sum_{i=0}^{k} (-1)^i \sigma_{k-i}^\infty(u,A)A^i. \tag{2.2}$$

If $u=0$, then $T_k^\infty(0,A) = T_k(A)$ is just the classical Newton transformations introduced and studied by R. C. Reilly [19].

Since A is self-adjoint operator, then T_k^∞ are self-adjoint as well and their eigenvectors are the same as those of A, and we have the following properties, whose proof can be found in [11].

Proposition 2.1. *For any reals $u_1,u_2\in\mathbb{R}$ and $\mu\in\mathbb{R}^n$, we have*

$$\sigma_k^\infty(u_1+u_2,\mu) = \sum_{i=0}^{k}\frac{u_1^i}{i!}\sigma_{k-i}^\infty(u_2,\mu).$$

In particular

$$\sigma_k^\infty(u_1,\mu) = \sum_{i=0}^{k}\frac{u_1^i}{i!}\sigma_{k-i}(\mu),$$

and

$$trace(AT_k^\infty(u,\mu)) = (k+1)\sigma_{k+1}^\infty(u,\mu) - u\sigma_k^\infty(u,\mu).$$

The i^{th} eigenvalue of $T_k^\infty(u,\mu)$ is equal to $\sigma_{k,i}^\infty(u,\mu)$, where

$$\sigma_{k,i}^\infty(u,\mu) = \sigma_k^\infty(u,\widehat{\mu}_i) = \sigma_k^\infty(u,\mu) - \mu_i\sigma_{k-1,i}^\infty(u,\mu),$$

where $\mu = (\mu_1,...,\mu_n)$, and $\mu_1,...,\mu_n$ are the eigenvalues of A.

Definition 2.2. Let M be an immersed hypersurface of $\overline{M}_f$ with shape operator A with relation to an unit normal vector field N. The weighted k-curvature σ_k^∞ of the hypersurface M is defined by,

$$\sigma_k^\infty = \sigma_k^\infty(u, A) \quad \text{for} \quad u = \langle \overline{\nabla} f, N \rangle.$$

The weighted Newton transformations will be denoted by,

$$T_k^\infty = T_k^\infty(u, A) \quad for \quad u = \langle \overline{\nabla} f, N \rangle.$$

And we have

$$T_k^\infty = \begin{cases} I, & \text{for } k = 0, \\ \sigma_k^\infty I - A T_{k-1}^\infty, & \text{for } \quad k \geq 1. \end{cases}$$

In particular, for $k = 1$, we get

$$\sigma_1^\infty = H_f = \sigma_1 + \langle \overline{\nabla} f, N \rangle,$$

which is the classical definition of the weighted mean curvature of the hypersurface M studied by Gromov [14].

The divergence of the weighted Newton transformations is define by

$$div\left(T_k^\infty\right) = \text{trace}\left(\nabla T_k^\infty\right) = \sum_{i=1}^{n} \left(\nabla_{e_i} T_k^\infty\right)(e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M^n.

Before to start our computations, we need the following results

Proposition 2.3. *1. For* $0 \leq k \leq n$, *we have*

$$T_k^\infty = \sum_{j=0}^{k} \frac{u^j}{j!} T_{k-j}, \quad \textit{where } u = \langle \overline{\nabla} f, N \rangle. \tag{2.3}$$

2. The divergence of the weighted Newton transformations is given by

$$divT_0^\infty = 0,$$

and for $1 \leq k \leq n$

$$divT_k^\infty = T_{k-1}^\infty \circ \nabla u + \sum_{j=1}^{k} \frac{u^j}{j!} divT_{k-j},$$

where $divT_{k-j}$ *is given by* (3.3) *in [4].*

In particular, if $\overline{M}^{n+1}$ has constant sectional curvature, then we have

$$divT_k^\infty = \sum_{j=0}^{k-1} (-1)^j \sigma_{k-1-j}^\infty A^j \circ \nabla u. \tag{2.4}$$

Proof. 1. The relation (2.3) can be deduced directly from the definition of T_k^∞.

In fact we have for $0 \leq k \leq n$

$$\begin{aligned} T_k^\infty &= \sum_{i=0}^{k} (-1)^i \sigma_{k-i}^\infty A^i, \\ &= \sum_{i=0}^{k} \sum_{l=0}^{k-j} (-1)^j \frac{u^l}{l!} \sigma_{k-j-l}^\infty A^j, \\ &= \sum_{l=0}^{k} \frac{u^l}{l!} T_{k-l}. \end{aligned}$$

Where the first equality combines equation (2.3) with the definition of T_k^{∞}, the second equality switches the order of the summation, and the third equality uses the definition of the (unweighted) Newton transformations.

