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Some inequalities for the Euclidean operator radius of an n-tuple of operators in Hilbert C^*-modules space

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Abstract This paper presents a new method for studying the Euclidean operator radius of an n-tuple of operators on Hilbert C^*-modules. Our method enables us to obtain new results and generalize some known theorems for an n-tuple of operators on Hilbert spaces to an n-tuple of adjointable operators on Hilbert C^*-module spaces.

1 Introduction

Let $(\mathcal{H};\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The numerical radius of $T\in\mathcal{B}(\mathcal{H})$ is defined by $w(T)=\sup\{|\langle Tx,x\rangle| : \ x\in\mathcal{H}, \ \|x\|=1\}$. The well-known operator norm $\|.\|$ is equivalent to the norm defined by $w(\cdot)$ on $\mathcal{B}(\mathcal{H})$. Namely, we have $\frac{1}{2}\|T\|\leq w(T)\leq\|T\|$, for any $T\in\mathcal{B}(\mathcal{H})$. For other results on the numerical radius (see [10, 16, 21, 22]).
In 2009, Popsecu [8] introduced the concept of Euclidean operator radius of an n-tuple $T=(T_1,\ldots,T_n)\in\mathcal{B}(\mathcal{H})^n:=\mathcal{B}(\mathcal{H})\times\ldots\times\mathcal{B}(\mathcal{H})$. Namely, for $T_1,\ldots,T_n\in\mathcal{B}(\mathcal{H})$, the Euclidean operator radius of $T_1,\ldots,T_n$ is defined by

$$w_e(T_1,\ldots,T_n)=\sup_{\|x\|=1}\left(\sum_{i=1}^{n}|\langle T_ix,x\rangle|^2\right)^{\frac{1}{2}}.$$

Popescu [8] proved that

$$\frac{1}{2\sqrt{n}}\left\|\sum_{i=1}^{n}T_iT_i^*\right\|^{\frac{1}{2}}\leq w_e(T_1,\ldots,T_n)\leq\left\|\sum_{i=1}^{n}T_iT_i^*\right\|^{\frac{1}{2}}. \tag{1.1}$$

In case, $n=2$ we have

$$\frac{\sqrt{2}}{4}\left\|T_1^*T_1+T_2^*T_2\right\|^{\frac{1}{2}}\leq w_e(T_1,T_2)\leq\left\|T_1^*T_1+T_2^*T_2\right\|^{\frac{1}{2}}. \tag{1.2}$$

For other results on the Euclidean operator radius (see [2, 13, 19, 20]).
We define a Hilbert C^*-module as a linear space having an inner product that accepts values from a C^*-algebra. The theory for commutative unital algebras was established by Kaplansky (see [9]), who originally introduced this concept. Paschke (see [25]) and Rieffel (see [11]) expanded the theory to general C^*-algebras. For further details, we direct the reader to [3]. While standard methods can be employed to prove some inequalities in Hilbert C^*-module spaces, the structure of Hilbert C^*-modules appears to require different definitions of some concepts. These concepts are merely extensions of some standard definitions to study some inequalities in Hilbert C^*-modules.
First, we recall some definitions and some inequalities that will be used in this paper.

Definition 1.1. ([14]). Let $\mathcal{A}$ be a C^*-algebra. An inner-product $\mathcal{A}$-module is a linear space E which is a right $\mathcal{A}$-module with compatible scalar multiplication:

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \text{ for all } x \in E,\ a \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}$$

together with a map $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathcal{A}$, which has the following properties:

(i) $\langle x, x \rangle \geqslant 0$, if $\langle x, x \rangle = 0$ then $x = 0$,

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$,

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$,

(iv) $\langle x, y \rangle^* = \langle y, x \rangle$,

for all $x, y, z \in E$, $a \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$.

We can define a norm on E by $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. An inner-product $\mathcal{A}$-module that is complete concerning its norm is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$. We define $\mathcal{L}(E)$ which is a C^*-algebra to be the set of all maps $T : E \longrightarrow E$ for which there is a map $T^* : E \longrightarrow E$ which satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in E$.

Definition 1.2. ([7, page 89]). A state on a C^*-algebra $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$ of norm one. We denote the state space of $\mathcal{A}$ by $S(\mathcal{A})$.

Definition 1.3. ([18]). Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the numerical radius of $T \in \mathcal{L}(E)$ by

$$w_{\mathcal{A}}(T) = \sup\{|\varrho\langle x, Tx \rangle| : \ x \in E,\ \varrho \in S(\mathcal{A}),\ \varrho\langle x, x \rangle = 1\}.$$

Definition 1.4. ([12]). Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the Euclidean operator radius of $B, C \in \mathcal{L}(E)$ by

$$w_e(B, C) = \sup\left\{ \left(|\varrho\langle x, Bx \rangle|^2 + |\varrho\langle x, Cx \rangle|^2 \right)^{\frac{1}{2}} : \ x \in E,\ \varrho \in S(\mathcal{A}),\ \varrho\langle x, x \rangle = 1 \right\}.$$

In order to drive our main results, we need the following lemmas:

Lemma 1.5. *([18]).* $w_{\mathcal{A}}(T) = \|T\|$ *for every self-adjoint element of* $\mathcal{L}(E)$.

Lemma 1.6. *([25]). Let* E *is a Hilbert* C^**-module and* $T \in \mathcal{L}(E)$*, we have*

$$\langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle \ \ for\ every\ x \in E.$$

Remark 1.7. It follows from Lemma 1.6 that for every positive linear functional ϱ,

$$\varrho\langle Tx, Tx \rangle \leq \|T\|^2 \varrho\langle x, x \rangle \ \ for\ every\ x \in E.$$

Lemma 1.8. *([7, page 88, Theorem 3.3.2]). Let* $\mathcal{A}$ *be a* C^**-algebra. If* ϱ *is a positive linear functional on* $\mathcal{A}$*, then*

$$\varrho(a^*) = \overline{\varrho(a)}, \ \ for\ \ all\ \ a \in \mathcal{A}.$$

Lemma 1.9. *([18]). Let* $T \in \mathcal{L}(E)$ *and* $\varrho \in S(\mathcal{A})$*. The following statements are equivalent:*
a) $\varrho\langle x, Tx \rangle = 0$ *for every* $x \in E$ *with* $\varrho\langle x, x \rangle = 1$,
b) $\varrho\langle x, Tx \rangle = 0$ *for every* $x \in E$.

Lemma 1.10. *([18]). Let* $T \in \mathcal{L}(E)$*, then* $T = 0$ *if and only if* $\varrho\langle x, Tx \rangle = 0$ *for every* $x \in E$ *and* $\varrho \in S(\mathcal{A})$.

For $T \in \mathcal{L}(E)$, then T is self-adjoint if and only $\varrho\langle x, Tx \rangle$ is positive for every $x \in E$ and $\varrho \in S(\mathcal{A})$.

Lemma 1.11. *([6]). For* $a, b \geq 0$ *and* $0 \leq \alpha \leq 1$,

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}} \ \ for\ \ r \geq 1.$$

Lemma 1.12. *([18]). Let $T \in \mathcal{L}(E)$, $T \geq 0$ and $x \in E$, then for every $\varrho \in S(\mathcal{A})$*

(i) $(\varrho\langle x, Tx\rangle)^r \leq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle$ *for* $r \geq 1$,

(ii) $(\varrho\langle x, Tx\rangle)^r \geq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle$ *for* $0 < r \leq 1$.

Lemma 1.13. *([18, page 13]). Let $(E; \varrho\langle\cdot,\cdot\rangle)$ is a semi-inner product space for every $\varrho \in S(\mathcal{A})$, then*

$$|\varrho\langle a, e\rangle \varrho\langle e, b\rangle| \leqslant \frac{1}{2}\Big((\varrho\langle a, a\rangle)^{\frac{1}{2}}(\varrho\langle b, b\rangle)^{\frac{1}{2}} + |\varrho\langle a, b\rangle|\Big),$$

for every $a, b, e \in E$ with $\varrho\langle e, e\rangle = 1$.

Lemma 1.14. *([5, Cauchy-Schwarz inequality]). Let $T \in \mathcal{B}(\mathcal{H})$ and $0 \leq \alpha \leq 1$, then*

$$|\langle x, Ty\rangle|^2 \leq \langle x, |T|^{2\alpha}x\rangle\langle y, |T^*|^{2(1-\alpha)}y\rangle,$$

for all any $x, y \in \mathcal{H}$.

The following result is a consequence of Lemma 1.14.

