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On Complex Intuitionistic Fuzzy Lie Sub-superalgebras

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Abstract This research provides a foundation for the study of complex intuitionistic fuzzy lie superalgebras, which can be useful in various applications such as quantum physics, mathematical physics, and other areas of mathematics where Lie superalgebras are relevant, The introduction of complex intuitionistic fuzzy sets in Lie superalgebras allows for a more flexible and a nuanced understanding of the structures and properties of these algebraic systems. Furthermore, the investigation of complex intuitionistic fuzzy lie sub-superalgebras and complex Intuitionistic fuzzy ideals provide valuable insights into the substructures and ideal structures of Lie superalgebras in the context of complex intuitionistic fuzziness. The study of anti-complex intuitionistic fuzzy lie sub-superalgebras and anti-complex intuitionistic fuzzy ideals under anti-homomorphisms expand the understanding of how these structures behave under certain transformations, which can have implications for the study and application of Lie superalgebras in various mathematical contexts. Overall, this research contributes to the development of a more comprehensive and sophisticated understanding of complex intuitionistic fuzzy lie superalgebras, opening up new avenues for further exploration and application of these algebraic structures, in various mathematical disciplines.

1 Introduction

The notion of intuitionistic fuzzy sets was introduced by Atanassov (see [6]). He presented in [6] the idea of intuitionistic fuzzy sets. He also, in [7], defined some properties of intuitionistic fuzzy sets. Atanassov presented in [8] interesting new operations about intuitionistic fuzzy sets. An intuitionistic fuzzy set is the generalization of a fuzzy set. Recently, Biswas applied the concepts of intuitionistic fuzzy sets to the theory. of groups and studied intuitionistic fuzzy subgroups of a group (see [10]); also, Banerjee studied intuitionistic fuzzy subrings and ideals of a ring (see [9]). Moreover, Jun investigated the concept of intuitionistic nilradicals of intuitionistic fuzzy ideals in rings (see [14]), and Davvaz, Dudek, and Jun applied the notion of intuitionistic fuzzy sets to certain types of modules (see [12]). Then, in [11], W. Chen and S. Zhang introduced the concept of intuitionistic fuzzy Lie superalgebras and intuitionistic fuzzy ideals. It is known that fuzzy sets are intuitionistic fuzzy sets, but the converse is not necessarily true (for more details, see [7]). More recently, Alkouri and Salleh [2] introduced the idea of complex intuitionistic fuzzy subsets, and then they expanded the basic properties of them. This concept became more effective and useful in the scientific field because it dealt with the degree of membership and non-membership in a complex plane. They also initiated the concept of complex intuitionistic fuzzy relations and developed the fundamental operation of complex intuitionistic fuzzy sets in [3, 4]. Then Garg and Rani made a huge effort to generalize the notion of complex intuitionistic fuzzy sets in decision-making problems (see [13]). In [20], S. Shaqaqha introduced the concepts of complex fuzzy sets to the theory of Lie algebras and studied complex fuzzy Lie subalgebras. Furthermore, in [1, 23], S. Shaqaqha and M. Al-Deiakeh introduced the concepts of complex intuitionistic fuzzy Lie algebras and complex intuitionistic fuzzy Lie ideals, and they studied the relation between complex intuitionistic fuzzy Lie subalgebras (ideals) and intuitionistic fuzzy Lie subalgebras (ideals). Also, in [19], S. Shaqaqha characterized the Noetherian and Artinian Gamma rings by complex, fuzzy ideals. Moreover, in [22], he introduced the notion of intuitionistic fuzzy Lie subalgebras and intuitionistic fuzzy Lie ideals of n-Lie algebras, which is a generalization of intuitionistic fuzzy Lie algebras. More recently, in [21], he introduced the concept of complex fuzzy G-rings and showed that there are isomorphism theorems concerning complex fuzzy G-rings as well as rings. I must point out here that the main idea of this article is to introduce the concepts of complex intuitionistic fuzzy Lie superalgebras and complex intuitionistic fuzzy ideals, which are generalizations of intuitionistic fuzzy Lie superalgebras and intuitionistic fuzzy ideals applied by W. Chen and S. Zhang in [11]. We prepared this paper as follows: In Section 2, we recall some basic definitions and notions that will be used in what follows. In Section 3, we introduce the definition of a $\mathbb{Z}_2$-graded complex intuitionistic fuzzy vector subspace, define complex intuitionistic fuzzy Lie sub-superalgebras, complex intuitionistic fuzzy ideals, and consider their characterization. Finally, in Section 4, we discuss the images and preimages of complex intuitionistic fuzzy Lie sub-superalgebras and complex intuitionistic fuzzy ideals under anti-homomorphisms.

2 Complex intuitionistic fuzzy sets

Let $X \neq \phi$. A complex intuitionistic fuzzy set on X is an object having the form $A = \{(x, \lambda_A(x), \rho_A(x)) \,|x \in X\}$, where the complex functions $\lambda_A : X \to \mathbb{C}$ and $\rho_A : X \to \mathbb{C}$ denote the degree of membership (namely $\lambda_A(x)$) and the degree of non-membership (namely $\rho_A(x)$) of each element $x \in X$ to the set A, respectively, that assigns to any element $x \in X$ complex numbers $\lambda_A(x)$, $\rho_A(x)$ lie within the unit circle with the property $|\lambda_A(x)| + |\rho_A(x)| \leq 1$. For the sake of simplicity, we shall use the symbol $A = (\lambda_A, \rho_A)$ for the complex intuitionistic fuzzy set $A = \{(x, \lambda_A(x), \rho_A(x)) \,|x \in X\}$.
We shall assume $\lambda_A(x)$ and $\rho_A(x)$ will be represented by $r_A(x)e^{i2\pi\omega_A(x)}$ and $\hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)}$, respectively, where $i = \sqrt{-1}$, $r_A(x), \hat{r}_A(x), \omega_A(x), \hat{\omega}_A(x) \in [0, 1]$. Thus the property of $|\lambda_A(x)| + |\rho_A(x)| \leq 1$ implies $r_A(x) + \hat{r}_A(x) \leq 1$. Note that the intuitionistic fuzzy set is a special case of complex intuitionistic fuzzy set with $\omega_A(x) = \hat{\omega}_A(x) = 0$. Also, if $\rho_A(x) = (1 - r_A(x))e^{i2\pi(1-\omega_A(x))}$, then we obtain a complex fuzzy set. Let $\alpha e^{i2\pi\beta}$ and $\gamma e^{i2\pi\delta}$ be two complex numbers, where $\alpha, \beta, \gamma, \delta \in [0, 1]$. By $\alpha e^{i2\pi\beta} \leq \gamma e^{i2\pi\delta}$ we mean $\alpha \leq \gamma$ and $\beta \leq \delta$. Through out this paper,we use the symbols $a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$. Let $A = (\lambda_A, \rho_A)$ be a complex intuitionistic fuzzy set on X with the degree of membership $\lambda_A(x) = r_A(x)e^{i2\pi\omega_A(x)}$ and the degree of non-membership $\rho_A(x) = \hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)}$. Then A is said to be a homogeneous complex intuitionistic fuzzy set if $\forall x, y \in X$, the following two conditions hold:
(1) $r_A(x) \leq r_A(y)$ if and only if $\omega_A(x) \leq \omega_A(y)$,
(2) $\hat{r}_A(x) \leq \hat{r}_A(y)$ if and only if $\hat{\omega}_A(x) \leq \hat{\omega}_A(y)$.
Let $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ be two complex intuitionistic fuzzy sets on the same set X. We say that A is homogeneous with B if $\forall x, y \in X$, the following conditions hold:
(1) $r_A(x) \leq r_B(y)$ if and only if $\omega_A(x) \leq \omega_B(y)$,
(2) $\hat{r}_A(x) \leq \hat{r}_B(y)$ if and only if $\hat{\omega}_A(x) \leq \hat{\omega}_B(y)$.

Definition 2.1. Let K be any field, and let V be a K-vector space. A complex intuitionistic fuzzy (CIF for short) set on V defined as an object having the form $A = \{(x, \lambda_A(x), \rho_A(x)) \,|x \in V\}$, where the complex functions $\lambda_A : V \to \mathbb{C}$ and $\rho_A : V \to \mathbb{C}$ denote the degree of membership (namely $\lambda_A(x)$) and the degree of non-membership (namely $\rho_A(x)$) of each element $x \in V$ to the set A, respectively, that assign to any element $x \in V$ complex numbers $\lambda_A(x)$, $\rho_A(x)$ lie within the unit circle with the property $|\lambda_A(x)| + |\rho_A(x)| \leq 1$.

We shall use the symbol $A = (\lambda_A, \rho_A)$ for the CIF set $A = \{(x, \lambda_A(x), \rho_A(x)) \,|x \in X\}$.

Definition 2.2. Let V be a K-vector space. A CIF set $A = (\lambda_A, \rho_A)$ of V is called a CIF vector subspace of V, if it satisfies the following conditions
for any $x, y \in V$, $\alpha \in K$
(1) $\lambda_A(x + y) \geq \lambda_A(x) \wedge \lambda_A(y)$, and $\rho_A(x + y) \leq \rho_A(x) \vee \rho_A(y)$
(2) $\lambda_A(\alpha x) \geq \lambda_A(x)$, and $\rho_A(\alpha x) \leq \rho_A(x)$.

From the above definition, we know that for any $x \in V$, $\lambda_A(0) \geq \lambda_A(x)$ and $\rho_A(0) \leq \rho_A(x)$. In this paper, we always assume that $\lambda_A(0) = 1e^{i2\pi} = 1$ and $\rho_A(0) = 0e^{i2(\pi)0} = 0$.

Definition 2.3. Let $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ be CIF vector subspaces of a vector space V. Then
(1) $A \subseteq B$ if $\lambda_A(x) \leq \lambda_B(x)$ and $\rho_A(x) \geq \rho_B(x)$,
(2) $A \cap B = \{x, \lambda_A(x) \wedge \lambda_B(x), \rho_A(x) \vee \rho_B(x) | x \in V\}$,
(3) $A \cup B = \{x, \lambda_A(x) \vee \lambda_B(x), \rho_A(x) \wedge \rho_B(x) | x \in V\}$.

Definition 2.4. Let $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ be CIF vector subspaces of a vector subspace V, where $\lambda_A = r_A e^{i2\pi\omega_A}, \lambda_B = r_B e^{i2\pi\omega_B}$ and $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}, \rho_B = \hat{r}_B e^{i2\pi\hat{\omega}_B}$. If A is homogenous with B. Then the complex intuitionistic sum of $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ is defined to the CIF set $A + B = (\lambda_{A+B}, \rho_{A+B})$ of V given by

$$\lambda_{A+B}(x) = \begin{cases} \sup_{x=a+b}\{(r_A(a) \wedge r_B(b))\} e^{i2\pi \sup_{x=a+b}\{(\omega_A(a) \wedge \omega_B(b))\}} & : \text{ if } x = a+b \\ 0 & : \ otherwise, \end{cases}$$

$$\rho_{A+B}(x) = \begin{cases} \inf_{x=a+b}\{(\hat{r}_A(a) \vee \hat{r}_B(b))\} e^{i2\pi \inf_{x=a+b}\{(\hat{\omega}_A(a) \vee \hat{\omega}_B(b))\}} & : \text{ if } x = a+b \\ 1 & : \ otherwise. \end{cases}$$

Furthermore, if $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$, where

$$\lambda_{A\cap B}(x) = \begin{cases} 0 & : \quad x \neq 0 \\ 1 & : \quad x = 0, \end{cases} \quad \text{and} \quad \rho_{A\cap B}(x) = \begin{cases} 1 & : \quad x \neq 0 \\ 0 & : \quad x = 0, \end{cases}.$$

Then $A + B$ is said to be the direct sum and denoted by $A \oplus B$.

Lemma 2.1. Let $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ be CIF vector subspaces of a vector space V such that A is homogenous with B. Then $A + B = (\lambda_{A+B}, \rho_{A+B})$ is also a CIF vector subspaces of V.

Proof. $A + B$ is homogenous, since A is homogenous with B. Let $x, y \in V$, then

$$\begin{aligned} \lambda_{A+B}(x) \wedge \lambda_{A+B}(y) &= r_{A+B}(x) e^{i2\pi\omega_{A+B}(x)} \wedge r_{A+B}(y) e^{i2\pi\omega_{A+B}(y)} \\ &= (r_{A+B}(x) \wedge r_{A+B}(y)) e^{i2\pi(\omega_{A+B}(x) \wedge \omega_{A+B}(x))}, \end{aligned}$$

and

$$\begin{aligned} r_{A+B}(x) \wedge r_{A+B}(y) &= \sup_{x=a+b}\{r_A(a) \wedge r_B(b)\} \wedge \sup_{y=c+d}\{r_A(c) \wedge r_B(d)\} \\ &= \sup_{x=a+b, y=c+d}\{(r_A(a) \wedge r_B(b)) \wedge (r_A(c) \wedge r_B(d))\} \\ &= \sup_{x=a+b, y=c+d}\{(r_A(a) \wedge r_A(c)) \wedge (r_B(b) \wedge r_B(d))\} \\ &\leq \sup_{x=a+b, y=c+d}\{r_A(a+c) \wedge r_B(b+d)\} \\ &= r_{A+B}(x+y). \end{aligned}$$

Similarly, $\omega_{A+B}(x) \wedge \omega_{A+B}(y) \leq \omega_{A+B}(x+y)$. Therefore,

$$\begin{aligned} \lambda_{A+B}(x) \wedge \lambda_{A+B}(y) &= (r_{A+B}(x) \wedge r_{A+B}(y)) e^{i2\pi(\omega_{A+B}(x) \wedge \omega_{A+B}(x))} \\ &\geq r_{A+B}(x+y) e^{i2\pi\omega_{A+B}(x+y)} = \lambda_{A+B}(x+y). \end{aligned}$$

Furthermore,

$$\begin{aligned} \rho_{A+B}(x) \vee \rho_{A+B}(y) &= \hat{r}_{A+B}(x) e^{i2\pi\hat{\omega}_{A+B}(x)} \vee \hat{r}_{A+B}(y) e^{i2\pi\hat{\omega}_{A+B}(y)} \\ &= (\hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y)) e^{i2\pi(\hat{\omega}_{A+B}(x) \vee \hat{\omega}_{A+B}(x))}, \end{aligned}$$

and

$$\begin{aligned} \hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y) &= \inf_{x=a+b}\{\hat{r}_A(a) \vee \hat{r}_B(b)\} \vee \inf_{y=c+d}\{\hat{r}_A(c) \vee \hat{r}_B(d)\} \\ &= \inf_{x=a+b,y=c+d}\{(\hat{r}_A(a) \vee \hat{r}_B(b)) \vee (\hat{r}_A(c) \vee \hat{r}_B(d))\} \\ &= \inf_{x=a+b,y=c+d}\{(\hat{r}_A(a) \vee \hat{r}_A(c)) \vee (\hat{r}_B(b) \vee \hat{r}_B(d))\} \\ &\geq \inf_{x=a+b,y=c+d}\{\hat{r}_A(a+c) \vee \hat{r}_B(b+d)\} \\ &= \hat{r}_{A+B}(x+y). \end{aligned}$$

