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The Marichev-Saigo-Maeda fractional calculus operators involving Mathieu-type series and generalized arbitrary order Mittag-Leffler-Type function

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Abstract Several fractional calculus operators have been introduced and studied. The main aim of this paper is to establish certain formulas of the Marichev-Saigo-Maeda fractional integral and fractional derivative operators involving the product of Mathieu-type series and generalized Mittag-Leffler function of arbitrary order. Corresponding assertions for the Saigo, Erdélyi–Kober and Riemann–Liouville fractional integral and differential operators are also obtained.

1 Introduction

Fractional calculus is a branch of mathematical analysis focused on the properties of functions and operators defined by arbitrary non-integer order derivatives and integrals. This field has gained increased attention across various fields of science and engineering, including physics, chemistry, biology, fluid dynamics, astrophysics, electrical engineering, image processing and others. For more details about the fractal calculus and its applications, interested readers can refer to [1, 2, 3, 4]. Recently, researchers have extensively studied fractional calculus, developing new fractional integral and derivative operators that have gained significant attention due to their wide applications in diverse fields. In particular, various fractional operators such as Riemann-Liouville, Erdélyi-Kober, Caputo, Saigo, Hilfer and Marichev-Saigo-Maeda fractional operators are present, see, for example, [5, 6, 7].

We recall here the generalized hypergeometric fractional integral and fractional derivative operators involving Appell's function F_3, introduced by Marichev [8] and later extended and studied by Saigo and Maeda [9]. These operators are known as the Marichev-Saigo-Maeda operators. The generalized fractional calculus operators with the Appell function F_3 in their kernel are defined as follows:

Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ with $\Re(\eta) > 0$, $x \in \mathbb{R}^+$, then the left and right fractional integral operators are defined as follows (see [9]):

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\acute{\nu}} F_3\left(\nu,\acute{\nu},\mu,\acute{\mu};\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt \tag{1.1}$$

and

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \frac{x^{-\acute{\nu}}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\nu} F_3\left(\nu,\acute{\nu},\mu,\acute{\mu};\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt, \tag{1.2}$$

where F_3 is defined as follows (see [10])

$$F_3\left(\nu,\acute{\nu},\mu,\acute{\mu};\eta;x,t\right) = \sum_{m,n=0}^{\infty} \frac{(\nu)_m(\acute{\nu})_n(\mu)_m(\acute{\mu})_n}{(\eta)_{m+n}} \frac{x^m}{m!}\frac{t^n}{n!}, \quad (max\{|x|,|t|\}<1)\,. \tag{1.3}$$

Here, $(\nu)_m$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function Γ by (see, e.g., [11])

$$(\nu)_m = \frac{\Gamma(\nu+m)}{\Gamma(\nu)} = \begin{cases} 1 & (m=0), \\ \nu(\nu+1)...(\nu+m-1) & (m \in \mathbb{N} := \{1,2,...\}). \end{cases}$$

These operators are reduced to the following Saigo fractional integral operators (see [12]):

$$\begin{aligned}\left(I_{0+}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) &= \left(I_{0+}^{\nu,\mu,\rho} f\right)(x) \\ &= \frac{x^{-\nu-\mu}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} {}_2F_1\left(\nu+\mu,-\rho;\nu;1-\frac{t}{x}\right) f(t)dt, \quad \rho \in \mathbb{C},\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}\left(I_{-}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) &= \left(I_{-}^{\nu,\mu,\rho} f\right)(x) \\ &= \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} t^{-\nu-\mu} {}_2F_1\left(\nu+\mu,-\rho;\nu;1-\frac{x}{t}\right) f(t)dt, \quad \rho \in \mathbb{C},\end{aligned} \tag{1.5}$$

where ${}_2F_1$ is the Gauss hypergeometric series defined by (see [10])

$$ {}_2F_1(\nu,\mu;\rho;x)=\sum_{m=0}^{\infty}\frac{(\nu)_m(\mu)_m}{(\rho)_m}\frac{x^m}{m!},\quad |x|<1. \tag{1.6}$$

Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ with $\Re(\eta)>0$, $x\in\mathbb{R}^+$, then the left and right generalized fractional differentiation operators involving the Appell function F_3 as a kernel are defined by (see [9])

$$\begin{aligned}\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}f\right)(x)&=\left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta}f\right)(x)\\&=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu}+m,-\mu,-\eta+m}f\right)(x)\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}f\right)(x)&=\left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta}f\right)(x)\\&=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu+m,-\eta+m}f\right)(x),\end{aligned}\tag{1.8}$$

where $m=[\Re(\eta)]+1$ and $[\Re(\eta)]$ denotes the integer part of $\Re(\eta)$.
These operators are reduced to the following Saigo fractional derivative operators (see [12]):

$$\left(D_{0+}^{\nu+\mu,0,-\rho,0,\nu}f\right)(x)=\left(D_{0+}^{\nu,\mu,\rho}f\right)(x),\quad \rho\in\mathbb{C},\tag{1.9}$$

and

$$\left(D_{-}^{\nu+\mu,0,-\rho,0,\nu}f\right)(x)=\left(D_{-}^{\nu,\mu,\rho}f\right)(x),\quad \rho\in\mathbb{C}.\tag{1.10}$$

