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On meromorphic solutions of $f^3(z) + f^3(qz + c) = h(z)$

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Abstract In this paper, we investigate the existence of meromorphic solutions of hyper-order strictly less than 1 to the Fermat-type q-shift equations $f^3(z) + f^3(qz + c) = h(z)$ over the complex plane $\mathbb{C}$, where $h(z) = a(z), e^{P(z)}$ for a small function $a(z)$ of f with two Borel exceptional values 0 and ∞, and for a polynomial $P(z)$. We have exhibited some examples for showing the accuracy of the results.

1 Introduction, Definitions and Results

By a meromorphic (resp. entire) function, we shall always mean meromorphic (resp. entire) function over the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results such as proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory (see e.g. [9, 13, 16]). A meromorphic function α is said to be a small function of f, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure. We denote the order and the hyper-order of a meromorphic function f respectively by $\rho(f)$ and $\rho_2(f)$ such that

$$\rho(f) = \limsup_{r\longrightarrow\infty} \frac{\log^+ T(r, f)}{\log r} \text{ and } \rho_2(f) = \limsup_{r\longrightarrow\infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

Next definitions are necessary in this paper.

Definition A. Let f be a transcendental meromorphic function. A complex number a is said to be a Borel exceptional value if $\limsup_{r\longrightarrow\infty} \frac{\log^+ N(r,a,f)}{\log r} < \rho(f)$.

Let f be of finite positive order ρ and has two Borel exceptional values 0 and ∞. If $\omega(z) = z^k \frac{P_0(z)}{P_\infty(z)}$ and $g(z) = \frac{f(z)}{\omega(z)}$, where $P_0(z)$ and $P_\infty(z)$ are respectively the canonical products of $f(z)$ formed with the zeros and poles of $f(z)$ in $\mathbb{C} \setminus \{0\}$, then we see that

$$\rho(P_0) = \rho_1(f) \le \limsup_{r\longrightarrow\infty} \frac{\log^+ N(r, 0, f)}{\log r} < \rho(f)$$

and

$$\rho(P_\infty) = \rho_1\left(\frac{1}{f}\right) \le \limsup_{r\longrightarrow\infty} \frac{\log^+ N(r, \infty, f)}{\log r} < \rho(f),$$

where we denote by $\rho_1(f)$ the exponent of convergence of zeros of f. And thus we have

$$\rho(\omega) \le \max\{\rho(z^k), \rho(P_0), \rho(P_\infty)\} = \max\{\rho(P_0), \rho(P_\infty)\} < \rho(f) = \rho(g).$$

Also for two distinct Borel exceptional values a_1 and a_2, one can consider the function $\frac{f-a_1}{f-a_2}$, and obtain the same result as previous.

Definition B. Given a meromorphic function $f(z)$, $f(z+c)$ (resp. $f(qz+c)$) is called a shift (resp. q-shift) of f, where $c, q \in \mathbb{C} \setminus \{0\}$. Also for given a meromorphic function $f(z)$, $f(qz)$ is called a q-difference of f, where $q \in \mathbb{C} \setminus \{0\}$.

Given three meromorphic functions $f(z)$, $g(z)$ and $h(z)$, $f^n(z) + g^n(z) = h^n(z)$ is called a Fermat-type functional equation on $\mathbb{C}$, where $n \in \mathbb{N}$. Actually the functional equation is due to the assertion in Fermat's Last Theorem in 1637 for the solutions of the Diophantine equation $x^n + y^n = z^n$ over some function fields, where $n \in \mathbb{N}$. For uniqueness related study see [11, 14].

We now consider the Fermat-type functional equation

$$f^3(z) + g^3(z) = 1. \tag{1.1}$$

Gross [7] and Baker [1] showed that the non-constant meromorphic solutions of (1.1) is as

$$f(z) = \frac{1}{2}\left\{1 + \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\} \Big/ \mathscr{P}(h(z)) \text{ and } g(z) = \frac{\eta}{2}\left\{1 - \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\} \Big/ \mathscr{P}(h(z)), \tag{1.2}$$

where $\eta^3 = 1$, $h(z)$ is a non-constant entire function and $\mathscr{P}(z)$ denotes the Weierstrass $\mathscr{P}$-function with periods ω_1 and ω_2 defined as

$$\mathscr{P}(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{\mu,\nu;\mu^2+\nu^2\neq 0} \left\{\frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2}\right\},$$

which is even and satisfies, after appropriately choosing ω_1 and ω_2,

$$(\mathscr{P}')^2 = 4\mathscr{P}^3 - 1. \tag{1.3}$$

Also Bank and Langley [2] indicates that

$$T(r, \mathscr{P}) = \frac{\pi}{A} r^2 (1 + o(1)) \text{ and } \rho(\mathscr{P}) = 2, \tag{1.4}$$

where A is the area of the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$.