2. For $0 \leq k \leq n$, we have

$$\begin{aligned} divT_k^{\infty} &= tr\left(\nabla T_k^{\infty}\right), \\ &= \sum_{i=1}^{n} \overline{\nabla}_{e_i}\left(T_k^{\infty}\right)(e_i), \\ &= \sum_{i=1}^{n} \overline{\nabla}_{e_i}\left(\sum_{j=0}^{k}\left(\frac{u^j}{j!}T_{k-j}\right)\right)(e_i), \\ &= \sum_{i=1}^{n} \overline{\nabla}_{e_i}\left(\sum_{j=0}^{k}\left(\frac{u^j}{j!}T_{k-j}\right)\right)(e_i), \\ &= \sum_{i=1}^{n} \overline{\nabla}_{e_i}\left(\sum_{j=0}^{k}\left(\frac{u^j}{j!}T_{k-j}\right)\right)(e_i), \\ &= \sum_{i=1}^{n}\sum_{j=0}^{k} \frac{1}{j!}\nabla_{e_i}\left(u^j T_{k-j}\right)(e_i), \\ &= divT_k + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{1}{j!}\nabla_{e_i}\left(u^j T_{k-j}\right)(e_i). \end{aligned}$$

And we have

$$\begin{aligned} \nabla_{e_i}\left(\left(u^j T_{k-j}\right)\right)(e_i) &= \nabla_{e_i}\left(\left(u^j T_{k-j}\right)e_i\right) - u^j T_{k-j}\left(\nabla_{e_i}e_i\right), \\ &= \left\langle \nabla u^j, e_i\right\rangle T_{k-j}(e_i) + u^j \nabla_{e_i}\left(\left(T_{k-j}\right)e_i\right) - u^j T_{k-j}\left(\nabla_{e_i}e_i\right), \\ &= ju^{j-1}\left\langle \nabla u, e_i\right\rangle T_{k-j}(e_i) + u^j\left[\nabla_{e_i}\left(T_{k-j}(e_i)\right) - T_{k-j}\left(\nabla_{e_i}e_i\right)\right], \\ &= ju^{j-1}\left\langle \nabla u, e_i\right\rangle T_{k-j}(e_i) + u^j\left[\nabla_{e_i}\left(T_{k-j}(e_i)\right) - T_{k-j}\left(\nabla_{e_i}e_i\right)\right], \\ &= ju^{j-1}\left\langle \nabla u, e_i\right\rangle T_{k-j}(e_i) + u^j \nabla_{e_i}\left(T_{k-j}\right)(e_i). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{1}{j!}ju^{j-1}\left\langle \nabla u, e_i\right\rangle T_{k-j}(e_i) &= \sum_{i=1}^{n}\left\langle \nabla u, e_i\right\rangle \sum_{j=1}^{k}\frac{u^{j-1}}{(j-1)!}T_{k-j}(e_i), \\ &= \sum_{i=1}^{n}\left\langle \nabla u, e_i\right\rangle \sum_{l=0}^{k-1}\frac{u^{l}}{l!}T_{k-l-1}(e_i), \\ &= \sum_{i=1}^{n}\left(T_{k-1}^{\infty}\right)\left(\left\langle \nabla u, e_i\right\rangle e_i\right), \\ &= T_{k-1}^{\infty} \circ \nabla u. \end{aligned}$$

And

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{1}{j!}u^j\nabla_{e_i}\left(T_{k-j}\right)(e_i) = \sum_{j=1}^{k}\frac{u^j}{j!}divT_{k-j}.$$

Summarize all these relations we obtain the desire formula.

In particular, if $\overline{M}^{n+1}$ has constant sectional curvature, then T_k are divergence-free, $divT_{k-j} = 0$ (see [4]). The desire relation can be deduced by a recursive argument. □

3 Main results

In this section we derive some integral formulae on manifolds with density. In order to derive our formulae,we consider the following configuration.

Let $p \in M^n$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM^n that diagonalizes A. Then it diagonalizes also T_k^∞. Denoting by $\mu_1, ..., \mu_n$ the eigenvalues of A with respect to the basis $\{e_1, \ldots, e_n\}$.