Similarly, $\hat{\omega}_{A+B}(x) \vee \hat{\omega}_{A+B}(y) \geq \hat{\omega}_{A+B}(x+y)$. Therefore,

$$\begin{aligned} \rho_{A+B}(x) \vee \rho_{A+B}(y) &= (\hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y))e^{i2\pi(\hat{\omega}_{A+B}(x)\vee\hat{\omega}_{A+B}(x))} \\ &\geq \hat{r}_{A+B}(x+y)e^{i2\pi\hat{\omega}_{A+B}(x+y)} = \rho_{A+B}(x+y). \end{aligned}$$

Also, if $\alpha \in K$, then $\lambda_{A+B}(x) = r_{A+B}(x)e^{i2\pi\omega_{A+B}(x)}$, and

$$\begin{aligned} r_{A+B}(x) &= \sup_{x=a+b}\{r_A(a) \wedge r_B(b)\} \\ &\leq \sup_{\alpha x=\alpha a+\alpha b}\{r_A(\alpha a) \wedge r_B(\alpha b)\} \\ &\leq \sup_{\alpha x=c+d}\{r_A(c) \wedge r_B(d)\} \\ &= r_{A+B}(\alpha x). \end{aligned}$$

Similarly, $\omega_{A+B}(x) \leq \omega_{A+B}(\alpha x)$. Hence,

$$\lambda_{A+B}(x) = r_{A+B}(x)e^{i2\pi\omega_{A+B}(x)} \leq r_{A+B}(\alpha x)e^{i2\pi\omega_{A+B}(\alpha x)} = \lambda_{A+B}(\alpha x).$$

Because $\rho_{A+B}(x) = \hat{r}_{A+B}(x)e^{i2\pi\hat{\omega}_{A+B}(x)}$, and

$$\begin{aligned} \hat{r}_{A+B}(x) &= \inf_{x=a+b}\{\hat{r}_A(a) \vee \hat{r}_B(b)\} \\ &\geq \inf_{\alpha x=\alpha a+\alpha b}\{\hat{r}_A(\alpha a) \vee \hat{r}_B(\alpha b)\} \\ &\geq \inf_{\alpha x=c+d}\{\hat{r}_A(c) \vee \hat{r}_B(d)\} \\ &= \hat{r}_{A+B}(\alpha x), \end{aligned}$$

then we also have $\hat{\omega}_{A+B}(x) \geq \hat{\omega}_{A+B}(\alpha x)$. As a result,

$$\rho_{A+B}(x) = \hat{r}_{A+B}(x)e^{i2\pi\hat{\omega}_{A+B}(x)} \geq \hat{r}_{A+B}(\alpha x)e^{i2\pi\hat{\omega}_{A+B}(\alpha x)} = \rho_{A+B}(\alpha x).$$

Consequently, $A + B = (\lambda_{A+B}, \rho_{A+B})$ is a CIF vector subspaces of V. □

Definition 2.5. Let $A = (\lambda_A, \rho_A)$ be a CIF vector subspace of a K-vector space V. For $\alpha \in K$ and $x \in V$, define $\alpha A = (\lambda_{\alpha A}, \rho_{\alpha A})$, where

$$\lambda_{\alpha A}(x) = \begin{cases} \lambda_A(\alpha^{-1}x) = r_A(\alpha^{-1}x)e^{i2\pi\omega_A(\alpha^{-1}x)} & : \quad \alpha \neq 0 \\ 1 & : \quad \alpha = 0, x = 0 \\ 0 & : \quad \alpha = 0, x \neq 0 \end{cases}$$

and

$$\rho_{\alpha A}(x) = \begin{cases} \rho_A(\alpha^{-1}x) = \hat{r}_A(\alpha^{-1}x)e^{i2\pi\hat{\omega}_A(\alpha^{-1}x)} & : \quad \alpha \neq 0 \\ 0 & : \quad \alpha = 0, x = 0 \\ 1 & : \quad \alpha = 0, x \neq 0 \end{cases}$$

Definition 2.6. Let V, V' be K-vector spaces and let $f : V \to V'$ be any map. If $A = (\lambda_A, \rho_A)$, $B = (\lambda_B, \rho_B)$ are CIF vector subspaces of V, V', respectively, then the preimage of B under f is defined to be a CIF set $f^{-1}(B) = (\lambda_{f^{-1}}, \rho_{f^{-1}})$, where $\lambda_{f^{-1}}(x) = \lambda_B(f(x))$ and $\rho_{f^{-1}}(x) = \rho_B(f(x))$ for any $x \in V$ and the image of $A = (\lambda_A, \rho_A)$ under f is defined to be the CIF set $f(A) = (\lambda_{f(A)}, \rho_{f(A)})$ where

$$\lambda_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\lambda_A(x)\} = \sup\limits_{x \in f^{-1}(y)} \{r_A(x)e^{i2\pi\omega_A(x)}\} & : \quad y \in f(V) \\ 0 & : \quad y \notin f(V) \end{cases}$$

and

$$\rho_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\rho_A(x)\} = \inf\limits_{x \in f^{-1}(y)} \{\hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)}\} & : \quad y \in f(V) \\ 1 & : \quad y \notin f(V). \end{cases}$$

The following results are easy to get. Here we omit the proofs.

Lemma 2.2. Let $A = (\lambda_A, \rho_A)$, where $\lambda_A = r_A e^{i2\pi\omega_A}$ and $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}$, be a CIF vector subspace of V. Then $\alpha A = (\lambda_{\alpha A}, \rho_{\alpha A})$ is also a CIF vector subspace of V.

Lemma 2.3. Let $A = (\lambda_A, \rho_A)$, where $\lambda_A = r_A e^{i2\pi\omega_A}$ and $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}$, be a CIF vector subspace of V' and $f : V \to V'$ any map. Then $f^{-1}(A) = (\lambda_{f^{-1}(A)}, \rho_{f^{-1}(A)})$ is also a CIF vector subspace of V, where $\lambda_{f^{-1}(A)}(x) = \lambda_A(f(x))$ and $\rho_{f^{-1}(A)}(x) = \rho_A(f(x))$ for all $x \in V$.

Lemma 2.4. Let $f : V \to V'$ be any map. If $A = (\lambda_A, \rho_A)$, where $\lambda_A = r_A e^{i2\pi\omega_A}$ and $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}$, is a CIF vector subspace of V, then $f(A) = (\lambda_{f(A)}, \rho_{f(A)})$ is a CIF vector subspace of V'.

Lemma 2.5. Let $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ be CIF vector subspace of V such that A is homogenous with B, where $\lambda_A = r_A e^{i2\pi\omega_A}$, $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}$ and $\lambda_B = r_B e^{i2\pi\omega_B}$, $\rho_B = \hat{r}_B e^{i2\pi\hat{\omega}_B}$. Then $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$ is a CIF vector subspace of V, where

$$\begin{aligned} \lambda_{A\cap B}(x) &= \lambda_A(x) \wedge \lambda_B(x) \\ &= (r_A(x) \wedge r_B(x))e^{i2\pi(\omega_A(x)\wedge\omega_B(x))} \end{aligned}$$

and

$$\begin{aligned} \rho_{A\cap B}(x) &= \rho_A(x) \vee \rho_B(x) \\ &= (\hat{r}_A(x) \vee \hat{r}_B(x))e^{i2\pi(\hat{\omega}_A(x)\vee\hat{\omega}_B(x))} \end{aligned}$$

Definition 2.7. A $\mathbb{Z}_2$-graded vector space $V = V_0 + V_1$ possessing the operation called the bilinear bracket product,

$$[\ ,\ ] : V \times V \overset{bilinear}{\longrightarrow} [x, y] \in V$$

is called a Lie superalgebra, if it satisfies the following conditions
(1) $[V_i, V_j] \subseteq V_{i+j}$
(2) $[x, y] = -(-1)^{|x||y|}[y, x]\ \forall x, y \in V_0 \cup V_1$
(3) $[x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] = [[x, y], z]$

3 Complex intuitionistic fuzzy lie sub-superalgebras and ideals

In this section we assume that V is a Lie superalgebra over a field K.

Definition 3.1. Let $V = V_0 + V_1$ be a $\mathbb{Z}_2$-graded vector space. Suppose that $A_0 = (\lambda_{A_0}, \rho_{A_0})$ and $A_1 = (\lambda_{A_1}, \rho_{A_1})$ are CIF vector subspaces of V_0 and V_1, respectively. Define $\mathfrak{a}_0 = (\lambda_{\mathfrak{a}_0}, \rho_{\mathfrak{a}_0})$, where

$$\lambda_{\mathfrak{a}_0}(x) = \begin{cases} \lambda_{A_0}(x) & : \quad x \in V_0 \\ 0 & : \quad x \notin V_0 \end{cases} \quad and \quad \rho_{\mathfrak{a}_0}(x) = \begin{cases} \rho_{A_0}(x) & : \quad x \in V_0 \\ 1 & : \quad x \notin V_0, \end{cases}$$

and define $\mathfrak{a}_1 = (\lambda_{\mathfrak{a}_1}, \rho_{\mathfrak{a}_1})$ where

$$\lambda_{\mathfrak{a}_1}(x) = \begin{cases} \lambda_{A_1}(x) & : \quad x \in V_1 \\ 0 & : \quad x \notin V_1 \end{cases} \quad and \quad \rho_{\mathfrak{a}_1}(x) = \begin{cases} \rho_{A_1}(x) & : \quad x \in V_1 \\ 1 & : \quad x \notin V_1. \end{cases}$$

Then $\mathfrak{a}_0 = (\lambda_{\mathfrak{a}_0}, \rho_{\mathfrak{a}_0})$ and $\mathfrak{a}_1 = (\lambda_{\mathfrak{a}_1}, \rho_{\mathfrak{a}_1})$ are the CIF vector subspaces of V. Moreover, we have $\mathfrak{a}_0 \cap \mathfrak{a}_1 = (\lambda_{\mathfrak{a}_0\cap\mathfrak{a}_1}, \rho_{\mathfrak{a}_0\cap\mathfrak{a}_1})$, where

$$\lambda_{\mathfrak{a}_0\cap\mathfrak{a}_1}(x) = \lambda_{\mathfrak{a}_0}(x) \wedge \lambda_{\mathfrak{a}_1}(x) = \begin{cases} 1 & : \quad x = 0 \\ 0 & : \quad x \neq 0, \end{cases}$$

and

$$\rho_{\mathfrak{a}_0\cap\mathfrak{a}_1}(x) = \rho_{\mathfrak{a}_0}(x) \vee \rho_{\mathfrak{a}_1}(x) = \begin{cases} 0 & : \quad x = 0 \\ 1 & : \quad x \neq 0. \end{cases}$$

So $\mathfrak{a}_0 + \mathfrak{a}_1$ is the direct sum and is denoted by $A_0 \oplus A_1$. If $A = (\lambda_A, \rho_A)$ is a CIF vector subspace of V and $A = A_0 \oplus A_1$, then $A = (\lambda_A, \rho_A)$ is called a $\mathbb{Z}_2$-graded CIF vector subspace of V.

Remark 3.2. (1)

$$\begin{aligned}
\lambda_A(x) &= \lambda_{A_0\oplus A_1}(x) \\
&= \sup_{x=\alpha+\beta} \{\lambda_{\mathfrak{a}_0}(\alpha) \wedge \lambda_{\mathfrak{a}_1}(\beta)\} \\
&= \sup_{x=\alpha+\beta} \{\lambda_{\mathfrak{a}_0}(\alpha_0+\alpha_1) \wedge \lambda_{\mathfrak{a}_1}(\beta_0+\beta_1)\},\ where\ \alpha = \alpha_0+\alpha_1, \beta = \beta_0+\beta_1 \\
&= \sup_{x=\alpha+\beta} \{\lambda_{\mathfrak{a}_0}(\alpha_0) \wedge \lambda_{\mathfrak{a}_1}(\beta_1)\} \\
&= \sup_{x=\alpha+\beta} \{r_{\mathfrak{a}_0}(\alpha_0)e^{i2\pi\omega_{\mathfrak{a}_0}(\alpha_0)} \wedge r_{\mathfrak{a}_1}(\beta_1)e^{i2\pi\omega_{\mathfrak{a}_1}(\beta_1)}\} \\
&= r_{\mathfrak{a}_0}(x_0)e^{i2\pi\omega_{\mathfrak{a}_0}(x_0)} \wedge r_{\mathfrak{a}_1}(x_1)e^{i2\pi\omega_{\mathfrak{a}_1}(x_1)} \\
&= r_{A_0}(x_0)e^{i2\pi\omega_{A_0}(x_0)} \wedge r_{A_1}(x_1)e^{i2\pi\omega_{A_1}(x_1)} \\
&= \lambda_{A_0}(x_0) \wedge \lambda_{A_1}(x_1).
\end{aligned}$$

(2) $A_0 = (\lambda_{A_0}, \rho_{A_0})$ and $A_1 = (\lambda_{A_1}, \rho_{A_1})$ are the even and odd parts of $A = (\lambda_A, \rho_A)$ (respectively).
(3) $\mathfrak{a}_0 = (\lambda_{\mathfrak{a}_0}, \rho_{\mathfrak{a}_0})$ and $\mathfrak{a}_1 = (\lambda_{\mathfrak{a}_1}, \rho_{\mathfrak{a}_1})$ are extensions of $A_0 = (\lambda_{A_0}, \rho_{A_0})$ and $A_1 = (\lambda_{A_1}, \rho_{A_1})$ (respectively).

Definition 3.3. Let $A = (\lambda_A, \rho_A)$ be CIF set of V. Then $A = (\lambda_A, \rho_A)$ is called a complex fuzzy lie sub-superalgebra of V, if it satisfies the following two conditions:
(1) $A = (\lambda_A, \rho_A)$ is a $\mathbb{Z}_2$-graded CIF vector space and
(2) $\lambda_A([x,y]) \geq \lambda_A(x) \wedge \lambda_A(y)$ and $\rho_A([x,y]) \leq \rho_A(x) \vee \rho_A(y)$.
If the condition(2) is replaced by
(3) $\lambda_A([x,y]) \geq \lambda_A(x) \vee \lambda_A(y)$ and $\rho_A([x,y]) \leq \rho_A(x) \wedge \rho_A(y)$,
then $A = (\lambda_A, \rho_A)$ is called a CIF ideal of V.