If set $\mu=-\nu$ in (1.4), (1.5), (1.9) and (1.10), the Saigo fractional integral and derivative operators reduce to the Riemann-Liouville fractional integral and derivative operators, which are defined as follows (see [2]):

$$\left(I_{0+}^{\nu,-\nu,\rho}f\right)=\left(I_{0+}^{\nu}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)dt,\tag{1.11}$$

$$\left(I_{-}^{\nu,-\nu,\rho}f\right)=\left(I_{-}^{\nu}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_x^{\infty}(t-x)^{\nu-1}f(t)dt,\tag{1.12}$$

$$\left(D_{0+}^{\nu,-\nu,\rho}f\right)=\left(D_{0+}^{\nu}f\right)(x)=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{m-\nu}f\right)(x)\tag{1.13}$$

and

$$\left(D_{-}^{\nu,-\nu,\rho}f\right)=\left(D_{-}^{\nu}f\right)(x)=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{m-\nu}f\right)(x),\tag{1.14}$$

where $\nu\in\mathbb{C}$, $\Re(\nu)>0$, $m=[\Re(\nu)]+1$ and $x\in\mathbb{R}^+$.
When $\mu=0$ in (1.4), (1.5), (1.9) and (1.10), the Saigo fractional integral and derivative operators reduce to the Erdélyi-Kober fractional integral and derivative operators, which are defined as follows (see [3]):

$$\left(I_{0+}^{\nu,0,\rho}f\right)(x)=\left(I_{\rho,\nu}^{+}f\right)(x)=\frac{x^{-\nu-\rho}}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}t^{\rho}f(t)dt,\tag{1.15}$$

$$\left(I_{-}^{\nu,0,\rho,}f\right)(x)=\left(K_{\rho,\nu}^{-}f\right)(x))=\frac{x^{\rho}}{\Gamma(\nu)}\int_x^{\infty}(t-x)^{\nu-1}t^{-\nu-\rho}f(t)dt,\tag{1.16}$$

$$\left(D_{0+}^{\nu,0,\rho}f\right)(x)=\left(D_{\rho,\nu}^{+}f\right)(x)=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{-\nu+m,-\nu,\nu+\rho-m}f\right)(x)\tag{1.17}$$

and

$$\left(D_{-}^{\nu,0,\rho}f\right)(x)=\left(D_{\rho,\nu}^{-}f\right)(x)=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{-\nu+m,-\nu,\nu+\rho}f\right)(x),\tag{1.18}$$

where $\nu\in\mathbb{C}$, $\Re(\nu)>0$, $m=[\Re(\nu)]+1$ and $x\in\mathbb{R}^+$.
Further, the image formulas for a power function, under operators (1.1), (1.2), (1.7) and (1.8) are given by [9]

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\nu-\acute{\nu}-\mu)\Gamma(\lambda+\acute{\mu}-\acute{\nu})}{\Gamma(\lambda+\acute{\mu})\Gamma(\lambda+\eta-\nu-\acute{\nu})\Gamma(\lambda+\eta-\acute{\nu}-\mu)}x^{\lambda-\nu-\acute{\nu}+\eta-1},\tag{1.19}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)>max\{0,\Re(\nu+\acute{\nu}+\mu-\eta),\Re(\acute{\nu}-\acute{\mu})\}$,

$$\begin{aligned}\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=&\frac{\Gamma(1-\lambda-\mu)\Gamma(1-\lambda-\eta+\nu+\acute{\nu})\Gamma(1-\lambda+\nu-\acute{\mu}-\eta)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\nu-\mu)\Gamma(1-\lambda+\nu+\acute{\nu}+\acute{\mu}-\eta)}\\&\times x^{\lambda-\nu-\acute{\nu}+\eta-1},\end{aligned}\tag{1.20}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)<1+max\{\Re(-\mu),\Re(\nu+\acute{\nu}-\eta),\Re(\nu+\acute{\mu}-\eta)\}$,

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)\Gamma(\lambda-\eta+\nu+\acute{\nu}+\acute{\mu})\Gamma(\lambda-\mu+\nu)}{\Gamma(\lambda-\mu)\Gamma(\lambda-\eta+\nu+\acute{\nu})\Gamma(\lambda-\eta+\nu+\acute{\mu})}x^{\lambda+\nu+\acute{\nu}-\eta-1}, \tag{1.21}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)>max\{0,\Re(\eta-\nu-\acute{\nu}-\acute{\mu}),\Re(\mu-\nu)\}$
and

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(1-\lambda+\acute{\mu})\Gamma(1-\lambda-\acute{\nu}-\mu+\eta)\Gamma(1-\lambda-\nu-\acute{\nu}+\eta)}{\Gamma(1-\lambda)\Gamma(1-\lambda-\acute{\nu}+\acute{\mu})\Gamma(1-\lambda-\nu-\acute{\nu}-\mu+\eta)} \times x^{\lambda+\nu+\acute{\nu}-\eta-1}, \tag{1.22}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)<1+max\{\Re(\acute{\mu}),\Re(\eta-\nu-\acute{\nu}),\Re(\eta-\acute{\nu}-\mu)\}$.