In [4] authors give some properties of the Weierstrass $\mathscr{P}$-function as follows

$$N(r, \mathscr{P}) = \frac{\pi}{k}(1 + o(1))r^2, \quad N\left(r, \frac{1}{\mathscr{P} - a}\right) = \frac{\pi}{k}(1 + o(1))r^2, \tag{1.5}$$

$$m(r, \mathscr{P}) = O(\log r), \quad T(r, \mathscr{P}) = m(r, \mathscr{P}) + N(r, \mathscr{P}) = \frac{\pi}{k}(1 + o(1))r^2, \tag{1.6}$$

$$N(r, \mathscr{P}') = \frac{3\pi}{2k}(1 + o(1))r^2, \quad N\left(r, \frac{1}{\mathscr{P}' - a}\right) = \frac{3\pi}{2k}(1 + o(1))r^2, \tag{1.7}$$

$$\begin{aligned} \overline{N}(r, \mathscr{P}) &= \frac{1}{2}N(r, \mathscr{P}) = \frac{\pi}{2k}(1 + o(1))r^2, \\ \overline{N}\left(r, \frac{1}{\mathscr{P}}\right) &= N\left(r, \frac{1}{\mathscr{P}}\right) = \frac{\pi}{k}(1 + o(1))r^2, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \overline{N}(r, \mathscr{P}') &= \frac{1}{3}N(r, \mathscr{P}') = \frac{\pi}{2k}(1 + o(1))r^2, \\ \overline{N}\left(r, \frac{1}{\mathscr{P}' - a}\right) &= N\left(r, \frac{1}{\mathscr{P}' - a}\right) = \frac{3\pi}{2k}(1 + o(1))r^2. \end{aligned} \tag{1.9}$$

In the same paper Bi and Lü [4] obtained the following two theorems:

Theorem A. [4] Let f be a non-constant meromorphic function of hyper-order strictly less than 1, and let a be a small meromorphic function of f with two Borel exceptional values 0 and ∞. Then f does not satisfy the following equation

$$f^3(z)+f^3(z+c)=a(z). \tag{1.10}$$

Theorem B. [4] Let P be a polynomial. Then the functional equation

$$f^3(z)+f^3(z+c)=e^{P(z)} \tag{1.11}$$

does not admit meromorphic solutions $f(\neq de^{\frac{\alpha z+\beta}{3}}$ with $d^3(1+e^{\alpha c})=1$, where $d(\neq 0), c(\neq 0), \alpha, \beta \in \mathbb{C})$ of hyper-order strictly less than 1.

Question 1. What happens if the function $f(z+c)$ in Theorem A and Theorem B is replaced by the q-shift function $f(qz+c)$, where $q(\neq 1), c \in \mathbb{C}\setminus\{0\}$?

In the above connection, for the difference equation, in 2017, Lü and Han [15] showed that the Fermat-type equation $f^3(z)+f^3(z+c)=1$ has no non-constant meromorphic solution of finite order on $\mathbb{C}$. In 2019, Han and Lü [8] showed that there is no meromorphic solution of finite order for the difference equation $f^3(z)+f^3(z+c)=e^{\alpha z+\beta}$ on $\mathbb{C}$, where $\alpha, \beta \in \mathbb{C}$.

2 Main results

For affirmative answer of Question 1, we obtain the following results.

Theorem 2.1. Let f be a non-constant meromorphic function of hyper-order strictly less than 1, and let a be a small function of f with two Borel exceptional values 0 and ∞, satisfying

$$f^3(z)+f^3(qz+c)=a(z), \tag{2.1}$$

where $q(\neq 1), c \in \mathbb{C}\setminus\{0\}$. Then $a(z)=\omega(z)e^{h(z)}$, where $h(z)$ is a polynomial with $\deg h(z)=n$, $\rho(\omega)<n$ and $q^n=1$.