Corollary 1.15. *Let $(E; \varrho\langle\cdot,\cdot\rangle)$ is a semi-inner product space for every $\varrho \in S(\mathcal{A})$. Suppose that $T \in \mathcal{L}(E)$ and $0 \leq \alpha \leq 1$, then*

$$|\varrho\langle x, Ty\rangle|^2 \leq \varrho\langle x, |T|^{2\alpha}x\rangle \varrho\langle y, |T^*|^{2(1-\alpha)}y\rangle,$$

for all any $x, y \in E$.

In this section, we present a new definition of the Euclidean operator radius for an n-tuple of operators on Hilbert C^*-modules, which naturally generalizes the concept to n-tuples of operators on Hilbert spaces. Using this definition alongside specific techniques, we establish fundamental inequalities concerning the Euclidean operator radius of an n-tuple of operators on Hilbert C^*-modules.

2 Main results

We start with the following definition.

Definition 2.1. Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the Euclidean operator radius of an n-tuple $(T_1, \dots, T_n) \in \mathcal{L}(E)^n := \mathcal{L}(E) \times \dots \times \mathcal{L}(E)$ by

$$w_{\mathcal{A},e}(T_1, \dots, T_n) = \sup\left\{\left(\sum_{i=1}^{n} |\varrho\langle x, T_i x\rangle|^2\right)^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}. \tag{2.1}$$

We can also consider the following norm on $\mathcal{L}(E)^n := \mathcal{L}(E) \times \cdots \times \mathcal{L}(E)$ by

$$\|(T_1, \dots, T_n)\| = \sup\left\{\left(\sum_{i=1}^{n} \varrho\langle T_i x, T_i x\rangle\right)^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}. \tag{2.2}$$

Our definition is a natural extension of the definition of the Euclidean operator radius of an n-tuple of operators on Hilbert spaces. In fact, in this case the C^*-algebra $\mathcal{A}$ is the set of complex numbers and $S(\mathcal{A})$ contains only the identity function on the set of complex numbers.
Hereafter, we assume that $\mathcal{A}$ is a C^*-algebra and E is an inner product $\mathcal{A}$-module.

Lemma 2.2. $\|(.,\dots,.)\|$ *is norm on $\mathcal{L}(E)^n$.*

Proof. If $T = (T_1, \dots, T_n) = 0$, then obviously $\|(T_1, \dots, T_n)\| = 0$. If $\|(T_1, \dots, T_n)\| = 0$, then for every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have $\varrho\langle T_i x, T_i x\rangle = 0$, for all $i = 1, \dots, n$. We want to show that $Tx = 0$ for each $x \in E$. Fix $x \in E$.
(i) If $\varrho\langle x, x\rangle = 0$, then by Remark 1.7, we have $\varrho\langle T_i x, T_i x\rangle \leq \|T_i\|^2\varrho\langle x, x\rangle = 0$,

and so $T_i = 0$ for all $i = 1, \dots, n$, hence $T = 0$.
(ii) If $\varrho\langle x, x\rangle \neq 0$, then by taking $y = \frac{x}{\varrho\langle x,x\rangle^{\frac{1}{2}}}$, then $\varrho\langle y, y\rangle = 1$. By Definition 2.1 (2.2),

$$\varrho\langle T_i y, T_i y\rangle = 0, \quad for\ all\ i = 1, \dots, n.$$

And so $\frac{1}{\varrho\langle x,x\rangle}\varrho\langle T_i x, T_i x\rangle = 0$ for all $i = 1, \dots, n$. Thus $\varrho\langle T_i x, T_i x\rangle = 0$ for all $i = 1, \dots, n$. Since for every $\varrho \in S(\mathcal{A})$, we have $\varrho\langle T_i x, T_i x\rangle = 0$ for all $i = 1, \dots, n$. We conclude that $\langle T_i x, T_i x\rangle = 0$ for each $x \in E$ and for all $i = 1, \dots, n$. So $T_1 = \ldots = T_n = 0$.
On the other hand $\mathcal{A}$ is an abelian C^*-algebra, then by [17, Theorem 3.6], $|x + y| \leq |x| + |y|$, for each $x, y \in E$. Thus

$$|Tx + Sx| \leq |Tx| + |Sx|, \ for\ each\ T, S \in \mathcal{L}(E)\ and\ x \in E. \tag{2.3}$$

It follows from (2.3) that for every positive linear functional ϱ,

$$\varrho\langle Tx + Sx, Tx + Sx\rangle^{\frac{1}{2}} \leq \varrho\langle Tx, Tx\rangle^{\frac{1}{2}} + \varrho\langle Sx, Sx\rangle^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}\left(\sum_{i=1}^{n}\varrho\langle (T_i + S_i)x, (T_i + S_i)x\rangle\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n}\left(\varrho\langle T_i x, T_i x\rangle^{\frac{1}{2}} + \varrho\langle S_i x, S_i x\rangle^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{i=1}^{n}\varrho\langle T_i x, T_i x\rangle\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}\varrho\langle S_i x, S_i x\rangle\right)^{\frac{1}{2}}\\ &\quad (by\ Minkowski's\ inequality).\end{aligned}$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\|(T_1, \dots, T_n) + (S_1, \dots, S_n)\| \leq \|(T_1, \dots, T_n)\| + \|(S_1, \dots, S_n)\|.$$

Clearly $\|\lambda(T_1, \dots, T_n)\| = |\lambda|\|(T_1, \dots, T_n)\|$, for $\lambda \in \mathbb{C}$. □

Lemma 2.3. *Let $T = (T_1, \dots, T_n) \in \mathcal{L}(E)^n$, then*

$$\|T\| = \sup\left\{\left(\sum_{i=1}^{n}|\varrho\langle x, T_i y\rangle|^2\right)^{\frac{1}{2}} : \ x, y \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = \varrho\langle y, y\rangle = 1\right\}.$$

Proof. Let

$$M = \sup\left\{\left(\sum_{i=1}^{n}|\varrho\langle x, T_i y\rangle|^2\right)^{\frac{1}{2}} : \ x, y \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = \varrho\langle y, y\rangle = 1\right\}.$$

If $\varrho \in S(\mathcal{A})$ and $x, y \in E$ with $\varrho\langle x, x\rangle = \varrho\langle y, y\rangle = 1$, then by using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left(\sum_{i=1}^{n}|\varrho\langle x, T_i y\rangle|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n}\varrho\langle x, x\rangle\varrho\langle T_i y, T_i y\rangle\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^{n}\varrho\langle T_i y, T_i y\rangle\right)^{\frac{1}{2}}\\ &\leq \|T\|.\end{aligned}$$

By taking supremum over $x, y \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = \varrho\langle y, y\rangle = 1$, we have

$$M \leq \|T\|.$$

For every $y \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle y, y\rangle = 1$, we have

$$\varrho\langle T_i y, T_i y\rangle^2 = \varrho\langle T_i y, T_i y\rangle \varrho\left\langle \frac{T_i y}{\varrho\langle T_i y, T_i y\rangle^{\frac{1}{2}}}, T_i y\right\rangle^2,$$

where we assume that $\varrho\langle T_i y, T_i y\rangle \neq 0$ and $T \in \mathcal{L}(E)^n$.
Thus

$$\begin{aligned}\left(\sum_{i=1}^n \varrho\langle T_i y, T_i y\rangle\right)^{\frac{1}{2}} &= \left(\sum_{i=1}^n \varrho\left\langle \frac{T_i y}{\varrho\langle T_i y, T_i y\rangle^{\frac{1}{2}}}, T_i y\right\rangle^2\right)^{\frac{1}{2}} \\ &\leq \sup\left\{\left(\sum_{i=1}^n |\varrho\langle x, T_i y\rangle|^2\right)^{\frac{1}{2}} : \ x, y \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = \varrho\langle y, y\rangle = 1\right\} = M.\end{aligned}$$

Taking the supremum over all $y \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle y, y\rangle = 1$, we have $\|T\| \leq M$. □

Theorem 2.4. *The Euclidean operator radius $w_{\mathcal{A},e}(\cdot) : \mathcal{L}(E) \times \ldots \times \mathcal{L}(E) \to [0, \infty)$ for an $n-$tuple of operators satisfies the following properties:*

(i) $w_{\mathcal{A},e}(T_1, \ldots, T_n) = 0$ *if and only if* $T_i = 0$ *for each* $i = 1, \ldots, n$,