Example 3.4. Let $N = N_0 \oplus N_1$, where $N_0 =< e >$, $N_1 =< a_1, ..., a_n, b_1, ..., b_n >$, and $[a_i, b_i] = e$, $i = 1, ..., n$, the remaining brackets being zero. Then, N is Lie superalgebra (see [15, page 11]). Define $A_0 = (\lambda_{A_0}, \rho_{A_0})$, where

$$\lambda_{A_0} : N_0 \to \mathbb{C}\ \ by\ \ \lambda_{A_0}(x) = \begin{cases} 0.7e^{i(1.4)\pi} & : \quad 0 \neq x \in N_0 \\ 1 & : \quad x = 0, \end{cases}$$

$$\rho_{A_0} : N_0 \to \mathbb{C}\ \ by\ \ \rho_{A_0}(x) = \begin{cases} 0.2e^{i(0.4)\pi} & : \quad 0 \neq x \in N_0 \\ 0 & : \quad x = 0. \end{cases}$$

Also, define $A_1 = (\lambda_{A_1}, \rho_{A_1})$, where

$$\lambda_{A_1} : N_1 \to \mathbb{C}\ \ by\ \ \lambda_{A_1}(x) = \begin{cases} 0.5e^{i\pi} & : \quad 0 \neq x \in N_1 \\ 1 & : \quad x = 0, \end{cases}$$

$$\rho_{A_1}: N_1 \to \mathbb{C} \ \ by \ \ \rho_{A_1}(x) = \begin{cases} 0.4e^{i(0.8)\pi} & : \quad 0 \neq x \in N_1 \\ 0 & : \quad x = 0. \end{cases}$$

Define $A = (\lambda_A, \rho_A)$. Then, by Definition 3.1, $A = A_0 \oplus A_1$, and so $A = (\lambda_A, \rho_A)$ is a $\mathbb{Z}_2$-graded CIF vector subspace of N. Moreover, it is easy to check that $A = (\lambda_A, \rho_A)$ is a CIF ideal of N.

For any complex fuzzy set $\lambda = re^{i2\pi\omega}$ of V, we define the image of λ by

$$\mathrm{Im}(\lambda) = \{(t,s) \in [0,1] \times [0,1] \ : \ \lambda(x) = r(x)e^{i2\pi\omega(x)} = te^{i2\pi s} \ for \ some \ x \in V\}$$

Definition 3.5. For any $t, s \in [0,1]$ and complex fuzzy subset $\lambda = re^{i2\pi\omega}$ of V, the set $U(\lambda,(t,s)) = \{x \in V \mid r(x) \geqslant t \ and \ \omega(x) \geqslant s\}$ is called an upper (t,s)-level cut of λ, $L(\lambda,(t,s)) = \{x \in V \mid r(x) \leqslant t \ and \ \omega(x) \leqslant s\}$ is called a lower (t,s)-level cut of λ.

Suppose that $\lambda_A(x) = r_A(x)e^{i2\pi\omega_A(x)}$ and $\rho_A(x) = \hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)}$, then the following result holds

Theorem 3.6. If $A = (\lambda_A, \rho_A)$ is a CIF lie sub-superalgebra (respectively CIF lie ideal) of V, then the sets $U(\lambda_A,(t,s))$ and $L(\rho_A,(t,s))$ are lie sub-superalgebras (respectively ideals) of V for every $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$.

Proof. Let $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A) \subseteq [0,1] \times [0,1]$, $x, y \in U(\lambda_A,(t,s))$. Let $\alpha \in K$. Because $A = (\lambda_A, \rho_A)$ is a CIF lie sub-superalgebra. Then $\lambda_A(x+y) = r_A(x+y)e^{i2\pi\omega(x+y)} \geqslant r_A(x)e^{i2\pi\omega_A(x)} \wedge r_A(y)e^{i2\pi\omega_A(y)} = \lambda_A(x) \wedge \lambda_A(y) \geqslant te^{i2\pi s}$, since $r_A(x+y) \geqslant r_A(x) \wedge r_A(y) \geqslant t$ and $\omega_A(x+y) \geqslant \omega_A(x) \wedge \omega_A(y) \geqslant s$. Moreover, $\lambda_A(\alpha x) = r_A(\alpha x)e^{i2\pi\omega_A(\alpha x)} \geqslant r_A(x)e^{i2\pi\omega_A(x)} \geqslant te^{i2\pi s}$, since $r_A(\alpha x) \geqslant r_A(x) \geqslant t$ and $\omega_A(\alpha x) \geqslant \omega_A(x) \geqslant s$. As a result, $x+y, \alpha x \in U(\lambda_A,(t,s))$. Let $x \in U(\lambda_A,(t,s)) \subseteq V$, then $x = x_0 + x_1$, where $x_0 \in V_0, x_1 \in V_1$. Hence,

$$\begin{aligned} \lambda_A(x) &= r_A(x)e^{i2\pi\omega_A(x)} \\ &= \lambda_{A_0}(x_0) \wedge \lambda_{A_1}(x_1) \\ &= r_{A_0}(x_0)e^{i2\pi\omega_{A_0}(x_0)} \wedge r_{A_1}(x_1)e^{i2\pi\omega_{A_1}(x_1)} \\ &\geqslant (r_{A_0}(x_0) \wedge r_{A_1}(x_1))e^{i2\pi(\omega_{A_0}(x_0)\wedge\omega_{A_1}(x_1))} \\ &\geqslant te^{i2\pi s}. \end{aligned}$$

As a result, if $\lambda_{A_0}(x_0) \geqslant \lambda_{A_1}(x_1)$, then

$$\begin{aligned} \lambda_A(x_1) &= r_A(x_1)e^{i2\pi\omega_A(x_1)} \\ &= r_{A_1}(x_1)e^{i2\pi\omega_{A_1}(x_1)} \\ &= \lambda_{A_1}(x_1) \\ &\geqslant te^{i2\pi s}, \end{aligned}$$

implies that $x_1 \in U(\lambda_A,(t,s))$ and

$$\begin{aligned} \lambda_A(x_0) &= \lambda_{A_0}(x_0) \\ &= r_{A_0}(x_0)e^{i2\pi\omega_{A_0}(x_0)} \\ &\geqslant \lambda_{A_1}(x_1) \\ &\geqslant te^{i2\pi s}, \end{aligned}$$

implies that $x_0 \in U(\lambda_A,(t,s))$. Similarly, if $\lambda_{A_1}(x_1) \geqslant \lambda_{A_0}(x_0)$, then $x_0, x_1 \in U(\lambda_A,(t,s))$. Consequently, $U(\lambda_A,(t,s))$ is a $\mathbb{Z}_2$-graded vector subspace of V for any $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$. Let $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$ and $x, y \in U(\lambda_A,(t,s))$. Then $\lambda_A(x) \geqslant te^{i2\pi s}$ implies that $r_A(x) \geqslant t, \omega_A(x) \geqslant s$. Also, $\lambda_A(y) \geqslant te^{i2\pi s}$ implies that $r_A(y) \geqslant t, \omega_A(y) \geqslant s$. As a result,

$\lambda_A([x,y]) \geqslant \lambda_A(x) \wedge \lambda_A(y) \geqslant te^{i2\pi s}$. Which implies that $[x,y] \in U(\lambda_A,(t,s))$. Therefore, $U(\lambda_A,(t,s))$ is a lie sub-superalgebra of V for any $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$. Similarly, if $x \in L(\rho_A,(t,s)) \subseteq V$, then $x = x_0 + x_1$, where $x_0 \in V_0, x_1 \in V_1$. Therefore,

$$\begin{aligned}
\rho_A(x) &= \hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)} \\
&= \rho_{A_0}(x_0) \vee \rho_{A_1}(x_1) \\
&= \hat{r}_{A_0}(x_0)e^{i2\pi\hat{\omega}_{A_0}(x_0)} \vee \hat{r}_{A_1}(x_1)e^{i2\pi\hat{\omega}_{A_1}(x_1)} \\
&\leqslant (\hat{r}_{A_0}(x_0) \vee \hat{r}_{A_1}(x_1))e^{i2\pi(\hat{\omega}_{A_0}(x_0)\vee\hat{\omega}_{A_1}(x_1))} \\
&\leqslant te^{i2\pi s}.
\end{aligned}$$

As a result, if $\rho_{A_0}(x_0) \geqslant \rho_{A_1}(x_1)$, then $\rho_A(x_0) = \rho_{A_0}(x_0) \leqslant te^{i2\pi s}$ implies that $x_0 \in L(\rho_A,(t,s))$, and $\rho_A(x_1) = \rho_{A_1}(x_1) \leqslant \rho_{A_0}(x_0) \leqslant te^{i2\pi s}$ implies that $x_1 \in L(\rho_A,(t,s))$. Similarly, if $\rho_{A_1}(x_1) \geqslant \rho_{A_0}(x_0)$, then $x_0, x_1 \in L(\rho_A,(t,s))$. Consequently, $L(\rho_A,(t,s))$ is a $\mathbb{Z}_2$-graded vector subspace of V for any $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$. Also, if $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$ and $x, y \in L(\rho_A,(t,s))$, then $\rho_A(x) \leqslant te^{i2\pi s}$ and $\rho_A(y) \leqslant te^{i2\pi s}$. Which implies that $\rho_A([x,y]) \leqslant \rho_A(x) \vee \rho_A(y) \leqslant te^{i2\pi s}$. As a result, $[x,y] \in L(\rho_A,(t,s))$. Accordingly, $L(\rho_A,(t,s))$ is a lie sub-superalgebra of V for any $(t,s) \in \mathrm{Im}(\lambda_A) \cap \mathrm{Im}(\rho_A)$. □

Suppose that $A = (\lambda_A, \rho_A)$ is a CIF set of V, where $\lambda_A(x) = r_A(x)e^{i2\pi\omega_A(x)}$ and $\rho_A(x) = \hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)}$. Then we define $\lambda_A^c(x)$ by $\lambda_A^c(x) = (1 - r_A(x))e^{i2\pi(1-\omega_A(x))}$ and ρ_A^c by $\rho_A^c = (1 - \hat{r}_A(x))e^{i2\pi(1-\hat{\omega}_A(x))}$.

Definition 3.7.
(1) $A^c = \{(x, \lambda_A(x), \lambda_A^c(x)) : x \in V\}$. Shortly $A^c = (\lambda_A, \lambda_A^c)$.
(2) $A^L = \{(x, \rho_A^c(x), \rho_A(x)) : x \in V\}$. Shortly $A^L = (\rho_A^c, \rho_A)$.

Theorem 3.8.
(1) If $A = (\lambda_A, \rho_A)$ is a CIF lie sub-suberalgebra (respectively CIF ideal) of V, then so is A^c.
(2) If $A = (\lambda_A, \rho_A)$ is a CIF lie sub-superalgebra (respectively CIF ideal) of V, then so is A^L.

Proof. (1) Since (λ_A, ρ_A) is a $\mathbb{Z}_2$-graded CIF vector subspace of V, $A = A_0 + A_1$, where $A_0 = (\lambda_{A_0}, \rho_{A_0})$, $A_1 = (\lambda_{A_1}, \rho_{A_1})$ are CIF vector subspaces of V_0 and V_1 (respectively). Also, if $x \in V$, then $\lambda_A(x) = \sup_{x=a+b}\{\lambda_{A_0}(a) \wedge \lambda_{A_1}(b)\}$. Define $A_0^c = (\lambda_{A_0}, \lambda_{A_0}^c)$, $A_1^c = (\lambda_{A_1}, \lambda_{A_1}^c)$ and define $\mathfrak{a}_0^c = (\lambda_{\mathfrak{a}_0}, \lambda_{\mathfrak{a}_0}^c)$ and $\mathfrak{a}_1^c = (\lambda_{\mathfrak{a}_1}, \lambda_{\mathfrak{a}_1}^c)$. Obviously, $(\mathfrak{a}_0^c, \mathfrak{a}_1^c)$ are the extensions of (A_0^c, A_1^c). In order to prove $A^c = \mathfrak{a}_0^c + \mathfrak{a}_1^c$ we only need to show that $\lambda_A^c(x) = \inf_{x=a+b}\{\lambda_{\mathfrak{a}_0}^c(a) \vee \lambda_{\mathfrak{a}_1}^c(b)\}$. Since A is homogeneous indeed,

$$\begin{aligned}
1 - \lambda_A^c(x) &= \sup_{x=a+b}\{(1 - \lambda_{\mathfrak{a}_0}^c(a)) \wedge (1 - \lambda_{\mathfrak{a}_1}^c(b))\} \\
&= \sup_{x=a+b}\{(1 - (\lambda_{\mathfrak{a}_0}^c(a) \vee \lambda_{\mathfrak{a}_1}^c(b))\} \\
&= \sup_{x=a+b}\{1 - ((1 - r_{\mathfrak{a}_0}(a))e^{i2\pi(1-w_{\mathfrak{a}_0}(a))} \vee (1 - r_{\mathfrak{a}_1}(b))e^{i2\pi(1-\omega_{\mathfrak{a}_1}(b))})\} \\
&= \sup_{x=a+b}\{1 - ((1 - r_{\mathfrak{a}_0}(a) \vee 1 - r_{\mathfrak{a}_1}(b))e^{i2\pi((1-\omega_{\mathfrak{a}_0}(a))\vee(1-\omega_{\mathfrak{a}_1}(b)))})\} \\
&= 1 - \inf_{x=a+b}\{1 - r_{\mathfrak{a}_0}(a) \vee 1 - r_{\mathfrak{a}_1}(b)\}e^{i2\pi(1-(1-\omega_{\mathfrak{a}_0}(a)\vee 1-w_{\mathfrak{a}_1}(b)))} \\
&= 1 - \inf_{x=a+b}\{(1 - r_{\mathfrak{a}_0}(a))e^{i2\pi(1-w_{\mathfrak{a}_0}(a))} \vee (1 - r_{\mathfrak{a}_1}(b))e^{i2\pi(1-\omega_{\mathfrak{a}_1}(b))}\} \\
&= 1 - \inf_{x=a+b}\{\lambda_{\mathfrak{a}_0}^c(a) \vee \lambda_{\mathfrak{a}_1}^c(b)\},
\end{aligned}$$

which implies that $\lambda_A^c(x) = \inf_{x=a+b}\{\lambda_{\mathfrak{a}_0}^c(a) \vee \lambda_{\mathfrak{a}_1}^c(b)\} = \lambda_{A_0}^c(x_0) \vee \lambda_{A_1}^c(x_1)$. Moreover, it is easy to see that $\lambda_{\mathfrak{a}_0}^c(x) \vee \lambda_{\mathfrak{a}_1}^c(x) = \begin{cases} 0 & : \quad x = 0 \\ 1 & : \quad x \neq 0 \end{cases}$. As a result, $A^c = A_0^c \oplus A_1^c$ is a $\mathbb{Z}_2$-graded CIF vector subspace of V. Let $x, y \in V$. Since $\lambda_A([x,y]) \geq \lambda_A(x) \wedge \lambda_A(y)$, $1 - \lambda_A^c([x,y]) \geq$

$(1-\lambda_A^c(x)) \wedge (1-\lambda_A^c(y))$, which implies that $1-\lambda_A^c([x,y]) \geq 1-(\lambda_A^c(x) \vee \lambda_A^c(y))$. Thus, $\lambda_A^c([x,y]) \leq \lambda_A^c(x) \vee \lambda_A^c(y)$. As a result, $A^c = (\lambda_A, \lambda_A^c)$ is a CIF lie sub-superalgebra of V. By using the same argument used above, we can prove the case of CIF ideal.
(2) The proof is similar to the proof in (1). □

By using the above result, it is not difficult to verify that the following theorem is valid.