Theorem 2.2. Let f be a non-constant meromorphic solution of

$$f^3(z)+f^3(qz+c)=e^{P(z)}, \tag{2.2}$$

with hyper-order strictly less than 1, where $q(\neq 1), c \in \mathbb{C}\setminus\{0\}$ and $P(z)$ is a polynomial with $\deg P(z)=m$. Then (2.2) has no solution for $q^m \neq 1$.

2.1 Some examples

Following examples are showing the sharpness of Theorems 2.1 and 2.2.

Example 2.1. Let $h(z)=z+1$, $q=-1$ and $c=-2$. Then $f(z)=\frac{1}{2\mathscr{P}(h(z))}\left\{1+\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\} e^{\frac{P(z)}{3}}$ is a meromorphic solution of finite order satisfying (2.2), where $P(z)=h^2(z)$. Clearly $q^{\deg P}=1$ and $\rho(\mathscr{P}(h(z)))=\deg P(z)$.

Example 2.2. Let $h(z)=(z+i)^7$, $q=-1$ and $c=-2i$. Then $f(z)$ is a meromorphic solution of finite order satisfying (2.2), where $P(z)=(z+i)^2$. Also $q^{\deg P}=1$ and $\rho(\mathscr{P}(h(z)))>\deg P(z)$.

Example 2.3. Let $h(z)=a_2z^2+a_1z+\frac{a_1^2}{4a_2}$, $q=i$, $a_1, a_2(\neq 0)\in\mathbb{C}$ and $c=\frac{a_1(1-i)}{-2a_2}$. Then $f(z)$ is a meromorphic solution of finite order satisfying (2.2), where $P(z)=h^{10}(z)$. Note that $q^{\deg P}=1$ and $\rho(\mathscr{P}(h(z)))<\deg P(z)$.

An infinite order solution of (2.2) has given below.

Example 2.4. Let $h(z)=\sin z$, $P(z)=(z-\pi)^2$ and $c=2\pi$ with $q=-1$, then $f(z)=\frac{1}{2\mathscr{P}(h(z))}\left\{1+\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\} e^{\frac{P(z)}{3}}$ is an infinite order meromorphic solution satisfying (2.2).

2.2 The technical lemmas

The following lemmas are necessary in this paper.

Lemma 2.5. *[10] Let f be a meromorphic function, and let $P(z) = a_m z^m + \cdots + a_1 z + a_0$ be a complex polynomial of degree $m \geq 1$. For a given $\delta \in (0, |a_m|)$, let $\mu = |a_m| - \delta$ and $\lambda = |a_m| + \delta$, then for $a \in \mathbb{C}$,*

$$\overline{N}\left(\mu r^m, \frac{1}{f-a}\right) + O(\log r) \leq \overline{N}\left(r, \frac{1}{f \circ P - a}\right) \leq \overline{N}\left(\lambda r^m, \frac{1}{f-a}\right) + O(\log r).$$

Lemma 2.6. *[12] Let f be a non-rational meromorphic function also let $\omega(z) = cz^n + p_{n-1}z^{n-1} + \cdots + p_0$ and $\phi(z) = cz^n + q_{n-1}z^{n-1} + \cdots + q_0$ be non-constant polynomials, where $c(\neq 0), p_i, q_j \in \mathbb{C}$ for $0 \leq i, j \leq n-1$. If*

$$\limsup_{r \longrightarrow \infty} \frac{\log\log T(r,f)}{\log r} < \frac{1}{n^2},$$

then

$$m\left(r, \frac{f \circ \omega}{f \circ \phi}\right) = o\left(T(|c|r^n, f)\right)$$

for all r outside of an exceptional set of finite logarithmic measure.

Lemma 2.7. *[3, 5] Let f be meromorphic and h be entire in $\mathbb{C}$. If $0 < \rho(f), \rho(h) < \infty$, then $\rho(f \circ h) = \infty$. If $\rho(f \circ h) < \infty$ and h is transcendental, then $\rho(f) = 0$.*

Lemma 2.8. *[6] Let F, f be two transcendental meromorphic functions, and g be a polynomial of degree m such that $F = f \circ g$. Then $\rho(F) = m\rho(f)$.*

Remark A. The conclusion of Lemma 2.5 still holds when $a = \infty$, one can check it by replacing g with $\frac{1}{f}$ in Lemma 2.5.