(ii) $w_{\mathcal{A},e}(\lambda T_1, \ldots, \lambda T_n) = |\lambda| w_{\mathcal{A},e}(T_1, \ldots, T_n)$,

(iii) $w_{\mathcal{A},e}(T_1 + S_1, \ldots, T_n + S_n) \leq w_{\mathcal{A},e}(T_1, \ldots, T_n) + w_{\mathcal{A},e}(S_1, \ldots, S_n)$,

(iv) $w_{\mathcal{A},e}(U^*T_1U, \ldots, U^*T_nU) = w_{\mathcal{A},e}(T_1, \ldots, T_n)$ *for any unitary operator* $U \in \mathcal{L}(E)$,

(v) $w_{\mathcal{A},e}(X^*T_1X, \ldots, X^*T_nX) \leq \|X\|^2 w_{\mathcal{A},e}(T_1, \ldots, T_n)$ *for any operator* $X \in \mathcal{L}(E)$,

(vi) $w_{\mathcal{A},e}(T_1, ..., T_n) = w_{\mathcal{A},e}(T_1^*, \ldots, T_n^*)$,

(vii) $w_{\mathcal{A},e}(T_1T_1^*, \ldots, T_nT_n^*) = w_{\mathcal{A},e}(T_1^*T_1, \ldots, T_n^*T_n)$,

for every $T_i, \ S_i \in \mathcal{L}(E) \ \ (1 \leq i \leq n)$.

Lemma 2.5. *If* $T_i \in \mathcal{L}(E) \ \ (i = 1, \ldots, n)$. *Then*

$$\left\|(T_1, \ldots, T_n)\right\| = \left\|\sum_{i=1}^n T_i^*T_i\right\|^{\frac{1}{2}}.$$

Proof. Let $T_i \in \mathcal{L}(E) \ \ (i = 1, \ldots, n)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$,

$$\left\|\sum_{i=1}^n T_i^*T_i\right\| \leq \sum_{i=1}^n \left\|T_i^*T_i\right\| \leq \sum_{i=1}^n \left\|T_i^*\right\|\left\|T_i\right\| = \sum_{i=1}^n \left\|T_i\right\|^2,$$

therefore,

$$\begin{aligned}\left\|\sum_{i=1}^n T_i^*T_i\right\| &\leq \sum_{i=1}^n \left\|T_i\right\|^2 \\ &= \sum_{i=1}^n \sup\left\{\varrho\langle T_i x, T_i x\rangle : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\} \\ &= \sup\left\{\sum_{i=1}^n \varrho\langle T_i x, T_i x\rangle : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}.\end{aligned}$$

Hence,

$$\left\|\sum_{i=1}^n T_i^*T_i\right\| \leq \left\|(T_1, \ldots, T_n)\right\|^2.$$

On the other hand, since $\sum_{i=1}^{n} T_i^* T_i \geq 0$, then it follows that

$$\begin{aligned}\sum_{i=1}^{n} \varrho\langle T_i x, T_i x\rangle &= \varrho\langle x, \sum_{i=1}^{n} T_i^* T_i x\rangle \\ &\leq \varrho\langle x, x\rangle^{\frac{1}{2}} \varrho\langle \sum_{i=1}^{n} T_i^* T_i x, \sum_{i=1}^{n} T_i^* T_i x\rangle^{\frac{1}{2}} \\ &\leq \varrho\langle x, x\rangle^{\frac{1}{2}} \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\| \varrho\langle x, x\rangle^{\frac{1}{2}} \\ &\leq \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|.\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\Big\| (T_1, \ldots, T_n) \Big\|^2 \leq \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|.$$

This proves the desired inequality. □

The following result improves and generalizes (1.1) for an n-tuple of operators on Hilbert C^*-modules.

Theorem 2.6. *If* $(T_1, \ldots, T_n) \in \mathcal{L}(E)^n$. *Then,*

$$\frac{\sqrt{2}}{2} \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|^{\frac{1}{2}} \leq w_{\mathcal{A},e}(T_1, \ldots, T_n) \leq \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|^{\frac{1}{2}}, \tag{2.4}$$

where the constants $\frac{\sqrt{2}}{2}$ *and* 1 *are best possible in (2.4).*

Proof. Let $(T_1, \ldots, T_n) \in \mathcal{L}(E)^n$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, by the Cauchy-Schwarz inequality we have

$$|\varrho\langle x, T_i x\rangle|^2 \leq \varrho\langle x, x\rangle \varrho\langle T_i x, T_i x\rangle.$$

Taking the sum over all i from 1 to n we get

$$\begin{aligned}\sum_{i=1}^{n} |\varrho\langle x, T_i x\rangle|^2 &\leq \varrho\langle x, x\rangle \varrho\langle x, \sum_{i=1}^{n} T_i^* T_i x\rangle \\ &\leq \varrho\langle x, x\rangle \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|.\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$w_{\mathcal{A},e}^2(T_1, \ldots, T_n) \leq \Big\| \sum_{i=1}^{n} T_i^* T_i \Big\|.$$

To prove the second inequality, let $T_k = P_k + iQ_k$ be the Cartesian decomposition of T_k for all $k = 1, \ldots, n$. We have

$$\begin{aligned}|\varrho\langle x, T_k x\rangle|^2 &= \varrho\langle x, P_k x\rangle^2 + \varrho\langle x, Q_k x\rangle^2 \\ &\geq \frac{1}{2}\big(|\varrho\langle x, P_k x\rangle| + |\varrho\langle x, Q_k x\rangle|\big)^2 \\ &\geq \frac{1}{2}|\varrho\langle x, (P_k \pm Q_k) x\rangle|^2.\end{aligned}$$

Summing over k and then taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$\begin{aligned} w^2_{\mathcal{A},e}(T_1, \ldots, T_n) &\geq \frac{1}{2} \sup_{\varrho\langle x,x\rangle=1} \sum_{k=1}^{n} |\varrho\langle x, (P_k \pm Q_k)x\rangle|^2 \\ &\geq \frac{1}{2}\left\| \sum_{k=1}^{n} (P_k \pm Q_k)^2 \right\|. \end{aligned}$$

Moreover,

$$\begin{aligned} 2w^2_{\mathcal{A},e}(T_1, \ldots, T_n) &\geq \frac{1}{2}\left\| \sum_{k=1}^{n} (P_k + Q_k)^2 \right\| + \frac{1}{2}\left\| \sum_{k=1}^{n} (P_k - Q_k)^2 \right\| \\ &\geq \frac{1}{2}\left\| \sum_{k=1}^{n} (P_k + Q_k)^2 + \sum_{k=1}^{n} (P_k - Q_k)^2 \right\| \\ &= \frac{1}{2}\left\| \sum_{k=1}^{n} (P_k + Q_k)^2 + (P_k - Q_k)^2 \right\| \\ &= \left\| \sum_{k=1}^{n} P_k^2 + Q_k^2 \right\| \qquad (2.5) \\ &= \left\| \sum_{k=1}^{n} T_k^* T_k \right\|. \end{aligned}$$

Thus, the desired result is obtained. □

Remark 2.7. (i) In particular, setting $n = 2$ in (2.4) we get

$$\frac{\sqrt{2}}{2}\left\| T_1^*T_1 + T_2^*T_2 \right\|^{\frac{1}{2}} \leq w_{\mathcal{A},e}(T_1, T_2) \leq \left\| T_1^*T_1 + T_2^*T_2 \right\|^{\frac{1}{2}}. \qquad (2.6)$$

The lower bound of $w_{\mathcal{A},e}(T_1, T_2)$ obtained in (2.6) is stronger than the lower bound in (1.2).

(ii) If we choose $T_1 = T_2 = T$ in (2.6) we get

$$\|T^*T\| \leq w^2_{\mathcal{A},e}(T, T) \leq 2\|T^*T\|. \qquad (2.7)$$

But $w^2_{\mathcal{A},e}(T, T) = 2w^2_{\mathcal{A}}(T)$ and Lemma 2.5, we have

$$\frac{\sqrt{2}}{2}\|T\| \leq w_{\mathcal{A}}(T) \leq \|T\|. \qquad (2.8)$$

Clearly, the bound in (2.8) is stronger than the first bound in [18, Theorem 2.13].