The Fox-Wright function is defined as [10]

$${}_p\Psi_q\left[\begin{array}{c}(d_1,D_1),...,(d_p,D_p)\\(e_1,E_1),...,(e_q,E_q)\end{array}\middle|z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(d_i+D_in)}{\prod_{j=1}^{q}\Gamma(e_j+E_jn)}\frac{z^n}{n!}, \tag{1.23}$$

where $d_i,D_i,e_j,E_j,z\in\mathbb{C}$, $\Re(d_i)>0,\Re(D_i)>0$, $i=1,...,p$, $\Re(e_i)>0,\Re(E_i)>0$, $j=1,...,q$ and $1+\Re\left(\sum_{j=1}^{q}E_j-\sum_{i=1}^{p}D_i\right)\geq 0$.

During the study of elasticity of solid bodies, Mathieu [13] introduced and investigated the famous infinite functional series called Mathieu series of the form

$$S(u)=\sum_{r=0}^{\infty}\frac{2r}{(r^2+u^2)},\quad \left(u\in\mathbb{R}^+\right). \tag{1.24}$$

Srivastava and Tomovski defined a family of generalized Mathieu series [14] as

$$S_{\iota}^{(\upsilon,\xi)}(u,d)=\sum_{r=0}^{\infty}\frac{2d_r^{\xi}}{(d_r^{\upsilon}+u^2)^{\iota}},\quad \left(u,d,\iota,\upsilon,\xi\in\mathbb{R}^+\right). \tag{1.25}$$

Various applications of the Mathieu series and its generalizations in special functions, number theory, probability theory, mathematical physics, quantum physics, etc., can be found in the book by Tomovski et al. [15]. For our present investigation, we consider the following family of generalized Mathieu series defined by Tomovski and Mehrez [16] as:

$$S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t)=\sum_{r=0}^{\infty}\frac{2d_r^{\xi}(\delta)_r}{r!\,(d_r^{\upsilon}+u^2)^{\iota}}\frac{t^r}{r!},\quad \left(u,d,\iota,\upsilon,\xi\in\mathbb{R}^+;|t|\leq 1\right). \tag{1.26}$$

The Mittag-Leffler function and its generalization appear in special functions as a solution of fractional integro-differential equations having the arbitrary order. The importance of such functions in applied mathematics and engineering sciences is steadily increasing. Some interesting applications of the Mittag-Leffler function are considered in the study of quantum mechanics, electric network, random walks, Levy flights and kinetic equation, interested readers can refer to the recent work of researchers [17, 18, 19, 20] and the references cited therein. Besides fractional calculus the Mittag-Leffler function also plays an important role in several branches of science and engineering like applied physics, statistics, quantum mechanics, mechanics, thermodynamics, telecommunications, electrical engineering and more.

In recent years, Mittag-Leffler functions have garnered significant attention in the field of special functions, prompting numerous researchers to explore their generalizations and applications. Pathan and Bin-Saad [21] introduced and studied the function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as

$$E_{\alpha,\beta}^{j,k}(z)=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma(\beta+\alpha(nj+k))},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,j\geq 1,k\geq 0. \tag{1.27}$$

Very recently, Bin-Saad and Younis [22] investigated a new generalization of the arbitrary order Mittag-Leffler-type function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as

$$E_{\alpha,\beta,\gamma}^{j,k,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}z^{nj+k}, \tag{1.28}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$.

Due to the great importance of fractional calculus operators involving various types of special functions, in this work, we establish several formulas for the Marichev-Saigo-Maeda fractional derivative and integral operators involving the product of Mathieu-type series and generalized Mittag-Leffler function. We also consider some interesting special cases of our main results.

2 Marichev-Saigo-Maeda Fractional Integral with the Function $E_{\alpha,\beta,\gamma}^{j,k,q}(z)$ and Mathieu-Type Series

In this section, we present the Marichev-Saigo-Maeda fractional integral formulas involving the product of generalized arbitrary order Mittag-Leffler-type function and Mathieu-type series, expressed using the Fox-Wright function.

Theorem 2.1. *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \tau, \alpha, \beta, \gamma, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda + r\tau + \sigma k) > max\{0, \Re(\nu + \acute{\nu} + \mu - \eta), \Re(\acute{\nu} - \acute{\mu})\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Let $I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the left-sided operator of Marichev-Saigo-Maeda fractional integral. Then*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x) = \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;x^\tau)$$
$$\times\, {}_4\Psi_4\left[\begin{array}{c} (\gamma,q),(\lambda+r\tau+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+r\tau+\sigma k,\sigma j),\\ (\beta+\alpha k,\alpha j),(\acute{\mu}+\lambda+r\tau+\sigma k,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+r\tau+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\nu-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k,\sigma j)\\ (-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^\sigma)^j\right]. \tag{2.1}$$