3 Proof of the theorems

Proof of Theorem 2.1. From (2.1), we have $f^3(qz+c) = -\left(f^3(z) - a(z)\right)$, which implies that the zeros of $f^3(z) - a(z)$ are of multiplicities at least 3. Similarly the zeros of $f^3(qz+c) - a(z)$ are of multiplicities at least 3. Thus the zeros of $f^3(z) - a\left(\frac{z-c}{q}\right)$ are of multiplicities at least 3. Set $G(z) = f^3(z)$. Let the functions $a(z)$ and $a\left(\frac{z-c}{q}\right)$ be distinct. Here a is non-constant, since a has Borel exceptional value. Then applying the second main theorem of Nevanlinna to G, one gets that

$$\begin{aligned}
2T(r,G) \quad &\leq \quad \overline{N}\left(r, \frac{1}{G(z) - a\left(\frac{z-c}{q}\right)}\right) + \overline{N}\left(r, \frac{1}{G(z) - a(z)}\right) \\
&\quad + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + O(\log rT(r,G)) \\
&\leq \quad \frac{4}{3}T(r,G) + O(\log rT(r,G)),
\end{aligned}$$

which implies G is a rational function, a contradiction. Thus we conclude that

$$a(z) = a\left(\frac{z-c}{q}\right) \Rightarrow a(qz+c) = a(z).$$

Since a has two Borel exceptional values 0 and ∞, then by definition of Borel exceptional value and Hadamard factorization theorem, one has

$$a(z) = \omega(z)e^{h(z)},$$

where $\omega(z)$ is a meromorphic function of finite order such that $\rho(\omega) < \rho(a) = \rho(e^h)$ and h is an entire function satisfying $\rho_2(a) = \rho(h) < 1$. Then the fact $a(z) = a(qz+c)$ yields that

$$\omega(z)e^{h(z)} = \omega(qz+c)e^{h(qz+c)}$$
$$\Rightarrow \quad \frac{\omega(z)}{\omega(qz+c)} = e^{h(qz+c)-h(z)}, \tag{3.1}$$

which implies that $\rho\left(e^{h(qz+c)-h(z)}\right) = \rho\left(\frac{\omega(z)}{\omega(qz+c)}\right) < \infty$. Thus $h(qz+c) - h(z)$ reduces to a polynomial, say Q. Differentiating $h(qz+c) - h(z) = Q(z)$, $s(\geq \deg Q + 1)$ times, we get

$$q^s h^{(s)}(qz+c) - h^{(s)}(z) = 0. \tag{3.2}$$

Now if $a(z)$ is of finite order, then $h(z)$ must be a polynomial of degree n, say. Let $q^n \neq 1$. Then $\rho(e^{h(qz+c)-h(z)}) = n$, but $\rho\left(\frac{\omega(z)}{\omega(qz+c)}\right) < n$, which contradicts (3.1).
And if $a(z)$ is of infinite order, then $h(z)$ is a transcendental entire function with $\rho(h) < 1$. Clearly, $h^{(s)}(z)$ is also a transcendental entire function with $\rho(h^{(s)}) < 1$. Now if $h^{(s)}$ does not have any zero, then $h^{(s)}(z) = e^{H(z)}$, where $H(z)$ is an entire function. Further since $\rho(h^{(s)}) < 1$, we must have $H(z)$ as constant i.e. $h(z)$ as a polynomial. Here we consider following cases.
Case 1. Let $|q| > 1$. If z_0 is a zero of $h^{(s)}$, from (3.2) we have, $\frac{z_0 - c\sum_{k=0}^{m-1} q^k}{q^m}$, $m = 1, 2, \cdots$ as a sequence of zeros of $h^{(s)}$. Then $\frac{c}{1-q}$ would be an essential singularity of $h^{(s)}$, which is a contradiction.
Case 2. Let $|q| < 1$. If z_0 is a zero of $h^{(s)}$, from (3.2) we get, $q^m z_0 + c\sum_{k=0}^{m-1} q^k$, $m = 1, 2, \cdots$ is a sequence of zeros of $h^{(s)}$. Which implies that $\frac{c}{1-q}$ would be an essential singularity of $h^{(s)}$, again a contradiction.
Case 3. Let $|q| = 1$. Then there exists $m \in \mathbb{R}^+$ such that $q^m = 1$. Since $q \neq 1$, from (3.2), it is clear that if z_0 is a zero of $h^{(s)}$, then there are exactly $[m]$ distinct zeros of $h^{(s)}$. Those are $z_0, qz_0 + c, q^2 z_0 + qc + c, q^3 z_0 + q^2 c + qc + c, \cdots, q^{m-1} z_0 + c\sum_{k=0}^{m-2} q^k$. Therefore, h must be a polynomial of degree $k(= [m] + \deg Q + 1)$, which is not the case. This completes the proof. □