(iii) We observe that, if T_1 and T_2 are self-adjoint operators, then (2.6) becomes

$$\frac{\sqrt{2}}{2}\left\| T_1^2 + T_2^2 \right\|^{\frac{1}{2}} \leq w_{\mathcal{A},e}(T_1, T_2) \leq \left\| T_1^2 + T_2^2 \right\|^{\frac{1}{2}}. \qquad (2.9)$$

(iv) We observe also that if $T \in \mathcal{L}(E)$ and $T = T_1 + iT_2$ is the Cartesian decomposition of T, then

$$w^2_{\mathcal{A},e}(T_1, T_2) = \sup_{\substack{\varrho\langle x,x\rangle=1, \\ \varrho\in S(\mathcal{A})}} \left(|\varrho\langle x, T_1x\rangle|^2 + |\varrho\langle x, T_2x\rangle|^2\right) = \sup_{\substack{\varrho\langle x,x\rangle=1, \\ \varrho\in S(\mathcal{A})}} |\varrho\langle x, Tx\rangle|^2 = w^2_{\mathcal{A}}(T).$$

and

$$T^*T + TT^* = 2(T_1^2 + T_2^2).$$

By the inequality (2.9), then we have

$$\frac{1}{4}\left\|T^*T+TT^*\right\| \leq w_{\mathcal{A}}^2(T) \leq \frac{1}{2}\left\|T^*T+TT^*\right\|. \tag{2.10}$$

In [4], Kittaneh has proved the inequality (2.10) in a Hilbert space.

Corollary 2.8. *If* $(T_1,\dots,T_n)\in\mathcal{L}(E)^n$. *Then*

$$\frac{1}{4}\left\|\sum_{i=1}^{n} T_i^*T_i+T_iT_i^*\right\| \leq w_{\mathcal{A},e}^2(T_1,\dots,T_n) \leq \frac{1}{2}\left\|\sum_{i=1}^{n} T_i^*T_i+T_iT_i^*\right\|. \tag{2.11}$$

Proof. Let $T_k=P_k+iQ_k$ be the Cartesian decomposition of T_k for all $k=1,\dots,n$. For every $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$, we have

$$\begin{aligned}\sum_{k=1}^{n}|\varrho\langle x,T_kx\rangle|^2 &= \sum_{i=1}^{n}(\varrho\langle x,P_kx\rangle^2+\varrho\langle x,Q_kx\rangle^2)\\ &\leq \sum_{k=1}^{n}(\varrho\langle x,P_k^2x\rangle+\varrho\langle x,Q_k^2x\rangle)\\ &= \varrho\langle x,\sum_{k=1}^{n}(P_k^2+Q_k^2)x\rangle.\end{aligned}$$

By taking the supremum over all $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$, we get

$$w_{\mathcal{A},e}^2(T_1,\dots,T_n)\leq\left\|\sum_{k=1}^{n}P_k^2+Q_k^2\right\|=\frac{1}{2}\left\|\sum_{k=1}^{n}T_k^*T_k+T_kT_k^*\right\|.$$

Conversely, According to the (2.5), we have

$$w_{\mathcal{A},e}^2(T_1,\dots,T_n)\geq\frac{1}{2}\left\|\sum_{k=1}^{n}P_k^2+Q_k^2\right\|=\frac{1}{4}\left\|\sum_{k=1}^{n}T_k^*T_k+T_kT_k^*\right\|,$$

as desired. □

Remark 2.9. (i) In particular, setting $n=2$ in (2.11) we get

$$\frac{1}{4}\left\|T_1^*T_1+T_1T_1^*+T_2^*T_2+T_2T_2^*\right\| \leq w_{\mathcal{A},e}^2(T_1,T_2) \leq \frac{1}{2}\left\|T_1^*T_1+T_1T_1^*+T_2^*T_2+T_2T_2^*\right\|. \tag{2.12}$$

We observe that, if T_i is normal for each $i=1,2$, then (2.12) becomes

$$\frac{1}{2}\left\|T_1^*T_1+T_2^*T_2\right\| \leq w_{\mathcal{A},e}^2(T_1,T_2) \leq \left\|T_1^*T_1+T_2^*T_2\right\|.$$

(ii) Lemma 2.5 together with the inequality (2.4) leads to the inequality

$$\frac{\sqrt{2}}{2}\left\|(T_1,\dots,T_n)\right\| \leq w_{\mathcal{A},e}(T_1,\dots,T_n) \leq \left\|(T_1,\dots,T_n)\right\|. \tag{2.13}$$

Theorem 2.10. *Let* $T_i\in\mathcal{L}(E)$ $(i=1,\dots,n)$. *Then*

$$w_{\mathcal{A},e}^2(T_1,\dots,T_n) \leqslant \frac{1}{4}\left\|\sum_{i=1}^{n}T_iT_i^*+T_i^*T_i\right\|+\frac{1}{2}w_{\mathcal{A},R}(T_1^2,\dots,T_n^2), \tag{2.14}$$

where

$$w_{\mathcal{A},R}(T_1,\dots,T_n)=\sup\left\{\sum_{i=1}^{n}|\varrho\langle x,T_ix\rangle| :\ x\in E,\ \varrho\in S(\mathcal{A}),\ \varrho\langle x,x\rangle=1\right\},$$

is called the Rhombic numerical radius [15].

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\varrho\langle x, T_i x\rangle|^2 &= \sum_{i=1}^{n} |\varrho\langle T_i^* x, x\rangle \varrho\langle x, T_i x\rangle| \\
&\leq \frac{1}{2}\left(\sum_{i=1}^{n} \varrho\langle T_i^* x, T_i^* x\rangle^{\frac{1}{2}} \varrho\langle T_i x, T_i x\rangle^{\frac{1}{2}} + |\varrho\langle T_i^* x, T_i x\rangle|\right) \\
&\leq \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{2}\varrho\langle T_i^* x, T_i^* x\rangle + \frac{1}{2}\varrho\langle T_i x, T_i x\rangle + |\varrho\langle x, T_i^2 x\rangle|\right) \\
&= \frac{1}{4}\varrho\langle x, (\sum_{i=1}^{n} T_i T_i^* + T_i^* T_i) x\rangle + \frac{1}{2}\sum_{i=1}^{n} |\varrho\langle x, T_i^2 x\rangle| \\
&\leq \frac{1}{4}\left\|\sum_{i=1}^{n} T_i T_i^* + T_i^* T_i\right\| + \frac{1}{2} w_{\mathcal{A},R}(T_1^2, \ldots, T_n^2).
\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get the desired result. □

Remark 2.11. In particular, setting $n = 2$ in (2.14) we get

$$w_{\mathcal{A},e}^2(T_1, T_2) \leq \frac{1}{4}\left\|T_1^* T_1 + T_1 T_1^* + T_2^* T_2 + T_2 T_2^*\right\| + \frac{1}{2} w_{\mathcal{A},R}(T_1^2, T_2^2). \tag{2.15}$$

Moreover, if we choose $T_1 = T_2 = T$ in (2.15) we get

$$w_{\mathcal{A},e}^2(T, T) \leq \frac{1}{2}\left\|T^* T + T T^*\right\| + \frac{1}{2} w_{\mathcal{A},R}(T^2, T^2). \tag{2.16}$$

By the inequality (2.16) and $w_{\mathcal{A},e}^2(T, T) = 2w_{\mathcal{A}}^2(T)$ and $w_{\mathcal{A},R}(T, T) = 2w_{\mathcal{A}}(T)$, then we have

$$w_{\mathcal{A}}^2(T) \leq \frac{1}{4}\left\|T^* T + T T^*\right\| + \frac{1}{2} w_{\mathcal{A}}(T^2). \tag{2.17}$$

We remark that in [1] the authors proved the inequality (2.17) in a Hilbert spaces.
By [18, page.7], we have $w_{\mathcal{A}}(T^2) \leq w_{\mathcal{A}}^2(T)$ and by adding the inequality in (2.17), we get

$$w_{\mathcal{A}}^2(T) \leq 2w_{\mathcal{A}}^2(T) - w_{\mathcal{A}}(T^2) \leq \frac{1}{2}\left\|T^* T + T T^*\right\|,$$

which shows that the inequality in (2.17) is sharper than the second inequality in (2.10).

Lemma 2.12. *([5]). Let $T \in \mathcal{B}(\mathcal{H})$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \|f(|T|)x\| \|g(|T^*|)y\|,$$

for all any $x, y \in \mathcal{H}$.

The following result is a consequence of Lemma 2.12.

Corollary 2.13. *For $\varrho \in S(\mathcal{A})$, $\varrho\langle \cdot, \cdot\rangle$ is a semi-inner product. Suppose that $T \in \mathcal{L}(E)$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \varrho\langle f(|T|)x, f(|T|)x\rangle^{\frac{1}{2}} \varrho\langle g(|T^*|)y, g(|T^*|)y\rangle^{\frac{1}{2}},$$

for all any $x, y \in E$.