Proof. In view of Equations (1.26) and (1.28), and then changing the orders of integration and summation, left hand side of (2.1) becomes

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x)$$
$$=\sum_{r=0}^{\infty}\frac{2d_r^{\xi}\,(\delta)_r}{r!\,(d_r^{\upsilon}+u^2)^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj+k}\,(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+r\tau+\sigma(nj+k)-1}\right)(x).$$

By using (1.19), we obtain

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x)$$
$$=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}\sum_{r=0}^{\infty}\frac{2d_r^{\xi}\,(\delta)_r}{r!\,(d_r^{\upsilon}+u^2)^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj}\,(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\frac{\Gamma(\lambda+r\tau+\sigma k+\sigma nj)}{\Gamma(\acute{\mu}+\lambda+r\tau+\sigma k+\sigma nj)}$$
$$\times\frac{\Gamma(-\acute{\nu}+\acute{\mu}+\lambda+r\tau+\sigma k+\sigma nj)\,\Gamma(-\nu-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k+\sigma nj)}{\Gamma(-\nu-\acute{\nu}+\eta+\lambda+r\tau+\sigma k+\sigma nj)\,\Gamma(-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k+\sigma nj)}x^{r\tau+\sigma nj}$$
$$=\frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)}\sum_{r=0}^{\infty}\frac{2d_r^{\xi}\,(\delta)_r}{(d_r^{\upsilon}+u^2)^{\iota}}\frac{(x^\tau)^r}{r!}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\,\Gamma(\lambda+r\tau+\sigma k+\sigma nj)}{\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(\acute{\mu}+\lambda+r\tau+\sigma k+\sigma nj)}$$
$$\times\frac{\Gamma(-\acute{\nu}+\acute{\mu}+\lambda+r\tau+\sigma k+\sigma nj)\,\Gamma(-\nu-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k+\sigma nj)}{\Gamma(-\nu-\acute{\nu}+\eta+\lambda+r\tau+\sigma k+\sigma nj)\,\Gamma(-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k+\sigma nj)}\frac{(\omega^j x^{\sigma j})^n}{n!}.$$

Finally, using definition (1.23), we arrive at the desired formula (2.1). □

Corollary 2.2. *Let $\nu, \mu, \rho, \lambda, \tau, \alpha, \beta, \gamma, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$, $\Re(\lambda+r\tau+\sigma k) > max\{0, \Re(\mu-\rho)\}$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Then the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x) = \frac{\omega^k x^{-\mu+\lambda+\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;x^\tau)$$
$$\times\, {}_3\Psi_3\left[\begin{array}{c} (\gamma,q),(\lambda+r\tau+\sigma k,\sigma j),(-\mu+\rho+\lambda+r\tau+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(-\mu+\lambda+r\tau+\sigma k,\sigma j),(\nu+\rho+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^\sigma)^j\right]. \tag{2.2}$$

Corollary 2.3. *Let $\nu, \rho, \lambda, \tau, \alpha, \beta, \gamma, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$, $\Re(\lambda + r\tau + \sigma k) > -\Re(\rho)$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Then the following left fractional integral formula holds true:*

$$\left(I_{\rho,\nu}^{+}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x) = \frac{\omega^k x^{\lambda+\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;x^\tau)$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c} (\gamma,q),(\rho+\lambda+r\tau+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(\rho+\nu+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^\sigma)^j\right]. \tag{2.3}$$

Corollary 2.4. *Let $\nu, \lambda, \tau, \alpha, \beta, \gamma, \omega, \sigma \in \mathbb{C}$ such that $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\tau) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\sigma) > 0$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Then the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;t^\tau)\,E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^\sigma)\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}(u,d;x^\tau)$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c} (\gamma,q),(\lambda+r\tau+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(\nu+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^\sigma)^j\right]. \tag{2.4}$$

If we take $\gamma = q = 1$ in the Theorem 2.1, we obtain the following new and interesting result concerning Marichev-Saigo-Maed fractional integral operator:

Corollary 2.5. *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \tau, \alpha, \beta, \sigma, \omega \in \mathbb{C}$ such that $\Re(\lambda+r\tau+\sigma k) > max\{0, \Re(\nu+\acute{\nu}+\mu-\eta), \Re(\acute{\nu}-\acute{\mu})\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$. Then the following left fractional integral formula* *holds true:*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)=\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(1,1),(\lambda+r\tau+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+r\tau+\sigma k,\sigma j),\\(\beta+\alpha k,\alpha j),(\acute{\mu}+\lambda+r\tau+\sigma k,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+r\tau+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\nu-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k,\sigma j)\\(-\acute{\nu}-\mu+\eta+\lambda+r\tau+\sigma k,\sigma nj)\end{array}\right|(\omega x^{\sigma})^{j}\Bigg].\tag{2.5}$$