Proof of Theorem 2.2. From (2.2), we have

$$\left(\frac{f(z)}{e^{\frac{P(z)}{3}}}\right)^3 + \left(\frac{f(qz+c)}{e^{\frac{P(z)}{3}}}\right)^3 = 1.$$

Using (1.2), one has

$$f(z) = \frac{1}{2}\frac{\left\{1 + \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\}}{\mathscr{P}(h(z))} e^{\frac{P(z)}{3}} \tag{3.3}$$
$$\text{and} \qquad f(qz+c) = \frac{\eta}{2}\frac{\left\{1 - \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\}}{\mathscr{P}(h(z))} e^{\frac{P(z)}{3}},$$

where h is an entire function and $P(z)$ is a polynomial of degree m. From (3.3), we get

$$\frac{\eta\left\{1 - \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\}}{\mathscr{P}(h(z))} = \frac{\left\{1 + \frac{\mathscr{P}'(h(qz+c))}{\sqrt{3}}\right\}}{\mathscr{P}(h(qz+c))} e^{\frac{P(qz+c)-P(z)}{3}}. \tag{3.4}$$

Then, from (1.3) and the first equation of (3.3), one has

$$\frac{f^2(z)\mathscr{P}^2(h(z))}{e^{\frac{2P(z)}{3}}} = \frac{1}{4}\left\{1 + \frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\}^2$$
$$\Rightarrow \quad \frac{3f^2(z)\mathscr{P}^2(h(z))}{e^{\frac{2}{3}P(z)}} - \frac{3f(z)\mathscr{P}(h(z))}{e^{\frac{1}{3}P(z)}} + 1 = \mathscr{P}^3(h(z))$$
$$\Rightarrow \quad \frac{3f(z)\mathscr{P}(h(z))}{e^{\frac{1}{3}P(z)}}\left\{\frac{3f(z)\mathscr{P}(h(z))}{e^{\frac{1}{3}P(z)}} - 1\right\} + 1 = \mathscr{P}^3(h(z))$$
$$\Rightarrow \quad T(r, \mathscr{P}(h)) \leq 2T(r, f) + 2T(r, e^{\frac{P(z)}{3}}) + O(1). \tag{3.5}$$

The first equation of (3.3) also yields that

$$T(r,f) \le 3T(r,\mathscr{P}(h)) + T(r,e^{\frac{P(z)}{3}}) + O(\log rT(r,\mathscr{P}(h))). \tag{3.6}$$

Now if h is a transcendental entire function, by Lemma 2.7, (3.5) and (3.6), we have $\rho(f) = \rho(\mathscr{P}(h)) = \infty$ and since P is a polynomial, e^P is a small function of f. Note that $q^m \neq 1$, then we can get a contradiction by Theorem 2.1. Thus, h is a polynomial. We assume $h(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n(\neq 0), a_{n-1}, \cdots, a_0 \in \mathbb{C}$. Now we consider following cases.

Case 1. Suppose that $\deg P < \rho(\mathscr{P}(h))$. Then $\rho(e^P) = \deg P < \rho(\mathscr{P}(h)) = \rho(f)$. Thus, e^P is a small function of f. By Theorem 2.1, we get a contradiction.