In [13, Theorem 3.2] with $p = 2$ and $r \geq 1$, if $(T_1, \ldots, T_n), (A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathcal{B}(\mathcal{H})^n$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$, Then

$$w_e^{2r}(A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leqslant \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left(B_i^* f^2(|T_i|) B_i \right)^{2r} + \left(A_i^* g^2(|T_i^*|) A_i \right)^{2r} \right\|.$$

The following theorem generalizes the above result of an n-tuple of operators on Hilbert C^*-modules. This result includes several inequalities as special cases, which significantly generalize the second inequality of (1.1) on Hilbert C^*-modules.

Theorem 2.14. *Let $(T_1, \ldots, T_n), (S_1, \ldots, S_n), (Q_1, \ldots, Q_n) \in \mathcal{L}(E)^n$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w_{\mathcal{A},e}^{2r}(S_1^* T_1 Q_1, \ldots, S_n^* T_n Q_n) \leqslant n^{r-1} \left\| \sum_{i=1}^{n} \alpha \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} + (1-\alpha) \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} \right\|,$$

for $0 < \alpha < 1$ and $r \geq 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\varrho\langle x, S_i^* T_i Q_i x\rangle|^2 &= \sum_{i=1}^{n} |\varrho\langle S_i x, T_i Q_i x\rangle|^2 \\
&\leqslant \sum_{i=1}^{n} \varrho\langle f(|T_i|) S_i x, f(|T_i|) S_i x\rangle \varrho\langle g(|T_i^*|) Q_i x, g(|T_i^*|) Q_i x\rangle \\
&= \sum_{i=1}^{n} \varrho\langle x, S_i^* f^2(|T_i|) S_i x\rangle \varrho\langle x, Q_i^* g^2(|T_i^*|) Q_i x\rangle \\
&= \sum_{i=1}^{n} \varrho\langle x, \left((S_i^* f^2(|T_i|) S_i)^{\frac{1}{\alpha}} \right)^{\alpha} x\rangle \varrho\langle x, \left((Q_i^* g^2(|T_i^*|) Q_i)^{\frac{1}{1-\alpha}} \right)^{1-\alpha} x\rangle \\
&\leqslant \sum_{i=1}^{n} \varrho\langle x, \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{1}{\alpha}} x\rangle^{\alpha} \varrho\langle x, \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{1}{1-\alpha}} x\rangle^{1-\alpha} \\
&\leqslant \sum_{i=1}^{n} \Big(\alpha \varrho\langle x, \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{1}{\alpha}} x\rangle^{r} + (1-\alpha) \varrho\langle x, \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{1}{1-\alpha}} x\rangle^{r} \Big)^{\frac{1}{r}} \\
&\leqslant \sum_{i=1}^{n} \Big(\alpha \varrho\langle x, \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} x\rangle + (1-\alpha) \varrho\langle x, \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} x\rangle \Big)^{\frac{1}{r}} \\
&= \sum_{i=1}^{n} \Big(\varrho\langle x, \big[\alpha \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} + (1-\alpha) \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} \big] x\rangle \Big)^{\frac{1}{r}} \\
&\leqslant n^{1-\frac{1}{r}} \Big(\varrho\langle x, \sum_{i=1}^{n} \big[\alpha \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} + (1-\alpha) \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} \big] x\rangle \Big)^{\frac{1}{r}}
\end{aligned}$$

(by the concavity of the function $f(t) = t^{\frac{1}{r}}$).

Thus,

$$\left(\sum_{i=1}^{n} |\varrho\langle x, S_i^* T_i Q_i x\rangle|^2 \right)^{r} \leqslant n^{r-1} \varrho\langle x, \sum_{i=1}^{n} \left[\alpha \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} + (1-\alpha) \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} \right] x\rangle.$$

Now, taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$w_{\mathcal{A},e}^{2r}(S_1^* T_1 Q_1, \ldots, S_n^* T_n Q_n) \leqslant n^{r-1} \left\| \sum_{i=1}^{n} \alpha \left(S_i^* f^2(|T_i|) S_i \right)^{\frac{r}{\alpha}} + (1-\alpha) \left(Q_i^* g^2(|T_i^*|) Q_i \right)^{\frac{r}{1-\alpha}} \right\|.$$

□

Remark 2.15. The particular case $\alpha = \frac{1}{2}$ produces the inequality

$$w_{\mathcal{A},e}^{2r}(S_1^*T_1Q_1,\ldots,S_n^*T_nQ_n) \leqslant \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}(S_i^*f^2(|T_i|)S_i)^{2r} + (Q_i^*g^2(|T_i^*|)Q_i)^{2r}\right\|, \tag{2.18}$$

for $r \geq 1$.

Corollary 2.16. *Let $(T_1,\ldots,T_n) \in \mathcal{L}(E)^n$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$w_{\mathcal{A},e}^{2r}(T_1,\ldots,T_n) \leqslant n^{r-1}\left\|\sum_{i=1}^{n}\alpha f^{\frac{2r}{\alpha}}(|T_i|) + (1-\alpha)g^{\frac{2r}{1-\alpha}}(|T_i^*|)\right\|, \tag{2.19}$$

for $0 < \alpha < 1$ and $r \geq 1$.

Letting $f(t) = g(t) = t^{\frac{1}{2}}$, we get

Corollary 2.17. *Let $(T_1,\cdots,T_n),(S_1,\ldots,S_n),(Q_1,\ldots,Q_n)$ be in $\mathcal{L}(E)^n$. Then*

$$w_{\mathcal{A},e}^{2r}(S_1^*T_1Q_1,\cdots,S_n^*T_nQ_n) \leqslant n^{r-1}\left\|\sum_{i=1}^{n}\alpha\Big(S_i^*|T_i|S_i\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(Q_i^*|T_i^*|Q_i\Big)^{\frac{r}{1-\alpha}}\right\|, \tag{2.20}$$

for $0 < \alpha < 1$ and $r \geq 1$.

Corollary 2.18. *Let $(S_1,\cdots,S_n),(Q_1,\ldots,Q_n) \in \mathcal{L}(E)^n$. Then*

$$w_{\mathcal{A},e}^{2r}(S_1^*Q_1,\ldots,S_n^*Q_n) \leqslant n^{r-1}\left\|\sum_{i=1}^{n}\alpha\Big(S_i^*S_i\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(Q_i^*Q_i\Big)^{\frac{r}{1-\alpha}}\right\|, \tag{2.21}$$

for $0 < \alpha < 1$ and $r \geq 1$.

In particular, setting $n = 2$ in (2.21) we get

$$\begin{aligned} w_{\mathcal{A},e}^{2r}(S_1^*Q_1,S_2^*Q_2) \leqslant 2^{r-1}\Big\|&\alpha\big(S_1^*S_1\big)^{\frac{r}{\alpha}} + \alpha\big(S_2^*S_2\big)^{\frac{r}{\alpha}} \\ &+ (1-\alpha)\big(Q_1^*Q_1\big)^{\frac{r}{1-\alpha}} + (1-\alpha)\big(Q_2^*Q_2\big)^{\frac{r}{1-\alpha}}\Big\|. \end{aligned} \tag{2.22}$$

Moreover, if we choose $S_1 = S_2 = S$ and $Q_1 = Q_2 = Q$, then

$$w_{\mathcal{A},e}^{2r}(S^*Q,S^*Q) \leqslant 2^r\left\|\alpha\Big(S^*S\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(Q^*Q\Big)^{\frac{r}{1-\alpha}}\right\|. \tag{2.23}$$

In particular, if we choose $w_{\mathcal{A},e}^2(S^*Q,S^*Q) = 2w_{\mathcal{A}}^2(S^*Q)$, we have

$$w_{\mathcal{A}}^{2r}(S^*Q,S^*Q) \leqslant \left\|\alpha\Big(S^*S\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(Q^*Q\Big)^{\frac{r}{1-\alpha}}\right\|. \tag{2.24}$$

We remark that in [23], Dragomir has proved the inequality (2.24) in a Hilbert space.

Corollary 2.19. *Let $(T_1,\ldots,T_n) \in \mathcal{L}(E)^n$. Then*

$$w_{\mathcal{A},e}^{2}(T_1,\cdots,T_n) \leq \left\|\sum_{i=1}^{n}\alpha|T_i|^{\frac{1}{\alpha}} + (1-\alpha)|T_i^*|^{\frac{1}{1-\alpha}}\right\|, \tag{2.25}$$

for $0 < \alpha < 1$.