Corollary 2.6. *If we set $j=1$ and $k=0$ in the Theorem 2.1, we obtain the following known result due to Khan et al. [23] concerning left Marichev-Saigo-Maeda fractional integral operator:*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{\gamma,q}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{x^{-\nu-\acute{\nu}+\eta+\lambda-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(\gamma,q),(\lambda+r\tau,\sigma),(-\acute{\nu}+\acute{\mu}+\lambda+r\tau,\sigma),(-\nu-\acute{\nu}-\mu+\eta+\lambda+r\tau,\sigma)\\(\beta,\alpha),(\acute{\mu}+\lambda+r\tau,\sigma),(-\nu-\acute{\nu}+\eta+\lambda+r\tau,\sigma),(-\acute{\nu}-\mu+\eta+\lambda+r\tau,\sigma)\end{array}\right|\omega x^{\sigma}\Bigg].\tag{2.6}$$

Theorem 2.7. *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\tau,\alpha,\beta,\gamma,\sigma,\omega\in\mathbb{C}$ with $\Re(\lambda+r\tau-\sigma k)<1+min\{\Re(-\mu),\Re(\nu+\acute{\nu}-\eta),\Re(\nu+\acute{\mu}-\eta)\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Let $I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of Marichev-Saigo-Maeda fractional integral. Then*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(\gamma,q),(-\mu-\lambda-r\tau+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda-r\tau+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j),(\nu-\mu-\lambda-r\tau+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+1,\sigma j)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\right|(\omega x^{-\sigma})^{j}\Bigg].\tag{2.7}$$

Proof. In view of Equations (1.26) and (1.28), and then changing the orders of integration and summation, left hand side of (2.7) becomes

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$=\sum_{r=0}^{\infty}\frac{2d_{r}^{\xi}\,(\delta)_{r}}{r!\,(d_{r}^{\upsilon}+u^{2})^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj+k}\,(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+r\tau-\sigma(nj+k)-1}\right)(x).$$

Applying (1.20), we have

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$=\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}\sum_{r=0}^{\infty}\frac{2d_{r}^{\xi}\,(\delta)_{r}}{r!\,(d_{r}^{\upsilon}+u^{2})^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj}\,(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\frac{\Gamma(-\mu-\lambda-r\tau+\sigma k+\sigma nj+1)}{\Gamma(-\lambda-r\tau+\sigma k+\sigma nj+1)}$$
$$\times\frac{\Gamma(\nu+\acute{\nu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)\,\Gamma(\nu+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)}{\Gamma(\nu-\mu-\lambda-r\tau+\sigma k+\sigma nj+1)\,\Gamma(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)}x^{r\tau-\sigma nj}$$
$$=\frac{\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{r=0}^{\infty}\frac{2d_{r}^{\xi}\,(\delta)_{r}}{(d_{r}^{\upsilon}+u^{2})^{\iota}}\frac{(x^{\tau})^{r}}{r!}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\,\Gamma(-\mu-\lambda-r\tau+\sigma k+\sigma nj+1)}{\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(-\lambda-r\tau+\sigma k+\sigma nj+1)}$$
$$\times\frac{\Gamma(\nu+\acute{\nu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)\,\Gamma(\nu+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)}{\Gamma(\nu-\mu-\lambda-r\tau+\sigma k+\sigma nj+1)\,\Gamma(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda-r\tau+\sigma k+\sigma nj+1)}\frac{(\omega^{j}x^{-\sigma j})^{n}}{n!}.$$

Thus, by using (1.23), we arrive at the desired formula (2.7). □

Corollary 2.8. *Let $\nu,\mu,\rho,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\lambda+r\tau-\sigma k)<1+min\{\Re(\mu),\Re(\rho)\}$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{-\mu+\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_3\Psi_3\left[\begin{array}{c}(\gamma,q),(\mu-\lambda-r\tau+\sigma k+1,\sigma j),(\rho-\lambda-r\tau+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j),(\nu+\mu+\rho-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\right|(\omega x^{-\sigma})^{j}\Bigg].\tag{2.8}$$

Corollary 2.9. *Let $\rho,\nu,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\lambda+r\tau-\sigma k)<1+\Re(\rho)$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following right fractional integral formula holds true:*

$$\left(K_{\rho,\nu}^{-}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(\rho-\lambda-r\tau+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho+\nu-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\right|(\omega x^{-\sigma})^{j}\Bigg].\tag{2.9}$$

Corollary 2.10. *Let $\nu, \lambda, \tau, \alpha, \beta, \gamma, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $0 < \Re(\nu) < 1 - \Re(\lambda + r\tau - \sigma k)$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Then the following right fractional integral formula holds true:*

$$\left(I_-^{\nu}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; x^{\tau}\right)$$

$$\times {}_2\Psi_2\left[\begin{array}{c}(\gamma, q), (-\nu - \lambda - r\tau + \sigma k + 1, \sigma j) \\ (\beta + \alpha k, \alpha j), (-\lambda - r\tau + \sigma k + 1, \sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{2.10}$$

If we take $\gamma = q = 1$ in the Theorem 2.7, we get the following new result involving the product of Mittag-Leffler function $E_{\alpha,\beta}^{j,k}$ and Mathieu-type series:

Corollary 2.11. *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \tau, \alpha, \beta, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda + r\tau - \sigma k) < 1 + min\{\Re(-\mu), \Re(\nu + \acute{\nu} - \eta), \Re(\nu + \acute{\mu} - \eta)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$. Let $I_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of Marichev-Saigo-Maeda fractional integral. Then*

$$\left(I_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; x^{\tau}\right)$$

$$\times {}_4\Psi_4\left[\begin{array}{c}(1,1), (-\mu - \lambda - r\tau + \sigma k + 1, \sigma j), (\nu + \acute{\nu} - \eta - \lambda - r\tau + \sigma k + 1, \sigma j), \\ (\beta + \alpha k, \alpha j), (-\lambda - r\tau + \sigma k + 1, \sigma j), (\nu - \mu - \lambda - r\tau + \sigma k + 1, \sigma j),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu + \acute{\mu} - \eta - \lambda - r\tau + \sigma k + 1, \sigma j) \\ (\nu + \acute{\nu} + \acute{\mu} - \eta - \lambda - r\tau + \sigma k + 1, \sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{2.11}$$

Corollary 2.12. *If we set $j = 1$ and $k = 0$ in the Theorem 2.7, we get the following known result due to Khan et al. [23]:*

$$\left(I_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta}^{\gamma,q}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{x^{-\nu-\acute{\nu}+\eta+\lambda-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; x^{\tau}\right)$$

$$\times {}_4\Psi_4\left[\begin{array}{c}(\gamma, q), (-\mu - \lambda - r\tau + 1, \sigma), (\nu + \acute{\nu} - \eta - \lambda - r\tau + 1, \sigma), (\nu + \acute{\mu} - \eta - \lambda - r\tau + 1, \sigma) \\ (\beta, \alpha), (-\lambda - r\tau + 1, \sigma), (\nu - \mu - \lambda - r\tau + 1, \sigma), (\nu + \acute{\nu} + \acute{\mu} - \eta - \lambda - r\tau + 1, \sigma)\end{array}\middle|\omega x^{-\sigma}\right]. \tag{2.12}$$

3 Marichev-Saigo-Maeda Fractional Derivative with the Function $E_{\alpha,\beta,\gamma}^{j,k,q}(z)$ and Mathieu-Type Series

Here, we present the Marichev-Saigo-Maeda fractional derivative formulas involving the product of generalized arbitrary order Mittag-Leffler-type function and Mathieu-type series, expressed using the Fox-Wright function.

Theorem 3.1. *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \tau, \alpha, \beta, \gamma, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda + r\tau + \sigma k) > max\{0, \Re(-\nu - \acute{\nu} - \acute{\mu} + \eta), \Re(\mu - \nu)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $\upsilon, \xi, \iota, \delta \in \mathbb{R}^+$, $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Let $D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the left-sided operator of Marichev-Saigo-Maeda fractional derivative. Then*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; x^{\tau}\right)$$

$$\times {}_4\Psi_4\left[\begin{array}{c}(\gamma, q), (\lambda + r\tau + \sigma k, \sigma j), (\nu - \mu + \lambda + r\tau + \sigma k, \sigma j), \\ (\beta + \alpha k, \alpha j), (-\mu + \lambda + r\tau + \sigma k, \sigma j), (\nu + \acute{\nu} - \eta + \lambda + r\tau + \sigma k, \sigma j),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu + \acute{\nu} + \acute{\mu} - \eta + \lambda + r\tau + \sigma k, \sigma j) \\ (\nu + \acute{\mu} - \eta + \lambda + r\tau + \sigma k, \sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{3.1}$$

Proof. In view of (1.26), (1.28) and the left-hand side of (3.1), we obtain

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x)$$

$$= \sum_{r=0}^{\infty} \frac{2d_r^{\xi}\,(\delta)_r}{r!\,(d_r^{\upsilon} + u^2)^{\iota}} \sum_{n=0}^{\infty} \frac{\omega^{nj+k}\,(\gamma)_{qn}}{n!\,\Gamma(\beta + \alpha(nj + k))} \left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} t^{\lambda+r\tau+\sigma(nj+k)-1}\right)(x).$$

Now, applying (1.21), we have

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1} S_{\iota,\delta}^{(\upsilon,\xi)}\left(u, d; t^{\tau}\right) E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x)$$

$$= \omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1} \sum_{r=0}^{\infty} \frac{2d_r^{\xi}\,(\delta)_r}{r!\,(d_r^{\upsilon} + u^2)^{\iota}} \sum_{n=0}^{\infty} \frac{\omega^{nj}\,(\gamma)_{qn}}{n!\,\Gamma(\beta + \alpha(nj + k))} \frac{\Gamma(\lambda + \sigma k + \sigma nj)}{\Gamma(-\mu + \lambda + \sigma k + \sigma nj)}$$

$$\times \frac{\Gamma(\nu - \mu + \lambda + \sigma k + \sigma nj)\,\Gamma(\nu + \acute{\nu} + \acute{\mu} - \eta + \lambda + \sigma k + \sigma nj)}{\Gamma(\nu + \acute{\nu} - \eta + \lambda + \sigma k + \sigma nj)\,\Gamma(\nu + \acute{\mu} - \eta + \lambda + \sigma k + \sigma nj)} x^{r\tau+\sigma nj}$$