Theorem 3.9. $A = (\lambda_A, \rho_A)$ is a CIF lie sub-superalgebra (respectively CIF ideal) of V If and only If A^c and A^L are CIF lie sub-superalgebras (respectively CIF ideals) of V.

Theorem 3.10. If $A = (\lambda_A, \rho_A)$ is a CIF set of V such that all non-empty level sets $U(\lambda_A, (t,s))$ and $L(\rho_A, (t,s))$ are lie sub-superalgebras (respectively ideals)of V, then $A = (\lambda_A, \rho_A)$ is CIF lie sub-superalgebra (respectively CIF ideal) of V.

Proof. Let $x, y \in V$ and let $\alpha \in K$. We may assume that $\lambda_A(y) \geq \lambda_A(x) = t_1 e^{i2\pi s_1}$ $\rho_A(y) \leq \rho_A(x) = t_0 e^{i2\pi s_0}$, where $t_1, s_1, t_0, s_0 \in [0,1]$. Then $x, y \in U(\lambda_A, (t_1, s_1))$ and $x, y \in L(\rho_A, (t_0, s_0))$. Because $U(\lambda_A, (t_1, s_1))$ and $L(\rho_A, (t_0, s_0))$ are vector subspaces of V, $x+y, \alpha x \in U(\lambda_A, (t_1, s_1))$ and $x+y, \alpha x \in L(\rho_A, (t_0, s_0))$. As a result, $\lambda_A(\alpha x) \geq \lambda_A(x) = t_1 e^{i2\pi s_1}$, $\lambda_A(x+y) \geq t_1 e^{i2\pi s_1} = \lambda_A(x) \wedge \lambda_A(y)$ and $\rho_A(\alpha x) \leqslant t_0 e^{i2\pi s_0} = \rho_A(x)$, $\rho_A(x+y) \leqslant t_0 e^{i2\pi s_0} = \rho_A(x) \vee \rho_A(y)$. Now, we show that $A = (\lambda_A, \rho_A)$ has a $\mathbb{Z}_2$-graded structure. Define $A_0 = (\lambda_{A_0}, \rho_{A_0})$, where $\lambda_{A_0} : V_0 \to \mathbb{C}$ by $x \mapsto \lambda_A(x)$, $\rho_{A_0} : V_0 \to \mathbb{C}$, by $x \mapsto \rho_A(x)$. Also, define $A_1 = (\lambda_{A_1}, \rho_{A_1})$, where $\lambda_{A_1} : V_1 \to \mathbb{C}$ by $x \mapsto \lambda_A(x)$, $\rho_{A_1} : V_1 \to \mathbb{C}$ by $x \mapsto \rho_A(x)$. We extend $A_0 = (\lambda_{A_0}, \rho_{A_0})$, $A_1 = (\lambda_{A_1}, \rho_{A_1})$ to $\mathfrak{a}_0 = (\lambda_{\mathfrak{a}_0}, \rho_{\mathfrak{a}_0})$, $\mathfrak{a}_1 = (\lambda_{\mathfrak{a}_1}, \rho_{\mathfrak{a}_1})$ as follows. Define $\mathfrak{a}_0 = (\lambda_{\mathfrak{a}_0}, \rho_{\mathfrak{a}_0})$ by

$$\lambda_{\mathfrak{a}_0}(x) = \begin{cases} \lambda_{A_0}(x) & if \quad x \in V_0 \\ 0 & if \quad x \notin V_0 \end{cases}, \ \rho_{\mathfrak{a}_0}(x) = \begin{cases} \rho_{A_0}(x) & if \quad x \in V_0 \\ 1 & if \quad x \notin V_0. \end{cases}$$

Also, define $\mathfrak{a}_1 = (\lambda_{\mathfrak{a}_1}, \rho_{\mathfrak{a}_1})$ by

$$\lambda_{\mathfrak{a}_1}(x) = \begin{cases} \lambda_{A_1}(x) & if \quad x \in V_1 \\ 0 & if \quad x \notin V_1 \end{cases}, \ \rho_{\mathfrak{a}_1}(x) = \begin{cases} \rho_{A_1}(x) & if \quad x \in V_1 \\ 1 & if \quad x \notin V_1. \end{cases}$$

Then it is obvious that $\mathfrak{a}_0$, $\mathfrak{a}_1$ are CIF vector subspaces of V, and for any $0 \neq x \in V$ we have $\mathfrak{a}_0 \cap \mathfrak{a}_1 = (\lambda_{\mathfrak{a}_0}(x) \wedge \lambda_{\mathfrak{a}_1}(x), \rho_{\mathfrak{a}_0}(x) \vee \rho_{\mathfrak{a}_1}(x)) = (0,1)$. To show that $A = A_0 \oplus A_1$, let $x \in V$. We may assume that $\lambda_A(x) = te^{i2\pi s}$, for some $t, s \in [0,1]$, then $x \in U(\lambda_A, (t,s))$. Because $U(\lambda_A, (t,s))$ is a $\mathbb{Z}_2$ -graded vector subspace of V, $x = x_0 + x_1$, where $x_0 \in V_0 \cap U(\lambda_A, (t,s))$ and $x_1 \in V_1 \cap U(\lambda_A, (t,s))$. Since $te^{i2\pi s} = \lambda_A(x) = \lambda_A(x_0 + x_1) \geq \lambda_A(x_0) \wedge \lambda_A(x_1)$, if $\lambda_A(x_0) \geq \lambda_A(x_1)$, then $te^{i2\pi s} \geq \lambda_A(x_1) \geq te^{i2\pi s}$. As a result, $\lambda_A(x_1) = te^{i2\pi s}$. Similarly, if $\lambda_A(x_1) \geq \lambda_A(x_0)$, then $\lambda_A(x_0) = te^{i2\pi s}$. Hence

$$\begin{aligned} \lambda_A(x) &= te^{i2\pi s} \\ &= \{\lambda_A(x_0) \wedge \lambda_A(x_1) | x = x_0 + x_1\} \\ &= \{\lambda_{A_0}(x_0) \wedge \lambda_{A_1}(x_1) | x = x_0 + x_1\} \\ &= \sup_{x=a+b} \{\lambda_{\mathfrak{a}_0}(a) \wedge \lambda_{\mathfrak{a}_1}(b)\} \\ &= \lambda_{\mathfrak{a}_0+\mathfrak{a}_1}(x) \\ &= \lambda_{A_0 \oplus A_1}(x) \qquad (\ Since\ \mathfrak{a}_0 \cap \mathfrak{a}_1 = (0,1)\ ). \end{aligned}$$

Similarly, if we assume that $\rho_A(x) = te^{i2\pi s}$, for some $t, s \in [0,1]$, then $x \in L(\rho_A, (t,s))$. Because $L(\rho_A, (t,s))$ is a $\mathbb{Z}_2$-graded vector subspace of V, $x = x_0 + x_1$, where $x_0 \in V_0 \cap L(\rho_A, (t,s))$ and $x_1 \in V_1 \cap L(\rho_A, (t,s))$. Since $te^{i2\pi s} = \rho_A(x) = \rho_A(x_0+x_1) \leq \rho_A(x_0) \wedge \rho_A(x_1)$, if $\rho_A(x_0) \geq \rho_A(x_1)$, then $te^{i2\pi s} \leq \rho_A(x_0) \leq te^{i2\pi s}$. As a result, $\rho_A(x_0) = te^{i2\pi s}$. Similarly, if

$\rho_A(x_1) \geq \rho_A(x_0)$, then $\rho_A(x_1) = te^{i2\pi s}$. Hence

$$\begin{aligned}\rho_A(x) &= te^{i2\pi s} \\ &= \{\rho_A(x_0) \vee \rho_A(x_1) | x = x_0 + x_1\} \\ &= \{\rho_{A_0}(x_0) \vee \rho_{A_1}(x_1) | x = x_0 + x_1\} \\ &= \inf_{x=a+b} \{\rho_{\mathfrak{a}_0}(a) \vee \rho_{\mathfrak{a}_1}(b)\} \\ &= \rho_{\mathfrak{a}_0+\mathfrak{a}_1}(x) \\ &= \rho_{A_0 \oplus A_1}(x).\end{aligned}$$

Therefore, $A = A_0 \oplus A_1$. Consequently, $A = (\lambda_A, \rho_A)$ is a CIF vector subspace of V. Let $x, y \in V$. Assume that $\lambda_A(y) \geq \lambda_A(x) \geq te^{i2\pi s}$, where $t, s \in [0, 1]$. Then $x, y \in U(\lambda_A, (t, s))$. Because $U(\lambda_A, (t, s))$ is a lie sub-superalgebra of V, $[x, y] \in U(\lambda_A, (t, s))$. Hence, $\lambda_A([x, y]) \geq te^{i2\pi s} = \lambda_A(x) \wedge \lambda_A(y)$.
Furthermore, assume that $\rho_A(x) \leq te^{i2\pi s} \leq \rho_A(y)$ for some $t, s \in [0, 1]$. Then $x, y \in L(\rho_A, (t, s))$. Because $L(\rho_A, (t, s))$ is a lie sub-superalgebra of V, $[x, y] \in L(\rho_A, (t, s))$. Hence, $\rho_A([x, y]) \leq te^{i2\pi s} \leq \rho_A(x) \vee \rho_A(y)$. Accordingly, $A = (\lambda_A, \rho_A)$ is CIF lie sub-superalgebra of V. The case of CIF ideal is similar to show. □

Let $A = (\lambda_A, \rho_A)$, $B = (\lambda_B, \rho_B)$ be CIF vector subspaces of V, where $\lambda_A = r_A e^{i2\pi\omega_A}$, $\lambda_B = r_B e^{i2\pi\omega_B}$ and $\rho_A = \hat{r}_A e^{i2\pi\hat{\omega}_A}$, $\rho_B = \hat{r}_B e^{i2\pi\hat{\omega}_B}$. We recall that if A is homogenous with B, then, by definition 2.4, the CIF set $A + B = (\lambda_{A+B}, \rho_{A+B})$ of V is defined by

$$\begin{aligned}\lambda_{A+B}(x) &= \sup_{x=a+b} \{(r_A(a) \wedge r_B(b))\} e^{i2\pi \sup_{x=a+b}\{(\omega_A(a) \wedge \omega_B(b))\}} \\ &= r_{A+B}(x) e^{i2\pi\omega_{A+B}(x)},\end{aligned}$$

and

$$\begin{aligned}\rho_{A+B}(x) &= \inf_{x=a+b} \{(\hat{r}_A(a) \vee \hat{r}_B(b))\} e^{i2\pi \inf_{x=a+b}\{(\hat{\omega}_A(a) \vee \hat{\omega}_B(b))\}} \\ &= \hat{r}_{A+B}(x) e^{i2\pi\hat{\omega}_{A+B}(x)}.\end{aligned}$$

We show that the following two results hold.

Theorem 3.11. If $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ are CIF lie sub-superalgebras (CIF ideals) of $V = V_0 + V_1$, respectively, then so is $A + B = (\lambda_{A+B}, \rho_{A+B})$.

Proof. For $\alpha = 0, 1$, define $(A + B)_\alpha = (\lambda_{(A+B)_\alpha}, \rho_{(A+B)_\alpha})$, where $\lambda_{(A+B)_\alpha} = \lambda_{A_\alpha} + \lambda_{B_\alpha}$ and $\rho_{(A+B)_\alpha} = \rho_{A_\alpha} + \rho_{B_\alpha}$. By Lemma 2.1, we know that they are CIF subspaces of V_α (respectively). Also, for $\alpha = 0, 1$, define $(\mathfrak{a} + \mathfrak{b})_\alpha = (\lambda_{(\mathfrak{a}+\mathfrak{b})_\alpha}, \rho_{(\mathfrak{a}+\mathfrak{b})_\alpha})$, where $\lambda_{(\mathfrak{a}+\mathfrak{b})_\alpha} = \lambda_{\mathfrak{a}_\alpha} + \lambda_{\mathfrak{b}_\alpha}$ and $\rho_{(\mathfrak{a}+\mathfrak{b})_\alpha} = \rho_{\mathfrak{a}_\alpha} + \rho_{\mathfrak{b}_\alpha}$. Obviously, $(\mathfrak{a} + \mathfrak{b})_\alpha$ are extensions of $(A+B)_\alpha$ for $\alpha = 0, 1$ (respectively).