In particular, $\alpha = \frac{1}{2}$

$$w^2_{\mathcal{A},e}(T_1,\dots,T_n) \leq \frac{1}{2}\Big\| \sum_{i=1}^{n} |T_i|^2 + |T_i^*|^2 \Big\|. \tag{2.26}$$

Proposition 2.20. *Let* $(T_1,\dots,T_n) \in \mathcal{L}(E)^n$. *Then*

$$w^2_{\mathcal{A},e}(T_1,\dots,T_n) \leq \frac{1}{4}\Big\| \sum_{i=1}^{n} \Big(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \Big)^2 \Big\|, \tag{2.27}$$

for $0 \leq \alpha \leq 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\varrho\langle x, T_i x\rangle|^2 &\leq \sum_{i=1}^{n} \Big(\varrho\langle x, |T_i|^{2\alpha} x\rangle^{\frac{1}{2}} \varrho\langle x, |T_i^*|^{2(1-\alpha)} x\rangle^{\frac{1}{2}} \Big)^2 \\
&\leq \frac{1}{4}\sum_{i=1}^{n} \Big(\varrho\langle x, |T_i|^{2\alpha} x\rangle + \varrho\langle x, |T_i^*|^{2(1-\alpha)} x\rangle \Big)^2 \\
&= \frac{1}{4}\sum_{i=1}^{n} \varrho\langle x, (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x\rangle^2 \\
&= \frac{1}{4}\varrho\langle x, \Big(\sum_{i=1}^{n} (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)})^2 \Big) x\rangle.
\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$w^2_{\mathcal{A},e}(T_1,\dots,T_n) \leqslant \frac{1}{4}\Big\| \sum_{i=1}^{n} \Big(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \Big)^2 \Big\|.$$

□

Proposition 2.21. *Let* $(T_1,\dots,T_n) \in \mathcal{L}(E)^n$. *Then*

$$w^2_{\mathcal{A},e}(T_1,\dots,T_n) \leqslant \Big\| \sum_{i=1}^{n} \alpha|T_i|^2 + (1-\alpha)|T_i^*|^2 \Big\|, \tag{2.28}$$

for $0 \leq \alpha \leq 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\varrho\langle x, T_i x\rangle|^2 &\leq \sum_{i=1}^{n} \varrho\langle x, |T_i|^{2\alpha} x\rangle \varrho\langle x, |T_i^*|^{2(1-\alpha)} x\rangle \\
&\leq \sum_{i=1}^{n} \varrho\langle x, |T_i|^2 x\rangle^{\alpha} \varrho\langle x, |T_i^*|^2 x\rangle^{1-\alpha} \\
&\leq \sum_{i=1}^{n} \alpha\varrho\langle x, |T_i|^2 x\rangle + (1-\alpha)\varrho\langle x, |T_i^*|^2 x\rangle \\
&= \varrho\langle x, (\sum_{i=1}^{n} \alpha|T_i|^2 + (1-\alpha)|T_i^*|^2) x\rangle.
\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we obtain

$$w^2_{\mathcal{A},e}(T_1,\dots,T_n) \leqslant \Big\| \sum_{i=1}^{n} \alpha|T_i|^2 + (1-\alpha)|T_i^*|^2 \Big\|.$$

□

Theorem 2.22. *Let* $(T_1,\ldots,T_n),(S_1,\ldots,S_n)\in\mathcal{L}(E)^n$ *and let* f *and* g *be non-negative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t)=t$ *for all* $t\in[0,\infty)$. *Then*

$$\begin{aligned}\frac{1}{2}\max\Big\{w^2_{\mathcal{A},e}(T_1+S_1,\ldots,T_n+S_n),\, w^2_{\mathcal{A},e}(T_1-S_1,\ldots,T_n-S_n)\Big\}\\ \leqslant w_{\mathcal{A},e}\Big(f^2(|T_1|)+if^2(|S_1|),\ldots,f^2(|T_n|)+if^2(|S_n|)\Big)\\ \cdot\, w_{\mathcal{A},e}\Big(g^2(|T_1^*|)+ig^2(|S_1^*|),\ldots,g^2(|T_n^*|)+ig^2(|S_n^*|)\Big).\end{aligned}\tag{2.29}$$

Proof. For every $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$, we have

$$\begin{aligned}|\varrho\langle x,(T_i+S_i)x\rangle|^2 &= |\varrho\langle x,T_ix\rangle+\varrho\langle x,S_ix\rangle|^2\\ &\le \Big(|\varrho\langle x,T_ix\rangle|+|\varrho\langle x,S_ix\rangle|\Big)^2\\ &\leqslant 2\Big(|\varrho\langle x,T_ix\rangle|^2+|\varrho\langle x,S_ix\rangle|^2\Big)\\ &\leqslant 2\Big(\varrho\langle f(|T_i|)x,f(|T_i|)x\rangle\,\varrho\langle g(|T_i^*|)x,g(|T_i^*|)x\rangle\\ &\quad+\varrho\langle f(|S_i|)x,f(|S_i|)x\rangle\,\varrho\langle g(|S_i^*|)x,g(|S_i^*|)x\rangle\Big)\\ &= 2\Big(\varrho\langle x,f^2(|T_i|)x\rangle\,\varrho\langle x,g^2(|T_i^*|)x\rangle+\varrho\langle x,f^2(|S_i|)x\rangle\,\varrho\langle x,g^2(|S_i^*|)x\rangle\Big)\\ &\leqslant 2\Big(\varrho\langle x,f^2(|T_i|)x\rangle^2+\varrho\langle x,f^2(|S_i|)x\rangle^2\Big)^{\frac{1}{2}}\\ &\quad\times\Big(\varrho\langle x,g^2(|T_i^*|)x\rangle^2+\varrho\langle x,g^2(|S_i^*|)x\rangle^2\Big)^{\frac{1}{2}}\\ &\quad\text{(by the inequality } (ab+cd)^2\leqslant(a^2+c^2)(b^2+d^2) \text{ for } a,b,c,d\in\mathbb{R})\\ &= 2\Big|\varrho\langle x,(f^2(|T_i|)+if^2(|S_i|))x\rangle\Big|\Big|\varrho\langle x,(g^2(|T_i^*|)+ig^2(|S_i^*|))x\rangle\Big|.\end{aligned}$$

Taking the sum over all i from 1 to n and the Minkowski inequality, we get

$$\sum_{i=1}^n|\varrho\langle x,(T_i{+}S_i)x\rangle|^2\leqslant 2\Big(\sum_{i=1}^n|\varrho\langle x,(f^2(|T_i|){+}if^2(|S_i|))x\rangle|^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^n|\varrho\langle x,(|T_i^*|){+}ig^2(|S_i^*|))x\rangle|^2\Big)^{\frac{1}{2}}.$$

By taking the supremum over all $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$, we get

$$\begin{aligned}w^2_{\mathcal{A},e}(T_1+S_1,\ldots,T_n+S_n)\leqslant 2w_{\mathcal{A},e}(f^2(|T_1|)+if^2(|S_1|),\ldots,f^2(|T_n|)+if^2(|S_n|)\\ w_{\mathcal{A},e}(g^2(|T_1^*|)+ig^2(|S_1^*|),\ldots,g^2(|T_n^*|)+ig^2(|S_n^*|)).\end{aligned}\tag{2.30}$$

Similarly, we can prove that:

$$\begin{aligned}w^2_{\mathcal{A},e}(T_1-S_1,\ldots,T_n-S_n)\leqslant 2w_{\mathcal{A},e}(f^2(|T_1|)+if^2(|S_1|),\ldots,f^2(|T_n|)+if^2(|S_n|)\\ w_{\mathcal{A},e}(g^2(|T_1^*|)+ig^2(|S_1^*|),\ldots,g^2(|T_n^*|)+ig^2(|S_n^*|)).\end{aligned}\tag{2.31}$$

Now the result follows by combining inequalities (2.30) and (2.31). □

Letting $f(t)=g(t)=t^{\frac{1}{2}}$, we get

Corollary 2.23. *Let* $(T_1,\cdots,T_n),(S_1,\ldots,S_n)\in\mathcal{L}(E)^n$. *Then*

$$\begin{aligned}\frac{1}{2}\max\Big\{w^2_{\mathcal{A},e}(T_1+S_1,\ldots,T_n+S_n),w^2_{\mathcal{A},e}(T_1-S_1,\ldots,T_n-S_n)\Big\}\\ \leqslant w_{\mathcal{A},e}\Big(|T_1|+i|S_1|,\ldots,|T_n|+i|S_n|\Big)w_{\mathcal{A},e}\Big(|T_1^*|+i|S_1^*|,\ldots,|T_n^*|+i|S_n^*|\Big).\end{aligned}$$

Corollary 2.24. *For any self-adjoint bounded linear operators $T_1, \ldots, T_n \in \mathcal{L}(E)$, we have*

$$\frac{1}{2}\max\left\{w^2_{\mathcal{A},e}(T_1+S_1,\ldots,T_n+S_n), w^2_{\mathcal{A},e}(T_1-S_1,\ldots,T_n-S_n)\right\} \leqslant w^2_{\mathcal{A},e}\Big(|T_1|+i|S_1|,\ldots,|T_n|+i|S_n|\Big).$$

Now we give another upper bound for the powers of $w_{\mathcal{A},e}$.