Case 2. Let $\deg P > \rho(\mathscr{P}(h))$. Then there exists $\epsilon > 0$ such that $\rho(\mathscr{P}(h)) + \epsilon < \deg P - \epsilon$. We rewrite (3.4) as

$$\begin{aligned} e^{\frac{P(z)-P(qz+c)}{3}} &= \frac{\mathscr{P}(h(z))}{\mathscr{P}(h(qz+c))} \frac{1+\frac{\mathscr{P}'(h(qz+c))}{\sqrt{3}}}{\eta\left(1-\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right)} \\ &= \frac{1}{\eta}\frac{\mathscr{P}(h(z))}{\mathscr{P}(h(qz+c))}\left[\frac{2\sqrt{3}}{\sqrt{3}-\mathscr{P}'(h(z))} - \frac{\sqrt{3}-\mathscr{P}'(h(qz+c))}{\sqrt{3}-\mathscr{P}'(h(z))}\right], \end{aligned} \tag{3.7}$$

which implies,

$$\begin{aligned} m(r,e^{\frac{P(z)-P(qz+c)}{3}}) &\le m\left(r,\frac{1}{\eta}\frac{\mathscr{P}(h(z))}{\mathscr{P}(h(qz+c))}\right) + m\left(r,\frac{2\sqrt{3}}{\sqrt{3}-\mathscr{P}'(h(z))}\right) \\ &\quad + m\left(r,\frac{\sqrt{3}-\mathscr{P}'(h(qz+c))}{\sqrt{3}-\mathscr{P}'(h(z))}\right) + O(1) \\ &= m\left(r,\frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) + O(r^{\rho(\mathscr{P}(h))+\epsilon}) \\ &\le T(r,\mathscr{P}'(h(z))) + O(r^{\rho(\mathscr{P}(h))+\epsilon}) \\ &\le 2T(r,\mathscr{P}(h(z))) + O(r^{\rho(\mathscr{P}(h))+\epsilon}) \\ &= O(r^{\rho(\mathscr{P}(h))+\epsilon}) = O(r^{\deg P-\epsilon}). \end{aligned} \tag{3.8}$$

Since, $P(z)$ is a polynomial of degree m and $q^m \neq 1$, we have

$$m\left(r,e^{\frac{P(z)-P(qz+c)}{3}}\right) = A(1+o(1))r^{\deg P}, \tag{3.9}$$

where A is a positive number. Thus, from (3.8) and (3.9), we get a contradiction.

Case 3. Let $\deg P = \rho(\mathscr{P}(h))$. Then using Lemma 2.8, we have $\deg P = \rho(\mathscr{P}(h)) = 2\deg h = 2n = m$.

Let ϕ be an argument of a_n. Then by (1.6), one has

$$\begin{aligned} m(r,\mathscr{P}(a_n z^n)) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\mathscr{P}(a_n(re^{i\theta})^n)|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\mathscr{P}(|a_n|r^n e^{i\phi}e^{in\theta})|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\mathscr{P}(|a_n|r^n e^{i\theta})|d\theta \\ &= m(|a_n|r^n,\mathscr{P}(z)) = O(\log|a_n|r^n) = O(\log r). \end{aligned} \tag{3.10}$$

By Lemma 2.6 and (3.10), we have

$$\begin{aligned} m(r, \mathscr{P}(h(z))) &\leq m\left(r, \frac{\mathscr{P}(h(z))}{\mathscr{P}(a_n z^n)}\right) + m(r, \mathscr{P}(a_n z^n)) \\ &= o(T(|a_n| r^n, \mathscr{P}(z))) + O(\log r) \\ &\leq o\left\{\frac{\pi}{k}(1+o(1))(|a_n| r^n)^2\right\} + O(\log r) \\ &= o(r^{2n}) + O(\log r) \\ &= o(r^m). \end{aligned} \tag{3.11}$$

Similarly, one can show

$$m(r, \mathscr{P}(h(qz+c))) = o(r^m). \tag{3.12}$$

Now from (3.4), we have

$$\begin{aligned} & e^{\frac{P(z)-P(qz+c)}{3}} \\ = & \frac{\mathscr{P}(h(z))}{\eta\left(1-\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right)} \frac{1+\frac{\mathscr{P}'(h(qz+c))}{\sqrt{3}}}{\mathscr{P}(h(qz+c))} \\ = & \frac{\mathscr{P}(h(z))}{\eta\left(1-\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right)} \left[\frac{1}{\mathscr{P}(h(qz+c))} + \frac{\mathscr{P}'(h(qz+c))h'(qz+c)q}{\sqrt{3}\mathscr{P}(h(qz+c))h'(qz+c)q}\right]. \end{aligned} \tag{3.13}$$