Let $x \in V$. Then

$$\lambda_{(A+B)}(x) = \sup_{x=a+b}\{\lambda_A(a) \wedge \lambda_B(b)\}$$
$$= \sup_{x=a+b}\{\lambda_{\mathfrak{a}_0+\mathfrak{a}_1}(a) \wedge \lambda_{\mathfrak{b}_0+\mathfrak{b}_1}(b)\}$$
$$= \sup_{x=a+b}\{\sup_{a=m+n}\{\lambda_{\mathfrak{a}_0}(m) \wedge \lambda_{\mathfrak{a}_1}(n)\} \wedge \sup_{b=k+l}\{\lambda_{\mathfrak{b}_0}(k) \wedge \lambda_{\mathfrak{b}_1}(l)\}\}$$
$$= \sup_{x=a+b}\{\sup_{a=m+n}\{r_{\mathfrak{a}_0}(m)e^{i2\pi\omega_{\mathfrak{a}_0}(m)} \wedge r_{\mathfrak{a}_1}(n)e^{i2\pi\omega_{\mathfrak{a}_1}(n)}\}$$
$$\wedge \sup_{b=k+l}\{r_{\mathfrak{b}_0}(k)e^{i2\pi\omega_{\mathfrak{b}_0}(k)} \wedge r_{\mathfrak{b}_1}(l)e^{i2\pi\omega_{\mathfrak{b}_1}(l)}\}\}$$
$$= \sup_{x=a+b}\{\sup_{a+b=m+n+k+l}\{r_{\mathfrak{a}_0}(m)e^{i2\pi\omega_{\mathfrak{a}_0}(m)} \wedge r_{\mathfrak{a}_1}(n)e^{i2\pi\omega_{\mathfrak{a}_1}(n)}$$
$$\wedge r_{\mathfrak{b}_0}(k)e^{i2\pi\omega_{\mathfrak{b}_0}(k)} \wedge r_{\mathfrak{b}_1}(l)e^{i2\pi\omega_{\mathfrak{b}_1}(l)}\}\}$$
$$= \sup_{a+b=m+n+k+l}\{r_{\mathfrak{a}_0}(m)e^{i2\pi\omega_{\mathfrak{a}_0}(m)} \wedge r_{\mathfrak{b}_0}(k)e^{i2\pi\omega_{\mathfrak{b}_0}(k)}\} \wedge \sup_{a+b=m+n+k+l}\{r_{\mathfrak{a}_1}(n)e^{i2\pi\omega_{\mathfrak{a}_1}(n)}$$
$$\wedge r_{\mathfrak{b}_1}(l)e^{i2\pi\omega_{\mathfrak{b}_1}(l)}\}$$
$$= \sup_{a+b=m+n+k+l}\{\sup_{m+k}\{(r_{\mathfrak{a}_0}(m) \wedge r_{\mathfrak{b}_0}(k))e^{i2\pi(\omega_{\mathfrak{a}_0}(m)\wedge\omega_{\mathfrak{b}_0}(k))}\}$$
$$\wedge \sup_{n+l}\{(r_{\mathfrak{a}_1}(n) \wedge r_{\mathfrak{b}_1}(l))e^{i2\pi(\omega_{\mathfrak{a}_1}(n)\wedge\omega_{\mathfrak{b}_1}(l))}\}\}$$
$$= \sup_{x=m+n+k+l}\{r_{\mathfrak{a}_0+\mathfrak{b}_0}(m+k)e^{i2\pi\omega_{\mathfrak{a}_0+\mathfrak{b}_0}(m+k)} \wedge r_{\mathfrak{a}_1+\mathfrak{b}_1}(n+l)e^{i2\pi\omega_{\mathfrak{a}_1+\mathfrak{b}_1}(n+l)}\}$$
$$= r_{\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{a}_1+\mathfrak{b}_1}(x)e^{i2\pi\omega_{\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{a}_1+\mathfrak{b}_1}(x)}$$
$$= \lambda_{(\mathfrak{a}+\mathfrak{b})_0+(\mathfrak{a}+\mathfrak{b})_1}(x),$$

and

$$\rho_{(A+B)}(x) = \inf_{x=a+b}\{\rho_A(a) \vee \rho_B(b)\}$$
$$= \inf_{x=a+b}\{\rho_{\mathfrak{a}_0+\mathfrak{a}_1}(a) \vee \rho_{\mathfrak{b}_0+\mathfrak{b}_1}(b)\}$$
$$= \inf_{x=a+b}\{\inf_{a=m+n}\{\rho_{\mathfrak{a}_0}(m) \vee \rho_{\mathfrak{a}_1}(n)\} \vee \inf_{b=k+l}\{\rho_{\mathfrak{b}_0}(k) \vee \rho_{\mathfrak{b}_1}(l)\}\}$$
$$= \inf_{x=a+b}\{\inf_{a=m+n}\{\hat{r}_{\mathfrak{a}_0}(m)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(m)} \vee \hat{r}_{\mathfrak{a}_1}(n)e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(n)}\}$$
$$\vee \inf_{b=k+l}\{\hat{r}_{\mathfrak{b}_0}(k)e^{i2\pi\hat{\omega}_{\mathfrak{b}_0}(k)} \vee \hat{r}_{\mathfrak{b}_1}(l)e^{i2\pi\hat{\omega}_{\mathfrak{b}_1}(l)}\}\}$$
$$= \inf_{x=a+b}\{\inf_{a+b=m+n+k+l}\{\hat{r}_{\mathfrak{a}_0}(m)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(m)} \vee \hat{r}_{\mathfrak{a}_1}(n)e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(n)}$$
$$\vee \hat{r}_{\mathfrak{b}_0}(k)e^{i2\pi\hat{\omega}_{\mathfrak{b}_0}(k)} \vee \hat{r}_{\mathfrak{b}_1}(l)e^{i2\pi\hat{\omega}_{\mathfrak{b}_1}(l)}\}\}$$
$$= \inf_{a+b=m+n+k+l}\{\hat{r}_{\mathfrak{a}_0}(m)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(m)} \vee \hat{r}_{\mathfrak{b}_0}(k)e^{i2\pi\hat{\omega}_{\mathfrak{b}_0}(k)}\}$$
$$\vee \inf_{a+b=m+n+k+l}\{\hat{r}_{\mathfrak{a}_1}(n)e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(n)} \vee \hat{r}_{\mathfrak{b}_1}(l)e^{i2\pi\hat{\omega}_{\mathfrak{b}_1}(l)}\}$$
$$= \inf_{a+b=m+n+k+l}\{\inf_{m+k}\{(\hat{r}_{\mathfrak{a}_0}(m) \vee \hat{r}_{\mathfrak{b}_0}(k))e^{i2\pi(\hat{\omega}_{\mathfrak{a}_0}(m)\vee\hat{\omega}_{\mathfrak{b}_0}(k))}\}$$
$$\vee \inf_{n+l}\{(\hat{r}_{\mathfrak{a}_1}(n) \vee \hat{r}_{\mathfrak{b}_1}(l))e^{i2\pi(\hat{\omega}_{\mathfrak{a}_1}(n)\vee\hat{\omega}_{\mathfrak{b}_1}(l))}\}\}$$
$$= \inf_{x=m+n+k+l}\{\hat{r}_{\mathfrak{a}_0+\mathfrak{b}_0}(m+k)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0+\mathfrak{b}_0}(m+k)} \vee \hat{r}_{\mathfrak{a}_1+\mathfrak{b}_1}(n+l)e^{i2\pi\hat{\omega}_{\mathfrak{a}_1+\mathfrak{b}_1}(n+l)}\}$$
$$= \hat{r}_{\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{a}_1+\mathfrak{b}_1}(x)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0+\mathfrak{b}_0+\mathfrak{a}_1+\mathfrak{b}_1}(x)}$$
$$= \rho_{(\mathfrak{a}+\mathfrak{b})_0+(\mathfrak{a}+\mathfrak{b})_1}(x).$$

Moreover, if $0 \neq x \in V$ then

$$\begin{aligned}\lambda_{(\mathfrak{a}+\mathfrak{b})_0}(x) \wedge \lambda_{(\mathfrak{a}+\mathfrak{b})_1}(x) &= \sup_{x=a+b}\{\lambda_{\mathfrak{a}_0}(a) \wedge \lambda_{\mathfrak{b}_0}(b)\} \wedge \sup_{x=a+b}\{\lambda_{\mathfrak{a}_1}(a) \wedge \lambda_{\mathfrak{b}_1}(b)\} \\ &= 0\end{aligned}$$

$$\begin{aligned}\rho_{(\mathfrak{a}+\mathfrak{b})_0}(x) \vee \rho_{(\mathfrak{a}+\mathfrak{b})_1}(x) &= \inf_{x=a+b}\{\rho_{\mathfrak{a}_0}(a) \vee \rho_{\mathfrak{b}_0}(b)\} \vee \inf_{x=a+b}\{\rho_{\mathfrak{a}_1}(a) \vee \rho_{\mathfrak{b}_1}(b)\} \\ &= 1.\end{aligned}$$

As a result, $A+B$ is a $\mathbb{Z}_2$-CIF vector subspaces of V.
(1) Let $x, y \in V$, we need to show that $\lambda_{A+B}([x,y]) \geq \lambda_{A+B}(x) \vee \lambda_{A+B}(y)$ and $\rho_{A+B}([x,y]) \leq \rho_{A+B}(x) \wedge \rho_{A+B}(y)$. Suppose that $\lambda_{A+B}([x,y]) < \lambda_{A+B}(x) \vee \lambda_{A+B}(y)$. Without loss of generality, we may assume that $\lambda_{A+B}([x,y]) < \lambda_{A+B}(x)$. Then

$$\lambda_{A+B}([x,y]) = r_{A+B}([x,y])e^{i2\pi\omega_{A+B}([x,y])} < r_{A+B}(x)e^{i2\pi\omega_{A+B}(x)}.$$

Hence, $r_{A+B}([x,y]) < r_{A+B}(x)$ or $\omega_{A+B}([x,y]) < \omega_{A+B}(x)$, since $A+B$ is homogenous. Again, without loss of generality, we may assume that $r_{A+B}([x,y]) < r_{A+B}(x)$. Choose a number $t \in [0,1]$, such that $r_{A+B}([x,y]) < t < r_{A+B}(x)$. Then there exist $a, b \in V$ with $x = a+b$ such that $r_A(a) > t$ and $r_B(b) > t$. So,

$$\begin{aligned}r_{A+B}([x,y]) &= \sup_{[x,y]=[a',y]+[b',y]}\{r_A([a',y]) \wedge r_B([b',y])\} \\ &\geq \sup_{[x,y]=[a',y]+[b',y]}\{r_A(a') \wedge r_B(b')\} \ (A, B \ are \ ideals) \\ &> t > r_{A+B}([x,y]),\end{aligned}$$

which is a contradiction. Similarly, suppose that $\rho_{A+B}([x,y]) > \rho_{A+B}(x) \wedge \rho_{A+B}(y)$. Then $\rho_{A+B}([x,y]) > \rho_{A+B}(x)$ or $\rho_{A+B}([x,y]) > \rho_{A+B}(y)$. Without loss of generality, we may assume that $\rho_{A+B}([x,y]) > \rho_{A+B}(x)$. Then $\hat{r}_{A+B}([x,y]) > \hat{r}_{A+B}(x)$ or $\hat{\omega}_{A+B}([x,y]) > \hat{\omega}_{A+B}(x)$, since $A+B$ is homogenous. Again, without loss of generality, we may assume that $\hat{r}_{A+B}([x,y]) > \hat{r}_{A+B}(x)$. Choose a number $t \in [0,1]$, such that $\hat{r}_{A+B}([x,y]) > t > \hat{r}_{A+B}(x)$. Then there exist $a, b \in V$ with $x = a+b$ such that $\hat{r}_A(a) < t$ and $\hat{r}_B(b) < t$. So,

$$\begin{aligned}\hat{r}_{A+B}([x,y]) &= \inf_{[x,y]=[a',y]+[b',y]}\{\hat{r}_A([a',y]) \vee \hat{r}_B([b',y])\} \\ &\leq \inf_{[x,y]=[a',y]+[b',y]}\{\hat{r}_A(a') \vee \hat{r}_B(b')\} \ (A, B \ are \ ideals) \\ &< t < \hat{r}_{A+B}([x,y]),\end{aligned}$$

which is a contradiction. Therefore, $A+B = (\lambda_{A+B}, \rho_{A+B})$ is a CIF ideal of V.
(2) Let $x, y \in V$, we need to show that $\lambda_{A+B}([x,y]) \geq \lambda_{A+B}(x) \wedge \lambda_{A+B}(y)$ and $\rho_{A+B}([x,y]) \leq \rho_{A+B}(x) \vee \rho_{A+B}(y)$. Suppose that $\lambda_{A+B}([x,y]) < \lambda_{A+B}(x) \wedge \lambda_{A+B}(y)$, then

$$r_{A+B}([x,y])e^{i2\pi\omega_{A+B}([x,y])} < r_{A+B}(x)e^{i2\pi\omega_{A+B}(x)} \wedge r_{A+B}(y)e^{i2\pi\omega_{A+B}(y)}$$

implies that $r_{A+B}([x,y])e^{i2\pi\omega_{A+B}([x,y])} < (r_{A+B}(x) \wedge r_{A+B}(y))e^{i2\pi(\omega_{A+B}(x) \wedge \omega_{A+B}(y))}$. Hence,

$$r_{A+B}([x,y]) < r_{A+B}(x) \wedge r_{A+B}(y) \quad \text{or} \quad \omega_{A+B}([x,y]) < \omega_{A+B}(x) \wedge \omega_{A+B}(y).$$

If $r_{A+B}([x,y]) < r_{A+B}(x) \wedge r_{A+B}(y)$, then $r_{A+B}([x,y]) < r_{A+B}(x)$ and $r_{A+B}([x,y]) < r_{A+B}(y)$. Choose a number $t \in [0,1]$ such that $r_{A+B}([x,y]) < t < r_{A+B}(x) \wedge r_{A+B}(y)$. Then there exist $a, b, c, d \in V$ with $x = a+b$ and $y = c+d$ such that $r_A(a) > t$, $r_B(b) > t$, $r_A(c) > t$,