Proposition 2.25. *Let $T_1, \ldots, T_n \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$w^2_{\mathcal{A},e}(T_1,\ldots,T_n) \leqslant w_{\mathcal{A},e}(f^2(|T_1|),\ldots,f^2(|T_n|))w_{\mathcal{A},e}(g^2(|T_1^*|),\ldots,g^2(|T_n^*|)).$$

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we have

$$\begin{aligned}\sum_{i=1}^{n}|\varrho\langle x,T_ix\rangle|^2 &\leq \sum_{i=1}^{n}\varrho\langle x,f^2(|T_i|)x\rangle\varrho\langle x,g^2(|T_i^*|)x\rangle\\ &\leq \left(\sum_{i=1}^{n}\varrho\langle x,f^2(|T_i|)x\rangle^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\varrho\langle x,g^2(|T_i^*|)x\rangle^2\right)^{\frac{1}{2}}.\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we get the desired result. □

The following lemma provides a simple yet useful extension of the Schwarz inequality for four operators:

Lemma 2.26. *([24]). Let $A,B,C,D \in \mathcal{B}(\mathcal{H})$, then*

$$|\langle x,DCBAy\rangle|^2 \leq \langle x,A^*|B|^2Ax\rangle\langle y,D|C^*|^2D^*y\rangle,$$

for all any $x,y \in \mathcal{H}$.

The following result is a consequence of Lemma 2.26.

Corollary 2.27. *Let $x \in E$ and $\varrho \in S(\mathcal{A})$, $\varrho\langle .,.\rangle$ is a semi-inner product. Suppose that $A,B,C,D \in \mathcal{L}(E)$, then*

$$|\varrho\langle x,DCBAy\rangle|^2 \leq \varrho\langle x,A^*|B|^2Ax\rangle\varrho\langle y,D|C^*|^2D^*y\rangle,$$

for all any $x,y \in E$.

In these results, several new inequalities, refinements, and generalizations are established for the Euclidean operator radius of an n-tuple of operators on Hilbert C^*-modules.

Theorem 2.28. *Let $D_i,C_i,B_i,A_i \in \mathcal{L}(E)$ $(i=1,...,n)$. Then*

$$w^2_{\mathcal{A},e}\Big(D_1C_1B_1A_1,\ldots,D_nC_nB_nA_n\Big) \leqslant w_{\mathcal{A},e}\Big(A_1^*|B_1|^2A_1,\ldots,A_n^*|B_n|^2A_n\Big) \times w_{\mathcal{A},e}\Big(D_1|C_1^*|^2D_1^*,\ldots,D_n|C_n^*|^2D_n^*\Big). \tag{2.32}$$

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we have

$$\begin{aligned}\sum_{i=1}^{n}|\varrho\langle x,D_iC_iB_iA_ix\rangle|^2 &\leq \sum_{i=1}^{n}\varrho\langle x,A_i^*|B_i|^2A_ix\rangle\varrho\langle x,D_i|C_i^*|^2D_i^*x\rangle\\ &\leq \left(\sum_{i=1}^{n}\varrho\langle x,A_i^*|B_i|^2A_ix\rangle^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\varrho\langle x,D_i|C_i^*|^2D_i^*x\rangle^2\right)^{\frac{1}{2}}.\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we get the desired result. □

Corollary 2.29. *Let $B_i \in \mathcal{L}(E)$ $(i = 1, \dots, n)$. Then*

$$w_{\mathcal{A},e}(B_1^2, \dots, B_n^2) \leqslant w_{\mathcal{A},e}(|B_1|^2, \dots, |B_n|^2). \tag{2.33}$$

Proof. Setting $A_i = U_i$, $D_i = U_i^*$ (U_i are unitaries for all $i = 1, \dots, n$) and $C_i = B_i$ in the previous result. Then

$$w_{\mathcal{A},e}^2\Big(U_1^* B_1^2 U_1, \dots, U_n^* B_n^2 U_n\Big) \leqslant w_{\mathcal{A},e}\Big(U_1^*|B_1|^2 U_1, \dots, U_n^*|B_n|^2 U_n\Big)$$
$$\times\, w_{\mathcal{A},e}\Big(U_1^*|B_1^*|^2 U_1, \dots, U_n^*|B_n^*|^2 U_n\Big).$$

Since $w_{\mathcal{A},e}(\cdot)$ is weakly unitarily invariant and

$$w_{\mathcal{A},e}(|B_1|^2, \dots, |B_n|^2) = w_{\mathcal{A},e}(|B_1^*|^2, \dots, |B_n^*|^2).$$

Thus, the desired result is obtained. □

Corollary 2.30. *Let $T_i \in \mathcal{L}(E)$ $(i = 1, \dots, n)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$. Then*

$$w_{\mathcal{A},e}^2\Big(T_1|T_1|^{\alpha+\beta-1}, \dots, T_n|T_n|^{\alpha+\beta-1}\Big) \leqslant w_{\mathcal{A},e}\Big(|T_1|^{2\alpha}, \dots, |T_n|^{2\alpha}\Big) w_{\mathcal{A},e}\Big(|T_1^*|^{2\beta}, \dots, |T_n^*|^{2\beta}\Big). \tag{2.34}$$

In particular, for $\alpha = \beta = \frac{1}{2}$ we have

$$w_{\mathcal{A},e}(T_1, \dots, T_n) \leqslant w_{\mathcal{A},e}(|T_1|, \dots, |T_n|). \tag{2.35}$$

Proof. Let U_i be unitaries for all $i = 1, \dots, n$, setting $D_i = U_i$, $B_i = I$, $C_i = |T_i|^\beta$ and $A_i = |T_i|^\alpha$ for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$ in (2.32), then we have

$$D_i C_i B_i A_i = U_i |T_i|^\beta |T_i|^\alpha = U_i |T_i| |T_i|^{\alpha+\beta-1} = |T_i| |T_i|^{\alpha+\beta-1},$$

also, we have $A_i^*|B_i|^2 A_i = |T_i|^{2\alpha}$ and $D_i|C_i^*|^2 D_i^* = U_i|T_i^*|^{2\beta} U_i^* = |T_i^*|^{2\beta}$ for all $i = 1, \cdots, n$. □

Letting $\alpha = \beta = 1$ in (2.34) and use the properties of $w_{\mathcal{A},e}(\cdot)$, we get

Corollary 2.31. *Let $T_i \in \mathcal{L}(E)$ $(i = 1, \dots, n)$. Then*

$$w_{\mathcal{A},e}(T_1|T_1|, \dots, T_n|T_n|) \leqslant w_{\mathcal{A},e}(|T_1|^2, \dots, |T_n|^2). \tag{2.36}$$