Then using (3.11), (3.12) into (3.13), we get

$$\begin{aligned} & m\left(r, e^{\frac{P(z)-P(qz+c)}{3}}\right) \\ \leq & m(r, \mathscr{P}(h(z))) + m\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) \\ & + m\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) + S(r, \mathscr{P}(h(qz+c))) \\ = & m\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) + m\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) \\ & + S(r, \mathscr{P}(h(z))) + S(r, \mathscr{P}(h(qz+c))) \\ \leq & T\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) - N\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) \\ & + T\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) - N\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) \\ & + S(r, \mathscr{P}(h(z))) + S(r, \mathscr{P}(h(qz+c))) \\ \leq & T(r, \mathscr{P}'(h(z))) - N\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) \\ & + T(r, \mathscr{P}(h(qz+c))) - N\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) \\ & + S(r, \mathscr{P}(h(z))) + S(r, \mathscr{P}(h(qz+c))) \\ \leq & m(r, \mathscr{P}'(h(z))) + N(r, \mathscr{P}'(h(z))) - N\left(r, \frac{1}{\sqrt{3}-\mathscr{P}'(h(z))}\right) \\ & + m(r, \mathscr{P}(h(qz+c))) + N(r, \mathscr{P}(h(qz+c))) - N\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) + o(r^m). \end{aligned} \tag{3.14}$$

Further by Lemma 2.5 and Remark A, for $\delta \in (0, |a_n q^n|)$, $\mu = |a_n q^n| - \delta$ and $\lambda = |a_n q^n| + \delta$, we get

$$\begin{aligned} & N\left(r, \mathscr{P}(h(qz+c))\right) - N\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) \\ \leq\ & 2\overline{N}\left(r, \mathscr{P}(h(qz+c))\right) - \overline{N}\left(r, \frac{1}{\mathscr{P}(h(qz+c))}\right) + O(\log r) \\ \leq\ & 2\overline{N}\left(\lambda r^n, \mathscr{P}(h(z))\right) - \overline{N}\left(\mu r^n, \frac{1}{\mathscr{P}(h(z))}\right) + O(\log r) \\ \leq\ & \frac{\pi}{k}(1+o(1))(\lambda r^n)^2 - \frac{\pi}{k}(1+o(1))(\mu r^n)^2 + O(\log r) \\ \leq\ & \frac{\pi}{k}[\lambda^2 - \mu^2](1+o(1))r^m. \end{aligned} \tag{3.15}$$

And similarly for $\delta' \in (0, |a_n|)$, $\mu' = |a_n| - \delta'$ and $\lambda' = |a_n| + \delta'$, we have

$$\begin{aligned} & N\left(r, \mathscr{P}'(h(z))\right) - N\left(r, \frac{1}{\sqrt{3} - \mathscr{P}'(h(z))}\right) \\ \leq\ & 3\overline{N}\left(r, \mathscr{P}(h(qz+c))\right) - \overline{N}\left(r, \frac{1}{\mathscr{P}'(h(qz+c))}\right) + O(\log r) \\ \leq\ & \frac{3\pi}{2k}[\lambda'^2 - \mu'^2](1+o(1))r^m. \end{aligned} \tag{3.16}$$

We can choose positive δ, δ' so small such that

$$\begin{aligned} & \frac{\pi}{k}[\lambda^2 - \mu^2] = \frac{\pi}{k}[(|a_n q^n| + \delta)^2 - (|a_n q^n| - \delta)^2] = \frac{\pi}{k}4|a_n q^n|\delta < \frac{A}{3} \\ & and \\ & \frac{3\pi}{2k}[\lambda'^2 - \mu'^2] = \frac{3\pi}{2k}[(|a_n| + \delta')^2 - (|a_n| - \delta')^2] = \frac{3\pi}{2k}4|a_n|\delta' < \frac{A}{3}. \end{aligned} \tag{3.17}$$

By (3.9),(3.14), (3.15), (3.16) and (3.17), we get

$$\begin{aligned} A(1+o(1))r^m \ & \leq \ \frac{\pi}{k}[\lambda^2 - \mu^2](1+o(1))r^m + \frac{3\pi}{2k}[\lambda'^2 - \mu'^2](1+o(1))r^m \\ & < \ \frac{A}{3}(1+o(1))r^m + \frac{A}{3}(1+o(1))r^m = \frac{2}{3}A(1+o(1))r^m, \end{aligned}$$

which is a contradiction. This completes the proof. □

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