$r_B(d) > t$. Thus,

$$\begin{aligned}
r_{A+B}([x,y]) &= \sup_{[x,y]=[a',y]+[b',y]} \{r_A([a',y]) \wedge r_B([b',y])\} \\
&= \sup_{[x,y]=[a'+b',c'+d']} \{r_A([a',c'+d']) \wedge r_B([b',c'+d'])\} \\
&= \sup_{[x,y]=[a'+b',c'+d']} \{r_A([a',c']) \wedge r_A([a',d']) \wedge r_B([b',c']) \wedge r_B([b',d'])\} \\
&= \sup_{x=a'+b',y=c'+d'} \{ \sup_{[a',c'+d']} \{r_A(a') \wedge r_A(c') \wedge r_A(d')\} \\
&\wedge \sup_{[a',c'+d']} \{r_A(a') \wedge r_A(c') \wedge r_A(d')\}\} \\
&= \sup_{x=a'+b'} \{r_A(a') \wedge r_B(b')\} \wedge \sup_{y=c'+d'} \{r_A(c') \wedge r_A(d')\} \\
&\wedge \sup_{y=c'+d'} \{r_B(c') \wedge r_B(d')\} \\
&\geq \sup_{x=a'+b'} \{r_A(a') \wedge r_B(b')\} \wedge r_A(c) \wedge r_B(d) \\
&\geq r_A(a) \wedge r_B(b) \wedge r_A(c) \wedge r_B(d) \\
&> t > r_{A+B}([x,y]),
\end{aligned}$$

a contradiction. The other case can be proved similarly, so $\lambda_{A+B}([x,y]) \geq \lambda_{A+B}(x) \wedge \lambda_{A+B}(y)$. Similarly, if we assume that $\rho_{A+B}([x,y]) > \rho_{A+B}(x) \vee \rho_{A+B}(y)$, then

$$\hat{r}_{A+B}([x,y])e^{i2\pi\hat{\omega}_{A+B}([x,y])} > (\hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y))e^{i2\pi(\hat{\omega}_{A+B}(x)\vee\hat{\omega}_{A+B}(y))},$$

which implies that $\hat{r}_{A+B}([x,y]) > \hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y)$ or $\hat{\omega}_{A+B}([x,y]) > \hat{\omega}_{A+B}(x) \vee \hat{\omega}_{A+B}(y)$. Without loss of generality we may assume that $\hat{r}_{A+B}([x,y]) > \hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y)$, since A, B are homogenous. Choose a number $t \in [0,1]$, such that $\hat{r}_{A+B}([x,y]) > t > \hat{r}_{A+B}(x) \vee \hat{r}_{A+B}(y)$. Then there exist $a, b, c, d \in V$ with $x = a + b$ and $y = c + d$ such that $\hat{r}_A(a) < t$, $\hat{r}_B(b) < t$ and $\hat{r}_A(c) < t$, $\hat{r}_B(d) < t$. Therefore,

$$\begin{aligned}
\hat{r}_{A+B}([x,y]) &= \inf_{[x,y]=[a',y]+[b',y]} \{\hat{r}_A([a',y]) \vee \hat{r}_B([b',y])\} \\
&= \inf_{[x,y]=[a'+b',c'+d']} \{\hat{r}_A([a',c'+d']) \vee \hat{r}_B([b',c'+d'])\} \\
&= \inf_{[x,y]=[a'+b',c'+d']} \{\hat{r}_A([a',c']) \vee \hat{r}_A([a',d']) \vee \hat{r}_B([b',c']) \vee \hat{r}_B([b',d'])\} \\
&= \inf_{x=a'+b',y=c'+d'} \{ \inf_{[a',c'+d']} \{\hat{r}_A(a') \vee \hat{r}_A(c') \vee \hat{r}_A(d')\} \\
&\vee \inf_{[a',c'+d']} \{\hat{r}_A(a') \vee \hat{r}_A(c') \vee \hat{r}_A(d')\}\} \\
&= \inf_{x=a'+b'} \{\hat{r}_A(a') \vee \hat{r}_B(b')\} \vee \inf_{y=c'+d'} \{\hat{r}_A(c') \vee \hat{r}_A(d')\} \\
&\vee \inf_{y=c'+d'} \{\hat{r}_B(c') \vee \hat{r}_B(d')\} \\
&\leq \inf_{x=a'+b'} \{\hat{r}_A(a') \vee \hat{r}_B(b')\} \vee \hat{r}_A(c) \vee \hat{r}_B(d) \\
&\leq \hat{r}_A(a) \vee \hat{r}_B(b) \vee \hat{r}_A(c) \vee \hat{r}_B(d) \\
&< t < \hat{r}_{A+B}([x,y]),
\end{aligned}$$

a contradiction. The other case can be proved similarly, so $\rho_{A+B}([x,y]) \leq \rho_{A+B}(x) \vee \rho_{A+B}(y)$. Accordingly, we conclude that $A + B = (\lambda_{A+B}, \rho_{A+B})$ is a CIF lie subsuperalgebra of V. □

Theorem 3.12. If $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ are CIF lie sub-superalgebras (respectively CIF ideals) of $V = V_0 + V_1$, then so is $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$.

Proof. By Definition 3.1 and Definition 3.3, $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$. For $\alpha = 0, 1$ define $(A\cap B)_\alpha = (\lambda_{(A\cap B)_\alpha}, \rho_{(A\cap B)_\alpha})$, where $\lambda_{(A\cap B)_\alpha} = \lambda_{A_\alpha} \cap \lambda_{B_\alpha}$ and $\rho_{(A\cap B)_\alpha} = \rho_{A_\alpha} \cap \rho_{B_\alpha}$. Then, by Lemma 2.5, they are CIF subspaces of V_α. Also, for $\alpha = 0, 1$, we can define $(\mathfrak{a}\cap\mathfrak{b})_\alpha = (\lambda_{(\mathfrak{a}\cap\mathfrak{b})_\alpha}, \rho_{(\mathfrak{a}\cap\mathfrak{b})_\alpha})$, where $\lambda_{(\mathfrak{a}\cap\mathfrak{b})_\alpha} = \lambda_{\mathfrak{a}_\alpha} \cap \lambda_{\mathfrak{b}_\alpha}$ and $\rho_{(\mathfrak{a}\cap\mathfrak{b})_\alpha} = \rho_{\mathfrak{a}_\alpha} \cap \rho_{\mathfrak{b}_\alpha}$. Obviously, $(\mathfrak{a}\cap\mathfrak{b})_\alpha$ are extensions of $(A \cap B)_\alpha$ for $\alpha = 0, 1$ (respectively). Moreover, it is easy to check that $(\mathfrak{a}\cap\mathfrak{b})_0 \cap (\mathfrak{a}\cap\mathfrak{b})_1 = (0, 1)$ for any nonzero $x \in V$. Let $x \in V$, as A, B are homogenous, then

$$
\begin{aligned}
(\lambda_{(\mathfrak{a}\cap\mathfrak{b})_0} &+ \lambda_{(\mathfrak{a}\cap\mathfrak{b})_1})(x) = \sup_{x=a+b}\{\lambda_{(\mathfrak{a}\cap\mathfrak{b})_0}(a) \wedge \lambda_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)\} \\
&= \sup_{x=a+b}\{r_{(\mathfrak{a}\cap\mathfrak{b})_0}(a)e^{i2\pi\omega_{(\mathfrak{a}\cap\mathfrak{b})_0}(a)} \wedge r_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)e^{i2\pi\omega_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)}\} \\
&= \sup_{x=a+b}\{(r_{\mathfrak{a}_0}(a) \wedge r_{\mathfrak{b}_0}(a))e^{i2\pi(\omega_{\mathfrak{a}_0}(a)\wedge\omega_{\mathfrak{b}_0}(a))} \wedge (r_{\mathfrak{a}_1}(b) \wedge r_{\mathfrak{b}_1}(b))e^{i2\pi(\omega_{\mathfrak{a}_1}(b)\wedge\omega_{\mathfrak{b}_1}(b))}\} \\
&= \sup_{x=a+b}\{(r_{\mathfrak{a}_0}(a) \wedge r_{\mathfrak{a}_1}(b))e^{i2\pi(\omega_{\mathfrak{a}_0}(a)\wedge\omega_{\mathfrak{a}_1}(b))}\} \\
&\wedge \sup_{x=a+b}\{(r_{\mathfrak{b}_0}(a) \wedge r_{\mathfrak{b}_1}(b))e^{i2\pi(\omega_{\mathfrak{b}_0}(a)\wedge\omega_{\mathfrak{b}_1}(b))}\} \\
&= r_A(x)e^{i2\pi\omega_A(x)} \wedge r_B(x)e^{i2\pi\omega_B(x)} \\
&= \lambda_A(x) \wedge \lambda_B(x) \\
&= \lambda_{A\cap B}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
(\rho_{(\mathfrak{a}\cap\mathfrak{b})_0} &+ \rho_{(\mathfrak{a}\cap\mathfrak{b})_1})(x) = \inf_{x=a+b}\{\rho_{(\mathfrak{a}\cap\mathfrak{b})_0}(a) \vee \rho_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)\} \\
&= \inf_{x=a+b}\{\hat{r}_{(\mathfrak{a}\cap\mathfrak{b})_0}(a)e^{i2\pi\hat{\omega}_{(\mathfrak{a}\cap\mathfrak{b})_0}(a)} \vee \hat{r}_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)e^{i2\pi\hat{\omega}_{(\mathfrak{a}\cap\mathfrak{b})_1}(b)}\} \\
&= \inf_{x=a+b}\{(\hat{r}_{\mathfrak{a}_0}(a) \vee \hat{r}_{\mathfrak{b}_0}(a))e^{i2\pi(\hat{\omega}_{\mathfrak{a}_0}(a)\vee\hat{\omega}_{\mathfrak{b}_0}(a))} \vee (\hat{r}_{\mathfrak{a}_1}(b) \vee \hat{r}_{\mathfrak{b}_1}(b))e^{i2\pi(\hat{\omega}_{\mathfrak{a}_1}(b)\vee\hat{\omega}_{\mathfrak{b}_1}(b))}\} \\
&= \inf_{x=a+b}\{(\hat{r}_{\mathfrak{a}_0}(a) \vee \hat{r}_{\mathfrak{a}_1}(b))e^{i2\pi(\hat{\omega}_{\mathfrak{a}_0}(a)\vee\hat{\omega}_{\mathfrak{a}_1}(b))}\} \\
&\vee \inf_{x=a+b}\{(\hat{r}_{\mathfrak{b}_0}(a) \vee \hat{r}_{\mathfrak{b}_1}(b))e^{i2\pi(\hat{\omega}_{\mathfrak{b}_0}(a)\vee\hat{\omega}_{\mathfrak{b}_1}(b))}\} \\
&= \hat{r}_A(x)e^{i2\pi\hat{\omega}_A(x)} \vee \hat{r}_B(x)e^{i2\pi\hat{\omega}_B(x)} \\
&= \rho_A(x) \vee \rho_B(x) \\
&= \rho_{A\cap B}(x).
\end{aligned}
$$

Hence, $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$ is a CIF vector subspace of V. To show that $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$ is a CIF lie subsuperalgebra of V, let $x, y \in V$. Then

$$
\begin{aligned}
\lambda_{A\cap B}([x,y]) &= \lambda_A([x,y]) \wedge \lambda_B([x,y]) \\
&= r_A([x,y])e^{i2\pi\omega_A([x,y])} \wedge r_B([x,y])e^{i2\pi\omega_B([x,y])} \\
&\geq (r_A(x) \wedge r_A(y))e^{i2\pi(\omega_A(x)\wedge\omega_A(y))} \wedge (r_B(x) \wedge r_B(y))e^{i2\pi(\omega_B(x)\wedge\omega_B(y))} \\
&= (r_A(x) \wedge r_B(x))e^{i2\pi(\omega_A(x)\wedge\omega_B(x))} \wedge (r_A(y) \wedge r_B(y))e^{i2\pi(\omega_A(y)\wedge\omega_B(y))} \\
&= \lambda_{A\cap B}(x) \wedge \lambda_{A\cap B}(y),
\end{aligned}
$$

and

$$\begin{aligned}\rho_{A\cap B}([x,y]) &= \rho_A([x,y]) \vee \rho_B([x,y]) \\ &= \hat{r}_A([x,y])e^{i2\pi\hat{\omega}_A([x,y])} \vee \hat{r}_B([x,y])e^{i2\pi\hat{\omega}_B([x,y])} \\ &\leq (\hat{r}_A(x) \vee \hat{r}_A(y))e^{i2\pi(\hat{\omega}_A(x)\vee\hat{\omega}_A(y))} \vee (\hat{r}_B(x) \vee \hat{r}_B(y))e^{i2\pi(\hat{\omega}_B(x)\vee\hat{\omega}_B(y))} \\ &= (\hat{r}_A(x) \vee \hat{r}_B(x))e^{i2\pi(\hat{\omega}_A(x)\vee\hat{\omega}_B(x))} \vee (\hat{r}_A(y) \vee \hat{r}_B(y))e^{i2\pi(\hat{\omega}_A(y)\vee\hat{\omega}_B(y))} \\ &= \rho_{A\cap B}(x) \vee \rho_{A\cap B}(y).\end{aligned}$$

Consequently, $A \cap B = (\lambda_{A\cap B}, \rho_{A\cap B})$ is a CIF lie subsuperalgebra of V. □

4 On Lie-superalgebra Anti-homomoiphisms

Recall that if $\phi : V \to V'$ is a linear map between lie superalgebras such that $\phi(a_\alpha b_\beta) = (-1)^{\alpha\beta}\phi(b_\beta)\phi(a_\alpha)$ for all $a_\alpha, b_\beta \in h(V)$, $\alpha, \beta = 0, 1$, then ϕ is called an anti-homomorphism from V into V'. In this case, if $a_\alpha, b_\beta \in h(V)$, where $\alpha, \beta = 0, 1$, then

$$\begin{aligned}\phi([a_\alpha, b_\beta]) &= \phi(a_\alpha b_\beta - (-1)^{\alpha\beta} b_\beta a_\alpha) \\ &= \phi(a_\alpha b_\beta) - (-1)^{\alpha\beta}\phi(b_\beta a_\alpha) \\ &= (-1)^{\alpha\beta}\phi(b_\beta)\phi(a_\alpha) - \phi(a_\alpha)\phi(b_\beta) \\ &= -(\phi(a_\alpha)\phi(b_\beta) - (-1)^{\alpha\beta}\phi(b_\beta)\phi(a_\alpha)) \\ &= -[\phi(a_\alpha), \phi(b_\beta)].\end{aligned}$$

As a result, if $x = x_0 + x_1, y = y_0 + y_1 \in V$, then

$$\begin{aligned}\phi([x,y]) &= \phi([x_0,y_0] + [x_0,y_1] + [x_1,y_0] + [x_1,y_1]) \\ &= \phi([x_0,y_0]) + \phi([x_0,y_1]) + \phi([x_1,y_0]) + \phi([x_1,y_1]) \\ &= -([\phi(x_0),\phi(y_0)] + [\phi(x_0),\phi(y_1)] + [\phi(x_1),\phi(y_0)] + [\phi(x_1),\phi(y_1)]) \\ &= -[\phi(x), \phi(y)].\end{aligned}$$

Therefore, the following definition is an equivalent definition of anti-homomorphism of lie superalgebras.

Definition 4.1. If $\phi : V \to V'$ is a linear map between lie superalgebras $V,\ V'$ which satisfies:

$$\phi(V_\alpha) \subseteq V'_\alpha,\ (\alpha = 0, 1), \tag{4.1}$$

$$\phi([x,y]) = -[\phi(x), \phi(y)] \tag{4.2}$$

Then ϕ is called an anti-homomorphism of lie-superalgebras.