We have

$$\left\langle divT_k^\infty, \overline{\nabla}_N N\right\rangle = div\left(T_k^\infty \overline{\nabla}_N N\right) - \sum_{i=1}^n \left\langle T_k^\infty(e_i), \overline{\nabla}_{e_i}\overline{\nabla}_N N\right\rangle.$$

This gives

$$\begin{aligned} div\left(T_k^\infty \overline{\nabla}_N N\right) &= \left\langle divT_k^\infty, \overline{\nabla}_N N\right\rangle + \sum_{i=1}^n \left\langle T_k^\infty(e_i), \overline{\nabla}_{e_i}\overline{\nabla}_N N\right\rangle, \qquad (3.1)\\ &= \left\langle divT_k^\infty, \overline{\nabla}_N N\right\rangle + \sum_{i=1}^n \sigma_{k,i}^\infty \left\langle e_i, \overline{\nabla}_{e_i}\overline{\nabla}_N N\right\rangle. \end{aligned}$$

Where $\sigma_{k,i}^\infty$ is the eigenvalue of T_k^∞ with respect to e_i.

On the other hand

$$-\left\langle e_i, \overline{\nabla}_{e_i}\overline{\nabla}_N N\right\rangle = \left\langle \overline{\nabla}_{e_i} e_i, \overline{\nabla}_N N\right\rangle + \left\langle \overline{\nabla}_{e_i} N, \overline{\nabla}_N e_i\right\rangle + \left\langle N, \overline{\nabla}_{e_i}\overline{\nabla}_N e_i\right\rangle.$$

Using now the Codazzi equation, we have

$$\begin{aligned} \left\langle e_i, \overline{\nabla}_{e_i}\overline{\nabla}_N N\right\rangle &= \left\langle \overline{\nabla}_N\left(\overline{\nabla}_{e_i} N\right), e_i\right\rangle + \left\langle \overline{\nabla}_{\overline{\nabla}_{e_i} N} N, e_i\right\rangle - \left\langle \overline{\nabla}_{\overline{\nabla}_N e_i} N, e_i\right\rangle - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= N\left\langle \overline{\nabla}_{e_i} N, e_i\right\rangle - \left\langle \overline{\nabla}_{e_i} N, \overline{\nabla}_N e_i\right\rangle + \left\langle A^2 e_i, e_i\right\rangle + \left\langle A\left(\overline{\nabla}_{e_i} N\right), e_i\right\rangle\\ &\quad - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= -\left\langle \overline{\nabla}\mu_i, N\right\rangle + \left\langle A^2 e_i, e_i\right\rangle - \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle + \left\langle \overline{\nabla}_N\left(A e_i\right), e_i\right\rangle\\ &\quad - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= -\left\langle \overline{\nabla}\mu_i, N\right\rangle + \left\langle A^2 e_i, e_i\right\rangle - \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle + \left\langle \overline{\nabla}_N\left(\mu_i e_i\right), e_i\right\rangle\\ &\quad - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= -\left\langle \overline{\nabla}\mu_i, N\right\rangle + \left\langle A^2 e_i, e_i\right\rangle - \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle\\ &\quad + \left(\mu_i\left\langle \overline{\nabla}_N e_i, e_i\right\rangle + \left\langle \left(\overline{\nabla}_N \mu_i\right) e_i, e_i\right\rangle\right) - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= -\left\langle \overline{\nabla}\mu_i, N\right\rangle + \left\langle A^2 e_i, e_i\right\rangle - \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle + \mu_i\left\langle \overline{\nabla}_N e_i, e_i\right\rangle + \left\langle \overline{\nabla}\mu_i, N\right\rangle\\ &\quad - \left\langle R(e_i, N)e_i, N\right\rangle,\\ &= \left\langle A^2 e_i, e_i\right\rangle - \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle - \left\langle R(e_i, N)e_i, N\right\rangle. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i=1}^n \sigma_{k,i}^\infty \left\langle A^2 e_i, e_i\right\rangle &= tr(A^2 T_k^\infty),\\ &= \sigma_{k+1}^\infty trA - tr(AT_{k+1}^\infty),\\ &= \sigma_{k+1}^\infty\left(\sigma_1^\infty - \left\langle \overline{\nabla} f, N\right\rangle\right) - (k+2)\,\sigma_{k+2}^\infty + \left\langle \overline{\nabla} f, N\right\rangle \sigma_{k+1}^\infty,\\ &= \sigma_1^\infty \sigma_{k+1}^\infty - (k+2)\,\sigma_{k+2}^\infty, \end{aligned}$$

and [2]

$$\sum_{i=1}^n \sigma_{k,i}^\infty \left\langle \left(\overline{\nabla}_N A\right) e_i, e_i\right\rangle = tr\left(T_k^\infty \circ \overline{\nabla}_N A\right) = N(\sigma_{k+1}^\infty) - N(\left\langle \overline{\nabla} f, N\right\rangle)\sigma_k^\infty.$$