Theorem 2.32. *Let $D_i, C_i, B_i, A_i \in \mathcal{L}(E)$ $(i = 1, \dots, n)$. Then*

$$w_{\mathcal{A},e}^r(D_1 C_1 B_1 A_1, \dots, D_n C_n B_n A_n) \leqslant \frac{n^{\frac{r}{2}-1}}{2} \left\| \sum_{i=1}^n (A_i^*|B_i|^2 A_i)^r + (D_i|C_i^*|^2 D_i^*)^r \right\|, \tag{2.37}$$

for all $r \geq 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
\sum_{i=1}^n |\varrho\langle x, D_i C_i B_i A_i x\rangle|^2 &\leq \sum_{i=1}^n \Big(\varrho\langle x, A_i^*|B_i|^2 A_i x\rangle^{\frac{1}{2}} \varrho\langle x, D_i|C_i^*|^2 D_i^* x\rangle^{\frac{1}{2}}\Big)^2 \\
&\leq \sum_{i=1}^n \Big(\frac{1}{2}\varrho\langle x, A_i^*|B_i|^2 A_i x\rangle^r + \frac{1}{2}\varrho\langle x, D_i|C_i^*|^2 D_i^* x\rangle^r\Big)^{\frac{2}{r}} \\
&\leq \sum_{i=1}^n \Big(\frac{1}{2}\varrho\langle x, (A_i^*|B_i|^2 A_i)^r x\rangle + \frac{1}{2}\varrho\langle x, (D_i|C_i^*|^2 D_i^*)^r x\rangle\Big)^{\frac{2}{r}} \\
&= 2^{-\frac{2}{r}} \sum_{i=1}^n \varrho\langle x, \Big((A_i^*|B_i|^2 A_i)^r + (D_i|C_i^*|^2 D_i^*)^r\Big) x\rangle^{\frac{2}{r}} \\
&\leq 2^{-\frac{2}{r}} n^{\frac{r-2}{r}} \varrho\langle x, \sum_{i=1}^n \Big((A_i^*|B_i|^2 A_i)^r + (D_i|C_i^*|^2 D_i^*)^r\Big) x\rangle^{\frac{2}{r}} \\
&\qquad (by\ the\ concavity\ of\ the\ function\ f(t) = t^{\frac{2}{r}}).
\end{aligned}$$

Thus

$$\left(\sum_{i=1}^{n}|\varrho\langle x, D_iC_iB_iA_ix\rangle|^2\right)^{\frac{r}{2}} \leq \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}(A_i^*|B_i|^2A_i)^r+(D_i|C_i^*|^2D_i^*)^r\right\|.$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$,we get the desired result. □

Corollary 2.33. *Let $B_i \in \mathcal{L}(E)$ $(i = 1, \ldots, n)$, $r \geq 1$. Then*

$$w^r_{\mathcal{A},e}(B_1^2, \ldots, B_n^2) \leqslant \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}|B_i|^{2r}+|B_i^*|^{2r}\right\|. \tag{2.38}$$

Proof. Setting $A_i = U_i$, $D_i = U_i^*$ (U_i are unitaries for all $i = 1, \ldots, n$) and $C_i = B_i$ in the previous result. Then

$$w^r_{\mathcal{A},e}\left(U_1^*B_1^2U_1, \cdots, U_n^*B_n^2U_n\right) \leq \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}(U_i^*|B_i|^2U_i)^r+(U_i^*|B_i^*|^2U_i)^r\right\|.$$

Since $w_{\mathcal{A},e}(\cdot)$ is weakly unitarily invariant and $U_i^*|B_i|^2U_i = |B_i|^2$, $U_i^*|B_i^*|^2U_i = |B_i^*|^2$. Thus, the desired result is obtained. □

Corollary 2.34. *Let $T_i \in \mathcal{L}(E)$ $(i = 1, \ldots, n)$, $r \geq 1$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$. Then*

$$w^r_{\mathcal{A},e}\left(T_1|T_1|^{\alpha+\beta-1}, \ldots, T_n|T_n|^{\alpha+\beta-1}\right) \leqslant \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}|T_i|^{2r\alpha}+|T_i^*|^{2r\beta}\right\|. \tag{2.39}$$

In particular, for $\alpha = \beta = \frac{1}{2}$ we have

$$w^r_{\mathcal{A},e}(T_1, \cdots, T_n) \leqslant \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}|T_i|^{r}+|T_i^*|^{r}\right\|. \tag{2.40}$$

Proof. Let $T_i = U_i|T_i|$ be the polar decomposition of the operator T_i, where U_i is partial isometry for all $i = 1, \ldots, n$, setting $D_i = U_i$, $B_i = I_i$, $C_i = |T_i|^\beta$ and $A_i = |T_i|^\alpha$ for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$ in (2.37), then we have

$$D_iC_iB_iA_i = U_i|T_i|^\beta|T_i|^\alpha = U_i|T_i||T_i|^{\alpha+\beta-1} = |T_i||T_i|^{\alpha+\beta-1},$$

also, we have $A_i^*|B_i|^2A_i = |T_i|^{2\alpha}$ and $D_i|C_i^*|^2D_i^* = U_i|T_i^*|^{2\beta}U_i^* = |T_i^*|^{2\beta}$ for all $i = 1, \cdots, n$. □

Letting $\alpha = \beta = 1$ in (2.39) and use the properties of $w_{\mathcal{A},e}(.)$, we get

Corollary 2.35. *Let $T_i \in \mathcal{L}(E)$ $(i = 1, \ldots, n)$, $r \geq 1$. Then*

$$w^r_{\mathcal{A},e}(T_1|T_1|, \cdots, T_n|T_n|) \leqslant \frac{n^{\frac{r}{2}-1}}{2}\left\|\sum_{i=1}^{n}|T_i|^{2r}+|T_i^*|^{2r}\right\|.$$

3 Concluding remarks

We presented a new method for studying the Euclidean operator radius of an n-tuple of operators on Hilbert C^*-modules. We proved that some results concerning the Euclidean operator radius of an n-tuple of operators on Hilbert spaces remain true when the operators are defined on a Hilbert C^*-module. It seems that our method is also applicable for proving other inequalities for an n-tuple of operators on Hilbert C^*-modules.

References

[1] A. Abu-Omar, F. Kittaneh, *Upper and lower bounds for the numerical radius with an application to involution operators*, The Rocky Mountain Journal of Mathematics, **45(4)**, 1055–1065, (2015).

[2] A. Sheikhhosseini, M. S. Moslehian, K. Shebrawi, *Inequalities for generalized Euclidean operator radius via Youngs inequality*, J. Math. Anal. Appl., **445(2)**, 1516–1529, (2017).

[3] E. C. Lance, *Hilbert C^*-Module: A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge, (1995).

[4] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Mathematica, **168(1)**, 73-80, (2005).

[5] F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci., **24(2)**, 283–293, (1988).

[6] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, (1988).

[7] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, (1990).

[8] G. Popescu, *Unitary invariants in multivariable operator theory*, American Mathematical Soc, (2009).

[9] I. Kaplansky, *Modules over operator algebras*, Am. J. Math., **75(4)**, 839–858, (1953).

[10] K. E. Gustafson, D. K. M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Universitext, Springer, New York, (1997).

[11] M. A. Rieffel, *Induced representations of C^*-algebras*, Adv. Math., **13(2)**, 176–257, (1974).

[12] M. Hassaouy, N. Bounader, *EUCLIDEAN OPERATOR RADIUS INEQUALITIES FOR HILBERT C^*-MODULES*, Palestine Journal of Mathematics., **13(4)**, 756–769, (2024).

[13] M. S. Moslehian, M. Sattari, K. Shebrawi, *Extensions of Euclidean operator radius inequalities*, Mathematica Scandinavica, 129-144, (2017).

[14] N. Bounader, *Selberg type inequalities in $2-*$-semi inner product space and its applications*, Palestine Journal of Mathematics, **11(1)**, (2022).

[15] N. Bounader, M. Hassaouy, *Rhombic numerical radius inequalities for Hilbert C^*-Modules*, (2024).

[16] P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on numerical radius inequalities*, Springer, (2022).

[17] R. Jiang, *A note on the tringular inequality for the C^*-valued norm on the Hilbert C^*-modules*, Math. Inequal. Appl., **16(3)**, 743–749, (2013).

[18] S. F. Moghaddam, A. K. Mirmostafaee, *Numerical radius inequalities for hilbert C^*-modules*, Mathematica Bohemica, **147(4)**, 547–566, (2022).

[19] S. Jana, P. Bhunia, K. Paul, *Euclidean operator radius inequalities of a pair of bounded linear operators and their applications*, Bulletin of the Brazilian Mathematical Society, New Series, **54(1)**, (2023).

[20] S. S. Dragomir, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear algebra and its applications, **419(1)**, 256-264, (2006).

[21] S. S. Dragomir, *A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces*, Banach Journal of Mathematical Analysis, **1(2)**, 154-175, (2007).

[22] S. S. Dragomir, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Springer, (2013).

[23] S. S. Dragomir, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Research report collection, **11(4)**, (2008).

[24] S. S. Dragomir, *Some inequalities generalizing Kato's and Furuta's results*, FILOMAT, **28(1)** , 179-195, (2014).

[25] W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Am. Math. Soc., **182**, 443–468, (1973).

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