For example, the transpose and inversion maps both give anti-homomorphisms in matrices.

Definition 4.2. Let $A = (\lambda_A, \rho_A)$ be a CIF set of V and let $x, y \in V$. Then $A = (\lambda_A, \rho_A)$ is called an anti-complex intuitionistic fuzzy lie sub-superalgebra of V (anti-CIF for short), if it satisfies the following conditions:
(1) $A = (\lambda_A, \rho_A)$ is a $\mathbb{Z}_2$-graded CIF vector subspace of V
(2) $\lambda_A(-[x,y]) \geq \lambda_A(x) \wedge \lambda_A(y)$ and $\rho_A(-[x,y]) \leq \rho_A(x) \vee \rho_A(y)$.
If the condition(2) is replaced by (3) $\lambda_A(-[x,y]) \geq \lambda_A(x) \vee \lambda_A(y)$ and $\rho_A(-[x,y]) \leq \rho_A(x) \wedge \rho_A(y)$, then $A = (\lambda_A, \rho_A)$ is called an anti-CIF ideal of V.

Proposition 4.3. Let $\phi : V \to V'$ be an anti-homomorphism of lie-superalgebras. If $A = (\lambda_A, \rho_A)$ is an anti-CIF lie sub-superalgebra (respectively an anti-CIF ideal) of V', then the CIF set $\phi^{-1}(A)$ of V is also an anti-CIF lie sub-superalgebra (respectively an anti-CIF ideal).

Proof. Let $x = x_0 + x_1 \in V$, then $\phi(x) = \phi(x_0) + \phi(x_1) \in V'$. Define $\phi^{-1}(A)_\alpha = (\lambda_{\phi^{-1}(A)_\alpha}, \rho_{\phi^{-1}(A)_\alpha})$, where $\lambda_{\phi^{-1}(A)_\alpha} = \phi^{-1}(\lambda_{A_\alpha})$ and $\rho_{\phi^{-1}(A)_\alpha} = \phi^{-1}(\rho_{A_\alpha})$, $\alpha = 0, 1$. Then, by Lemma 2.1, they are CIF subspaces of V_α, $\alpha = 0, 1$ (respectively). Define $\phi^{-1}(\mathfrak{a})_\alpha = (\lambda_{\phi^{-1}(\mathfrak{a})_\alpha}, \rho_{\phi^{-1}(\mathfrak{a})_\alpha})$, where $\lambda_{\phi^{-1}(\mathfrak{a})_\alpha} = \phi^{-1}(\lambda_{\mathfrak{a}_\alpha})$ and $\rho_{\phi^{-1}(\mathfrak{a})_\alpha} = \phi^{-1}(\rho_{\mathfrak{a}_\alpha})$, $\alpha = 0, 1$. Clearly,

$$\lambda_{\phi^{-1}(\mathfrak{a})_\alpha}(x) = \begin{cases} \lambda_{\phi^{-1}(A)_\alpha}(x) & : \quad x \in V_\alpha \\ 0 & : \quad x \notin V_\alpha \end{cases} \quad \textit{and} \quad \rho_{\phi^{-1}(\mathfrak{a})_\alpha}(x) = \begin{cases} \rho_{\phi^{-1}(A)_\alpha}(x) & : \quad x \in V_\alpha \\ 1 & : \quad x \notin V_\alpha \end{cases},$$

for $\alpha = 0, 1$. Which implies that $\phi^{-1}(\mathfrak{a})_\alpha$ are CIF vector subspace of V, for $\alpha = 0, 1$. Moreover, if $0 \neq x \in V$, then $\lambda_{\phi^{-1}(\mathfrak{a})_0}(x) \wedge \lambda_{\phi^{-1}(\mathfrak{a})_1}(x) = \phi^{-1}(\lambda_{\mathfrak{a}_0})(x) \wedge \phi^{-1}(\lambda_{\mathfrak{a}_1})(x) = \lambda_{\mathfrak{a}_0}(\phi(x)) \wedge \lambda_{\mathfrak{a}_1}(\phi(x)) = 0$. Similarly, if $0 \neq x \in V$, then $\rho_{\phi^{-1}(\mathfrak{a})_0}(x) \vee \rho_{\phi^{-1}(\mathfrak{a})_1}(x) = 1$. Furthermore, if $0 \neq x \in V$, then

$$\begin{aligned}
\lambda_{\phi^{-1}(\mathfrak{a})_0 + \phi^{-1}(\mathfrak{a})_1}(x) &= \sup_{x=a+b} \{\lambda_{\phi^{-1}(\mathfrak{a})_0}(a) \wedge \lambda_{\phi^{-1}(\mathfrak{a})_1}(b)\} \\
&= \sup_{x=a+b} \{\lambda_{\mathfrak{a}_0}(\phi(a)) \wedge \lambda_{\mathfrak{a}_1}(\phi(b))\}, \ (a = a_0 + a_1, \ b = b_0 + b_1) \\
&= \sup_{x=a+b} \{\lambda_{\mathfrak{a}_0}(\phi(a_0)) \wedge \lambda_{\mathfrak{a}_1}(\phi(b_1))\} \\
&= \sup_{x=a+b} \{r_{\mathfrak{a}_0}(\phi(a_0)) e^{i2\pi\omega_{\mathfrak{a}_0}(\phi(a_0))} \wedge r_{\mathfrak{a}_1}(\phi(b_1)) e^{i2\pi\omega_{\mathfrak{a}_1}(\phi(b_1))}\} \\
&= r_{\mathfrak{a}_0}(\phi(x_0)) e^{i2\pi\omega_{\mathfrak{a}_0}(\phi(x_0))} \wedge r_{\mathfrak{a}_1}(\phi(x_1)) e^{i2\pi\omega_{\mathfrak{a}_1}(\phi(x_1))} \\
&= \lambda_{\mathfrak{a}_0}(\phi(x_0)) \wedge \lambda_{\mathfrak{a}_1}(\phi(x_1)) \\
&= \lambda_{\mathfrak{a}_0 + \mathfrak{a}_1}(\phi(x)), \ (x = x_0 + x_1) \\
&= \lambda_A(\phi(x)) \\
&= \lambda_{\phi^{-1}(A)}(x),
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\phi^{-1}(\mathfrak{a})_0 + \phi^{-1}(\mathfrak{a})_1}(x) &= \inf_{x=a+b} \{\rho_{\phi^{-1}(\mathfrak{a})_0}(a) \vee \rho_{\phi^{-1}(\mathfrak{a})_1}(b)\} \\
&= \inf_{x=a+b} \{\rho_{\mathfrak{a}_0}(\phi(a)) \vee \rho_{\mathfrak{a}_1}(\phi(b))\}, \ (a = a_0 + a_1, \ b = b_0 + b_1) \\
&= \inf_{x=a+b} \{\rho_{\mathfrak{a}_0}(\phi(a_0)) \vee \rho_{\mathfrak{a}_1}(\phi(b_1))\} \\
&= \inf_{x=a+b} \{\hat{r}_{\mathfrak{a}_0}(\phi(a_0)) e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(\phi(a_0))} \vee \hat{r}_{\mathfrak{a}_1}(\phi(b_1)) e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(\phi(b_1))}\} \\
&= \hat{r}_{\mathfrak{a}_0}(\phi(x_0)) e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(\phi(x_0))} \vee \hat{r}_{\mathfrak{a}_1}(\phi(x_1)) e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(\phi(x_1))} \\
&= \rho_{\mathfrak{a}_0}(\phi(x_0)) \vee \rho_{\mathfrak{a}_1}(\phi(x_1)) \\
&= \rho_{\mathfrak{a}_0 + \mathfrak{a}_1}(\phi(x)), \ (x = x_0 + x_1) \\
&= \rho_A(\phi(x)) \\
&= \rho_{\phi^{-1}(A)}(x).
\end{aligned}$$

Hence, $\phi^{-1}(A) = \phi^{-1}(A)_0 \oplus \phi^{-1}(A)_1$ is a $\mathbb{Z}_2$-graded CIF vector subspace of V. Let $x, y \in V$. Then

$$\begin{aligned}
\lambda_{\phi^{-1}(A)}(-[x, y]) = \lambda_A(-\phi([x, y])) &= \lambda_A([\phi(x), \phi(y)]) \\
&\geq \lambda_A(\phi(x)) \wedge \lambda_A(\phi(y)) \\
&= \lambda_{\phi^{-1}(A)}(x) \wedge \lambda_{\phi^{-1}(A)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\phi^{-1}(A)}(-[x, y]) = \rho_A(-\phi([x, y])) &= \rho_A([\phi(x), \phi(y)]) \\
&\leq \rho_A(\phi(x)) \vee \rho_A(\phi(y)) \\
&= \rho_{\phi^{-1}(A)}(x) \vee \rho_{\phi^{-1}(A)}(y).
\end{aligned}$$

As a result, $\phi^{-1}(A)$ is an anti-CIF Lie subsuperalgebra of V. Finally, we show that $\phi^{-1}(A)$ is an anti-CIF ideal of V. Let $x, y \in V$, then

$$\begin{aligned}\lambda_{\phi^{-1}(A)}(-[x,y]) = \lambda_A(-\phi([x,y])) &= \lambda_A([\phi(x),\phi(y)]) \\ &\geq \lambda_A(\phi(x)) \vee \lambda_A(\phi(y)) \\ &= \lambda_{\phi^{-1}(A)}(x) \vee \lambda_{\phi^{-1}(A)}(y),\end{aligned}$$

and

$$\begin{aligned}\rho_{\phi^{-1}(A)}(-[x,y]) = \rho_A(-\phi([x,y])) &= \rho_A([\phi(x),\phi(y)]) \\ &\leq \rho_A(\phi(x)) \wedge \rho_A(\phi(y)) \\ &= \rho_{\phi^{-1}(A)}(x) \wedge \rho_{\phi^{-1}(A)}(y).\end{aligned}$$

Consequently, $\phi^{-1}(A)$ is an anti-CIF ideal of V. □

Proposition 4.4. Let $\phi : V \to V'$ be a surjective anti-homomorphism of lie-superalgebras. If $A = (\lambda_A, \rho_A)$ is an anti-CIF lie sub-superalgebra (respectively an anti-CIF ideal) of V, then the CIF set $\phi(A)$ of V' is also an anti-CIF lie sub-superalgebra (respectively an anti-CIF ideal).

Proof. By Definition 3.1 and Definition 3.3, $A = A_0 \oplus A_1$ where $A_0 = (\lambda_{A_0}, \rho_{A_0})$, $A_1 = (\lambda_{A_1}, \rho_{A_1})$ are CIF vector subspaces of V_0, V_1 (respectively). For $\alpha = 0, 1$, define $\phi(A)_\alpha = (\lambda_{\phi(A)_\alpha}, \rho_{\phi(A)_\alpha})$, where $\lambda_{\phi(A)_\alpha} = \phi(\lambda_{A_\alpha})$, $\rho_{\phi(A)_\alpha} = \phi(\rho_{A_\alpha})$. Then, by Lemma 2.4, $\phi(A)_\alpha$ is an anti-CIF subspace of V_α. Define $\phi(\mathfrak{a})_\alpha = (\lambda_{\phi(\mathfrak{a})_\alpha}, \rho_{\phi(\mathfrak{a})_\alpha})$, as an extension of $\phi(A)_\alpha = (\lambda_{\phi(A)_\alpha}, \rho_{\phi(A)_\alpha})$, where $\lambda_{\phi(\mathfrak{a})_\alpha} = \phi(\lambda_{\mathfrak{a}_\alpha})$, $\rho_{\phi(\mathfrak{a})_\alpha} = \phi(\rho_{\mathfrak{a}_\alpha})$. Clearly

$$\lambda_{\phi(\mathfrak{a})_\alpha}(x) = \begin{cases}\lambda_{\phi(A)_\alpha}(x) & : \quad x \in V'_\alpha \\ 0 & : \quad x \notin V'_\alpha\end{cases} \text{ and } \rho_{\phi(\mathfrak{a})_\alpha}(x) = \begin{cases}\rho_{\phi(A)_\alpha}(x) & : \quad x \in V'_\alpha \\ 1 & : \quad x \notin V'_\alpha\end{cases}.$$

Let $0 \neq x \in V'$. Then

$$\begin{aligned}\lambda_{\phi(\mathfrak{a})_0}(x) \wedge \lambda_{\phi(\mathfrak{a})_1}(x) &= \phi(\lambda_{\mathfrak{a}_0})(x) \wedge \phi(\lambda_{\mathfrak{a}_1})(x) \\ &= \sup_{x=\phi(a)} \{\lambda_{\mathfrak{a}_0}(a)\} \wedge \sup_{x=\phi(a)} \{\lambda_{\mathfrak{a}_1(a)}\} \\ &= \sup_{x=\phi(a)} \{\lambda_{\mathfrak{a}_0}(a) \wedge \lambda_{\mathfrak{a}_1(a)}\} = 0,\end{aligned}$$

and

$$\begin{aligned}\rho_{\phi(\mathfrak{a})_0}(x) \vee \rho_{\phi(\mathfrak{a})_1}(x) &= \phi(\rho_{\mathfrak{a}_0})(x) \vee \phi(\rho_{\mathfrak{a}_1})(x) \\ &= \inf_{x=\phi(a)} \{\rho_{\mathfrak{a}_0}(a)\} \vee \inf_{x=\phi(a)} \{\rho_{\mathfrak{a}_1(a)}\} \\ &= \inf_{x=\phi(a)} \{\rho_{\mathfrak{a}_0}(a) \vee \rho_{\mathfrak{a}_1(a)}\} = 1.\end{aligned}$$