$$= \frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \frac{2d_r^{\xi}\,(\delta)_r}{(d_r^{\upsilon} + u^2)^{\iota}} \frac{(x^{\tau})^r}{r!} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + qn)\,\Gamma(\lambda + \sigma k + \sigma nj)}{\Gamma(\beta + \alpha k + \alpha nj)\,\Gamma(-\mu + \lambda + \sigma k + \sigma nj)}$$

$$\times \frac{\Gamma(\nu - \mu + \lambda + \sigma k + \sigma nj)\,\Gamma(\nu + \acute{\nu} + \acute{\mu} - \eta + \lambda + \sigma k + \sigma nj)}{\Gamma(\nu + \acute{\nu} - \eta + \lambda + \sigma k + \sigma nj)\,\Gamma(\nu + \acute{\mu} - \eta + \lambda + \sigma k + \sigma nj)} \frac{(\omega^j x^{\sigma j})^n}{n!},$$

which, in terms of (1.23), leads to the right side of (3.1). □

Corollary 3.2. *Let $\nu,\mu,\rho,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda+r\tau+\sigma k)>-min\{0,\Re(\nu+\mu+\rho)\}$, $\Re(\alpha)>0$, $\Re(\beta)>0$ $\Re(\gamma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^+$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{\mu+\lambda+\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_3\Psi_3\left[\begin{array}{c}(\gamma,q),(\lambda+r\tau+\sigma k,\sigma j),(\nu+\mu+\rho+\lambda+r\tau+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(\mu+\lambda+r\tau+\sigma k,\sigma j),(\rho+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{3.2}$$

Corollary 3.3. *Let $\rho,\nu,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda+r\tau+\sigma k)>-\Re(\rho+\nu)$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^+$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following left fractional derivative formula holds true:*

$$\left(D_{\rho,\nu}^{+}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{\lambda+\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(\rho+\nu+\lambda+r\tau+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{3.3}$$

Corollary 3.4. *Let $\nu,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\tau)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\sigma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^+$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{-\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(\lambda+r\tau+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(-\nu+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{3.4}$$

If we put $\gamma=q=1$ in the Theorem 3.1, we get the following new result involving the product of Mittag-Leffler function $E_{\alpha,\beta}^{j,k}$ and Mathieu-type series:

Corollary 3.5. *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\tau,\alpha,\beta,\sigma,\omega\in\mathbb{C}$ such that $\Re(\lambda+r\tau+\sigma k)>max\{0,\Re(-\nu-\acute{\nu}-\acute{\mu}+\eta),\Re(\mu-\nu)\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^+$, $j\geq 1$, $k\geq 0$. Let $D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the left-sided operator of Marichev-Saigo-Maeda fractional derivative. Then*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)=\omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(1,1),(\lambda+r\tau+\sigma k,\sigma j),(\nu-\mu+\lambda+r\tau+\sigma k,\sigma j),\\(\beta+\alpha k,\alpha j),(-\mu+\lambda+r\tau+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+r\tau+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+r\tau+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+r\tau+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{3.5}$$

Corollary 3.6. *If we set $j=1$ and $k=0$ in the Theorem 3.1, we obtain the following known result due to Khan et al. [23] concerning left Marichev-Saigo-Maeda fractional derivative operator:*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{\gamma,q}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{x^{\nu+\acute{\nu}-\eta+\lambda-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(\gamma,q),(\lambda+r\tau,\sigma),(\nu-\mu+\lambda+r\tau,\sigma),(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+r\tau,\sigma)\\(\beta,\alpha),(-\mu+\lambda+r\tau,\sigma),(\nu+\acute{\nu}-\eta+\lambda+r\tau,\sigma),(\nu+\acute{\mu}-\eta+\lambda+r\tau,\sigma)\end{array}\middle|\omega x^{\sigma}\right]. \tag{3.6}$$

Theorem 3.7. *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ with $\Re(\lambda+r\tau-\sigma k)<1+min\{\Re(\acute{\mu}),\Re(-\nu-\acute{\nu}+\eta),\Re(-\acute{\nu}-\mu+\eta)\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^+$, $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Let $D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of Marichev-Saigo-Maeda fractional derivative. Then*

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times\,{}_4\Psi_4\left[\begin{array}{c}(\gamma,q),(\acute{\mu}-\lambda-r\tau+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda-r\tau+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda-r\tau+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta-\lambda-r\tau+\sigma k+1,\sigma j)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{3.7}$$

Proof. In view of (1.26), (1.28) and the left-hand side of (3.7), we get

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$=\sum_{r=0}^{\infty}\frac{2d_r^{\xi}\,(\delta)_r}{r!\,(d_r^{\upsilon}+u^2)^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj+k}\,(\gamma)_{qn}}{n!\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+r\tau-\sigma(nj+k)-1}\right)(x).$$