It is not difficult to see that

$$\begin{aligned} div(\sigma_{k+1}^{\infty}N) &= N(\sigma_{k+1}^{\infty}) + \sigma_{k+1}^{\infty}.divN, \\ &= N(\sigma_{k+1}^{\infty}) - \sigma_1^{\infty}\sigma_{k+1}^{\infty} + \langle \overline{\nabla} f, N\rangle \sigma_{k+1}^{\infty}. \end{aligned}$$

Finally

$$\sum_{i=1}^{n} \sigma_{k,i}^{\infty} \langle (\overline{\nabla}_N A)\, e_i, e_i\rangle = div(\sigma_{k+1}^{\infty}N) + \sigma_1^{\infty}\sigma_{k+1}^{\infty} - \langle \overline{\nabla} f, N\rangle \sigma_{k+1}^{\infty} - N(\langle \overline{\nabla} f, N\rangle)\sigma_k^{\infty}.$$

Replacing in (3.6) we get

$$\begin{aligned} div\left(T_k^{\infty}\overline{\nabla}_N N + \sigma_{k+1}^{\infty}N\right) &= \langle divT_k^{\infty}, \overline{\nabla}_N N\rangle - (k+2)\,\sigma_{k+2}^{\infty} + \langle \overline{\nabla} f, N\rangle \sigma_{k+1}^{\infty} + N(\langle \overline{\nabla} f, N\rangle)\sigma_k^{\infty} \\ &\quad + tr(R(N)T_k^{\infty}. \end{aligned}$$

Integrating the two sides of the above relation and applying the divergence theorem, we obtain

Theorem 3.1. *Let $\psi : M^n \longrightarrow \overline{M}_f^{n+1}$ be a closed oriented hypersurface of a weighted manifold $\overline{M}_f^{n+1}$. Denoting by N a unit vector field normal to M^n in $\overline{M}_f^{n+1}$. Then for every $0 \leq k \leq n-2$, we have :*

$$\int_{M^n} \left[\langle divT_k^{\infty}, \overline{\nabla}_N N\rangle - (k+2)\,\sigma_{k+2}^{\infty} + \langle \overline{\nabla} f, N\rangle \sigma_{k+1}^{\infty} + N(\langle \overline{\nabla} f, N\rangle)\sigma_k^{\infty} + tr(R(N)T_k^{\infty}\right] dv_f = 0. \tag{3.2}$$

Taking $k = 0$, we have

Lemma 3.2. *Under the hypothesis of the above theorem, we have*

$$\int_{M^n} 2\sigma_2^{\infty} dv_f = \int_{M^n} \langle \overline{\nabla} f, N\rangle \sigma_1^{\infty} dv_f + \int_{M^n} \langle \overline{\nabla} f, \nabla_N N\rangle dv_f + \int_{M^n} Ric_f(N,N) dv_f. \tag{3.3}$$

Where Ric_f is the Bakry–Émery–Ricci tensor define in []equation $Ric_f = Ric + Hessf$.*

In particular if $\overline{M}^{n+1}$ is non weighted, then (3.3) reduced to the well known formula

$$\int_{M^n} 2\sigma_2 dv = {}_M Ric(N,N) dv.$$

Let consider the case where $\overline{M}^{n+1}$ is a non weighted Riemannian manifold, then we have

Theorem 3.3. *Let $\psi : M^n \longrightarrow \overline{M}^{n+1}$ be a closed oriented hypersurface of $\overline{M}^{n+1}$. Denoting by N a unit vector field normal to M^n in $\overline{M}^{n+1}$. Then for every $0 \leq k \leq n-2$, we have :*

$$\int_{M^n} \left[\langle divT_k, \overline{\nabla}_N N\rangle - (k+2)\,\sigma_{k+2} + tr(R(N)T_k\right] dv = 0 \tag{3.4}$$

In this case if $\overline{M}^{n+1}$ has constant sectional curvature c, then T_k are divergence-free and

$$tr(R(N)T_k = c.(n-k)\sigma_k.$$

Hence

$$(k+2)\int_{M^n} \sigma_{k+2} dv = c.(n-k)\int_{M^n} \sigma_k dv.$$

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