Let $0 \neq y \in V'$. Then

$$\begin{aligned}
\lambda_{\phi(\mathfrak{a})_0+\phi(\mathfrak{a})_1}(y) &= \sup_{y=a+b}\{\lambda_{\phi(\mathfrak{a})_0}(a) \wedge \lambda_{\phi(\mathfrak{a})_1(b)}\} \\
&= \sup_{y=a+b}\{\phi(\lambda_{\mathfrak{a}_0})(a) \wedge \phi(\lambda_{\mathfrak{a}_1})(b)\} \\
&= \sup_{y=a+b}\{\sup_{a=\phi(m)}\{\lambda_{\mathfrak{a}_0}(m)\} \wedge \sup_{b=\phi(n)}\{\lambda_{\mathfrak{a}_1}(n)\}\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=m+n}\{\lambda_{\mathfrak{a}_0}(m) \wedge \lambda_{\mathfrak{a}_1}(n)\}\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=m+n}\{r_{\mathfrak{a}_0}(m)e^{i2\pi\omega_{\mathfrak{a}_0}(m)} \wedge r_{\mathfrak{a}_1}(n)e^{i2\pi\omega_{\mathfrak{a}_1}(n)}\}\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=m+n}\{r_{\mathfrak{a}_0}(m) \wedge r_{\mathfrak{a}_1}(n)\}e^{i2\pi \sup_{x=m+n}\{\omega_{\mathfrak{a}_0}(m)\wedge\omega_{\mathfrak{a}_1}(n)\}}\} \\
&= \sup_{y=\phi(x)}\{r_{\mathfrak{a}_0+\mathfrak{a}_1}(x)e^{i2\pi\omega_{\mathfrak{a}_0+\mathfrak{a}_1}(x)}\} \\
&= \sup_{y=\phi(x)}\{\lambda_{\mathfrak{a}_0+\mathfrak{a}_1}(x)\} = \sup_{y=\phi(x)}\{\lambda_A(x)\} = \lambda_{\phi(A)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\phi(\mathfrak{a})_0+\phi(\mathfrak{a})_1}(y) &= \inf_{y=a+b}\{\rho_{\phi(\mathfrak{a})_0}(a) \vee \rho_{\phi(\mathfrak{a})_1(b)}\} \\
&= \inf_{y=a+b}\{\phi(\rho_{\mathfrak{a}_0})(a) \vee \phi(\rho_{\mathfrak{a}_1})(b)\} \\
&= \inf_{y=a+b}\{\inf_{a=\phi(m)}\{\rho_{\mathfrak{a}_0}(m)\} \vee \inf_{b=\phi(n)}\{\rho_{\mathfrak{a}_1}(n)\}\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=m+n}\{\rho_{\mathfrak{a}_0}(m) \vee \rho_{\mathfrak{a}_1}(n)\}\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=m+n}\{\hat{r}_{\mathfrak{a}_0}(m)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0}(m)} \vee \hat{r}_{\mathfrak{a}_1}(n)e^{i2\pi\hat{\omega}_{\mathfrak{a}_1}(n)}\}\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=m+n}\{\hat{r}_{\mathfrak{a}_0}(m) \vee \hat{r}_{\mathfrak{a}_1}(n)\}e^{i2\pi \inf_{x=m+n}\{\hat{\omega}_{\mathfrak{a}_0}(m)\vee\hat{\omega}_{\mathfrak{a}_1}(n)\}}\} \\
&= \inf_{y=\phi(x)}\{\hat{r}_{\mathfrak{a}_0+\mathfrak{a}_1}(x)e^{i2\pi\hat{\omega}_{\mathfrak{a}_0+\mathfrak{a}_1}(x)}\} \\
&= \inf_{y=\phi(x)}\{\rho_{\mathfrak{a}_0+\mathfrak{a}_1}(x)\} = \inf_{y=\phi(x)}\{\rho_A(x)\} = \rho_{\phi(A)}(y).
\end{aligned}$$

So, $\phi(A) = \phi(A)_0 \oplus \phi(A)_1$ is a $\mathbb{Z}_2$ graded CIF vector subspace of V'.
Let $x, y \in V'$. We show that $\phi(A)$ is an anti-CIF ideal of V'. That is we need to prove that $\lambda_{\phi(A)}(-[x,y]) \geq \lambda_{\phi(A)}(x) \vee \lambda_{\phi(A)}(y)$ and $\rho_{\phi(A)}(-[x,y]) \leq \rho_{\phi(A)}(x) \wedge \rho_{\phi(A)}(y)$. On contrary, suppose that $\lambda_{\phi(A)}(-[x,y]) < \lambda_{\phi(A)}(x) \vee \lambda_{\phi(A)}(y) = r_{\phi(A)}(x)e^{i2\pi\omega_{\phi(A)}(x)} \vee r_{\phi(A)}(y)e^{i2\pi\omega_{\phi(A)}(y)}$. Then

$$\lambda_{\phi(A)}(-[x,y]) = r_{\phi(A)}(-[x,y])e^{i2\pi\omega_{\phi(A)}(-[x,y])} < \{r_{\phi(A)}(x) \vee r_{\phi(A)}(y)\}e^{i2\pi\{\omega_{\phi(A)}(x)\vee\omega_{\phi(A)}(y)\}}.$$

Since $\phi(A)$ is homogeneous, $r_{\phi(A)}(-[x,y]) < r_{\phi(A)}(x) \vee r_{\phi(A)}(y)$ or $\omega_{\phi(A)}(-[x,y]) < \omega_{\phi(A)}(x) \vee \omega_{\phi(A)}(y)$. If $r_{\phi(A)}(-[x,y]) < r_{\phi(A)}(x) \vee r_{\phi(A)}(y)$, then $r_{\phi(A)}(-[x,y]) < r_{\phi(A)}(x)$ or $r_{\phi(A)}(-[x,y]) < r_{\phi(A)}(y)$. Suppose $r_{\phi(A)}(-[x,y]) < r_{\phi(A)}(x)$, then choose $t \in [0,1]$ such that $r_{\phi(A)}(-[x,y]) < t < r_{\phi(A)}(x)$, so there exists $a \in \phi^{-1}(x)$ such that $r_A(a) > t$. Let $b \in \phi^{-1}(y)$, since ϕ is onto. As $\phi([a,b]) = -[\phi(a), \phi(b)] = -[x,y]$, we get

$$\begin{aligned}
r_{\phi(A)}(-[x,y]) &= \sup_{-[x,y]=\phi([a',b'])}\{r_A([a',b'])\} \\
&= \sup_{-[x,y]=\phi([a',b'])}\{r_A(a') \vee r_A(b')\} \\
&\geq r_A(a) \vee r_A(b) \\
&> t > r_{\phi(A)}(-[x,y]),
\end{aligned}$$

which is a contradiction. We can use the same argument above for the case $\omega_{\phi(A)}(-[x,y]) < \omega_{\phi(A)}(x) \vee \omega_{\phi(A)}(y)$. Therefore, $\lambda_{\phi(A)}(-[x,y]) \geq \lambda_{\phi(A)}(x) \vee \lambda_{\phi(A)}(y)$. Again, on contrary, suppose that $\rho_{\phi(A)}(-[x,y]) > \rho_{\phi(A)}(x) \wedge \rho_{\phi(A)}(y)$. Then

$$\hat{r}_{\phi(A)}(-[x,y])e^{i2\pi\hat{\omega}_{\phi(A)}(-[x,y])} > \{\hat{r}_{\phi(A)}(x) \wedge \hat{r}_{\phi(A)}(y)\}e^{i2\pi\{\hat{\omega}_{\phi(A)}(x)\wedge\hat{\omega}_{\phi(A)}(y)\}}.$$

Since $\phi(A)$ is homogenous, $\hat{r}_{\phi(A)}(-[x,y]) > \hat{r}_{\phi(A)}(x) \wedge \hat{r}_{\phi(A)}(y)$ or $\hat{\omega}_{\phi(A)}(-[x,y]) > \hat{\omega}_{\phi(A)}(x) \wedge \hat{\omega}_{\phi(A)}(y)$. If $\hat{r}_{\phi(A)}(-[x,y]) > \hat{r}_{\phi(A)}(x) \wedge \hat{r}_{\phi(A)}(y)$, then $\hat{r}_{\phi(A)}(-[x,y]) > \hat{r}_{\phi(A)}(x)$ or $\hat{r}_{\phi(A)}(-[x,y]) > \hat{r}_{\phi(A)}(y)$. Without loss of generality, we may assume that $\hat{r}_{\phi(A)}(-[x,y]) > \hat{r}_{\phi(A)}(x)$, choose $t \in [0,1]$ such that $\hat{r}_{\phi(A)}(-[x,y]) > t > \hat{r}_{\phi(A)}(x)$, then there exists $a \in \phi^{-1}(x)$ such that $\hat{r}_A(a) < t$. Let $b \in \phi^{-1}(y)$. As $\phi([a,b]) = -[x,y]$, we get

$$\begin{aligned}
\hat{r}_{\phi(A)}(-[x,y]) &= \inf_{-[x,y]=\phi([a',b'])}\{\hat{r}_A([a',b'])\} \\
&= \inf_{-[x,y]=\phi([a',b'])}\{\hat{r}_A(a') \wedge \hat{r}_A(b')\} \\
&\leq \hat{r}_A(a) \wedge \hat{r}_A(b) \\
&< t < \hat{r}_{\phi(A)}(-[x,y]),
\end{aligned}$$

which is a contradiction. We can use the same argument above for the case $\hat{\omega}_{\phi(A)}(-[x,y]) > \hat{\omega}_{\phi(A)}(x) \wedge \hat{\omega}_{\phi(A)}(y)$. Therefore, $\phi(A)$ is an anti-CIF ideal of V'. The case of anti-CIF lie sub-superalgebra is similar to prove. □

Theorem 4.5. Let $\phi : V \to V'$ be a surjective anti-homomorphism of lie-superalgebras. If $A = (\lambda_A, \rho_A)$ and $B = (\lambda_B, \rho_B)$ are anti-CIF ideals of V, then $\phi(A+B) = \phi(A) + \phi(B)$ is an anti-CIF ideal of V'.

Proof. We already proved in Proposition 4.4 that $\phi(A+B)$ is an anti-CIF ideal of V'. Therefore the only thing we need to prove is that $\phi(A+B) = \phi(A) + \phi(B)$. Let $y \in V'$, then

$$\begin{aligned}
\lambda_{\phi(A+B)}(y) &= \sup_{y=\phi(x)}\{\lambda_{A+B}(x)\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=a+b}\{\lambda_A(a) \wedge \lambda_B(b)\}\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=a+b}\{r_A(a)e^{i2\pi\omega_A(a)} \wedge r_B(b)e^{i2\pi\omega_B(b)}\}\} \\
&= \sup_{y=\phi(x)}\{\sup_{x=a+b}\{(r_A(a) \wedge r_B(b))e^{i2\pi(\omega_A(a)\wedge\omega_B(b))}\}\} \\
&= \sup_{y=\phi(a)+\phi(b)}\{(r_A(a) \wedge r_B(b))e^{i2\pi(\omega_A(a)\wedge\omega_B(b))}\} \ (A, B \textit{ are homogenous}) \\
&= \sup_{y=m+n}\{(\sup_{m=\phi(a)}\{r_A(a)\} \wedge \sup_{n=\phi(b)}\{r_B(b)\})e^{i2\pi(\sup_{m=\phi(a)}\{\omega_A(a)\}\wedge\sup_{n=\phi(b)}\{\omega_B(b)\})}\} \\
&= \sup_{y=m+n}\{(r_{\phi(A)}(m) \wedge r_{\phi(B)}(n))e^{i2\pi(\omega_{\phi(A)}(m)\wedge\omega_{\phi(B)}(n))}\} \\
&= \sup_{y=m+n}\{r_{\phi(A)}(m)e^{i2\pi\omega_{\phi(A)}(m)} \wedge r_{\phi(B)}(n)e^{i2\pi\omega_{\phi(B)}(n)}\} \\
&= \sup_{y=m+n}\{\lambda_{\phi(A)}(m) \wedge \lambda_{\phi(B)}(n)\} \\
&= \lambda_{\phi(A)+\phi(B)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\phi(A+B)}(y) &= \inf_{y=\phi(x)}\{\rho_{A+B}(x)\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=a+b}\{\rho_A(a)\vee\rho_B(b)\}\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=a+b}\{\hat{r}_A(a)e^{i2\pi\hat{\omega}_A(a)}\vee\hat{r}_B(b)e^{i2\pi\hat{\omega}_B(b)}\}\} \\
&= \inf_{y=\phi(x)}\{\inf_{x=a+b}\{(\hat{r}_A(a)\vee\hat{r}_B(b))e^{i2\pi(\hat{\omega}_A(a)\vee\hat{\omega}_B(b))}\}\} \\
&= \inf_{y=\phi(a)+\phi(b)}\{(\hat{r}_A(a)\vee\hat{r}_B(b))e^{i2\pi(\hat{\omega}_A(a)\vee\hat{\omega}_B(b))}\}\ \ (A, B\ are\ homogenous) \\
&= \inf_{y=m+n}\{(\inf_{m=\phi(a)}\{\hat{r}_A(a)\}\vee\inf_{n=\phi(b)}\{\hat{r}_B(b)\})e^{i2\pi(\inf_{m=\phi(a)}\{\hat{\omega}_A(a)\}\vee\inf_{n=\phi(b)}\{\hat{\omega}_B(b)\})}\} \\
&= \inf_{y=m+n}\{(\hat{r}_{\phi(A)}(m)\vee\hat{r}_{\phi(B)}(n))e^{i2\pi(\hat{\omega}_{\phi(A)}(m)\vee\hat{\omega}_{\phi(B)}(n))}\} \\
&= \inf_{y=m+n}\{\hat{r}_{\phi(A)}(m)e^{i2\pi\hat{\omega}_{\phi(A)}(m)}\vee\hat{r}_{\phi(B)}(n)e^{i2\pi\hat{\omega}_{\phi(B)}(n)}\} \\
&= \inf_{y=m+n}\{\rho_{\phi(A)}(m)\vee\rho_{\phi(B)}(n)\} \\
&= \rho_{\phi(A)+\phi(B)}(y).
\end{aligned}$$

Hence, $\phi(A+B)=\phi(A)+\phi(B)$ is an anti-CIF ideal of V'. □

5 Conclusion

This extension of intuitionistic fuzzy Lie sub-superalgebras and ideals to a complex setting opens up new possibilities for studying the algebraic structures in a more generalized and comprehensive manner. The study of complex intuitionistic fuzzy Lie sub-superalgebras and ideals not only provides a deeper understanding of the underlying algebraic structures, but it also allows for a more flexible and versatile approach in handling fuzzy information in Lie superalgebras. The concepts introduced in this article can be applied to various fields such as mathematical physics, quantum mechanics, and control theory, where Lie superalgebras play a crucial role in modeling and analyzing complex systems. By incorporating fuzzy logic and intuitionistic fuzzy sets into the framework of Lie superalgebras, we are able to capture and represent uncertain and imprecise information in a more efficient and effective manner. Future research can focus on exploring the applications of complex intuitionistic fuzzy Lie sub-superalgebras and ideals in different mathematical and scientific disciplines. By further investigating the properties and relationships of these concepts, we can gain deeper insights into the behavior and structure of complex systems that are characterized by uncertainty and vagueness. Ultimately, the development of more advanced and sophisticated algebraic tools based on complex intuitionistic fuzzy logic can lead to new breakthroughs in our understanding and analysis of complex systems in the modern world.

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