Applying (1.22), we find

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}\sum_{r=0}^{\infty}\frac{2d_{r}^{\xi}\,(\delta)_{r}}{r!\,(d_{r}^{\upsilon}+u^{2})^{\iota}}\sum_{n=0}^{\infty}\frac{\omega^{nj}\,(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\frac{\Gamma(\acute{\mu}-\lambda+\sigma k+\sigma nj+1)}{\Gamma(-\lambda+\sigma k+\sigma nj+1)}$$
$$\times\frac{\Gamma(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1)\,\Gamma(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1)}{\Gamma(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1)\,\Gamma(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1)}x^{r\tau-\sigma nj}$$
$$=\frac{\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{r=0}^{\infty}\frac{2d_{r}^{\xi}\,(\delta)_{r}}{(d_{r}^{\upsilon}+u^{2})^{\iota}}\frac{(x^{\tau})^{r}}{r!}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\,\Gamma(\acute{\mu}-\lambda+\sigma k+\sigma nj+1)}{\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(-\lambda+\sigma k+\sigma nj+1)}$$
$$\times\frac{\Gamma(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1)\,\Gamma(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1)}{\Gamma(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1)\,\Gamma(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1)}\frac{(\omega^{j}x^{-\sigma j})^{n}}{n!}.$$

Now by using (1.23) in the above equation, we get the required result. □

Corollary 3.8. *Let $\nu,\mu,\rho,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda+r\tau-\sigma k)<1+min\{\Re(-\mu)-m,\Re(\nu+\rho)\}$, $m=[\Re(\nu)]+1$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq1$, $k\geq0$, $q\in(0,1)\cup\mathbb{N}$. Then the following right fractional derivative formula holds true:*

$$\left(D_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{\mu+\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times{}_{3}\Psi_{3}\left[\begin{array}{c}(\gamma,q),(-\mu-\lambda-r\tau+\sigma k+1,\sigma j),(\nu+\rho-\lambda-r\tau+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j),(-\mu+\rho-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right].\tag{3.8}$$

Corollary 3.9. *Let $\rho,\nu,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\lambda+r\tau-\sigma k)<\Re(\rho+\nu)-[\Re(\nu)]$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq1$, $k\geq0$, $q\in(0,1)\cup\mathbb{N}$. Then the following right fractional derivative formula holds true:*

$$\left(D_{\rho,\nu}^{-}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times{}_{2}\Psi_{2}\left[\begin{array}{c}(\gamma,q),(\rho+\nu-\lambda-r\tau+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right].\tag{3.9}$$

Corollary 3.10. *Let $\nu,\lambda,\tau,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\lambda+r\tau-\sigma k)<\Re(\nu)-[\Re(\nu)]$ and $j\geq1$, $k\geq0$. Then the following right fractional derivative formula holds true:*

$$\left(D_{-}^{\nu}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^{k}x^{-\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times{}_{2}\Psi_{2}\left[\begin{array}{c}(\gamma,q),(\nu-\lambda-r\tau+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right].\tag{3.10}$$

If we set $\gamma=q=1$ in the Theorem 3.7, we have the following new result involving the product of Mittag-Leffler function $E_{\alpha,\beta}^{j,k}$ and Mathieu-type series:

Corollary 3.11. *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\tau,\alpha,\beta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda+r\tau-\sigma k)<1+min\{\Re(\acute{\mu}),\Re(-\nu-\acute{\nu}+\eta),\Re(-\acute{\nu}-\mu+\eta)\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\upsilon,\xi,\iota,\delta\in\mathbb{R}^{+}$, $j\geq1$, $k\geq0$. Let $D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of Marichev-Saigo-Maeda fractional derivative. Then*

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times{}_{4}\Psi_{4}\left[\begin{array}{c}(1,1),(\acute{\mu}-\lambda-r\tau+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda-r\tau+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda-r\tau+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda-r\tau+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta-\lambda-r\tau+\sigma k+1,\sigma j)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda-r\tau+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right].\tag{3.11}$$

Corollary 3.12. *If we take $j=1$ and $k=0$ in the Theorem 3.7, we obtain the following known result due to Khan et al. [23] concerning right Marichev-Saigo-Maeda fractional derivative operator:*

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;t^{\tau}\right)E_{\alpha,\beta}^{\gamma,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{x^{\nu+\acute{\nu}-\eta+\lambda-1}}{\Gamma(\gamma)}S_{\iota,\delta}^{(\upsilon,\xi)}\left(u,d;x^{\tau}\right)$$
$$\times{}_{4}\Psi_{4}\left[\begin{array}{c}(\gamma,q),(\acute{\mu}-\lambda-r\tau+1,\sigma),(-\nu-\acute{\nu}+\eta-\lambda-r\tau+1,\sigma),(-\acute{\nu}-\mu+\eta-\lambda-r\tau+1,\sigma)\\(\beta,\alpha),(-\lambda-r\tau+1,\sigma),(-\acute{\nu}+\acute{\mu}-\lambda-r\tau+1,\sigma),(-\nu-\acute{\nu}-\mu+\eta-\lambda-r\tau+1,\sigma)\end{array}\middle|\omega x^{-\sigma}\right].\tag{3.12}$$

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