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On P-essential Submodules and P-uniform Modules

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Abstract.
Significant areas of study in module theory include expansions of the concepts of essential submodules and uniform modules. This paper focuses on two well-known concepts: the P-essential submodule and the P-uniform module, which were introduced by Nada M. Al-Thani and Maria M. Baher with Muna A. Ahmed, respectively. They did not emphasize these concepts in their original work, as they were not central to their research.

A thorough exploration and development of the P-essential submodule is advantageous because it enables researchers to create and discover various types of modules, serving as a foundation for their structures. Consequently, this paper is dedicated to advancing the understanding of the P-essential submodule. We introduce and discuss numerous characteristics and other characterizations of this type of submodule. Additionally, we provide proof under certain conditions that the P-socle of a module, denoted as $Soc_p(M)$, is a finitely pure cogenerated and P-essential submodule of M if and only if M is finitely pure cogenerated. Furthermore, we examined the P uniform module structurally dependent on the P-essential submodule in greater detail. The relationships between P-essential and P-uniform modules with semi-essential submodules and semi-uniform modules are also investigated.

1 Introduction

A submodule N of M is called essential (briefly, $N \leq_e M$), if $\forall\ K \leq M$, the condition $N \cap K = 0$ implies $K = 0$, [15]. A non-zero module M is named uniform if each non-zero submodule of M is essential, [12, 15]. A submodule T of M is called pure (for short, $T \leq_{pu} M$) if $T \cap IM = IT$ for every ideal I of R, [7]. A submodule T of M is P-essential (briefly, $N \leq_{pe} M$) if, for all $S \leq_{pu} M$, $T \cap S = 0$, implying that $S = 0$, [2, 5], it is denoted by $T \leq_{pe} M$. Al-Thani introduced the concept of P-essential in 1997, but she didn't discuss all its properties. P-essential submodule forms the nucleus and basis for the structure of several modules, such as the pure closed submodule which appeared in [5].

No one developed or used P-essential in their work except Maria M. Baher and Muna A. Ahmed, [9]. They used it to introduce the notion of P-uniform module, where a non-zero module M is P-uniform if each non-zero submodule of M is P-essential, [1, 2, 9]. Every uniform module is P-uniform.

This paper is interested in a comprehensive study of P-essential submodules and P-uniform modules. We will endeavour to discuss and develop these types of submodules in more detail to answer the following question:

How can we find results about the P-essential submodule and P-uniform module corresponding to those results known in the essential submodule and uniform module?

This article consists of nine sections. Section 2 presents several features of P-essential submodules that had appeared in [5]. In Section 3, more properties about P-essential submodules are introduced, for example, if N' is a pure relative complement of a non-zero submodule N in M, then $N \oplus N' \leq_{pe} M$, see Proposition 3.16. Other characterizations of P-essential submodules are considered, see Propositions 3.1 and 3.19. Also, we will show in Proposition 3.25 that when an R-module M satisfies GPSP, every P-essential R-submodule of M is finitely generated if and only if M is Noetherian. In Section 4, the direct sum of P-essential submodules is studied, see Propositions 4.1, 4.2, and 4.3. In Section 5, the P-socle of any module is established and studied, see Proposition 5.3. Furthermore, under certain conditions, Theorem 5.8 proved that $Soc_p(M)$ is a finitely pure cogenerated and P-essential in M if and only if M is a finitely pure cogenerated. Section 6 investigates the relationship between P-essential and semi-essential submodules, see Propositions 6.2 and 6.4. Section 7 develops the concept of a P-uniform module; for instance, the hereditary property of P-uniform property between M and its submodules is studied, as shown in Propositions 7.2 and 7.4. Besides that, Theorem 7.3 gives another characterization of the P-uniform module. Additionally, the relationship between P-uniform and semi-uniform modules is discussed, see Propositions 7.6 and 7.8.

Finally, it should be noted that in this study, the symbols R and M denote commutative rings with unity and unitary left R-modules, respectively.

2 Preliminaries

This section investigates the main characteristics of P-essential submodules, which appeared in [5]. Before that, let us highlight some known notes.

Remark 2.1.

1. Each module is a P-essential submodule of itself.
2. Every essential submodule is P-essential. The converse holds when M is uniform.
3. In any F-regular module, there is no difference between P-essential and essential submodules, where an R-module M is F-regular if every submodule of M is pure, [13].

The following proposition appeared in [5]; we give the proof for completeness.

Proposition 2.2. ([5], Theorem 4.2)

The following statements are equivalent for any module M:

i. $K \leq_{pe} M$.

ii. $\forall\ N \leq M$ *with* $h \in PH_R(M, N)$, *if* $\ker h \cap K = 0$ *then h is a monomorphism, where* $PH_R(M, N)$=*{*$h|h\colon M \to N$ *such that ker h is a pure submodule of M}.*

Proof.

$\Rightarrow$) is followed directly by the definition of the P-essential submodule.

$\Leftarrow$) Because each $N \leq_{pu} M$ is the kernel of some $h \in PH_R(M, A)$ and some R-module A, then $L \cap K = 0$, and by (ii), L is zero. Thus, $K \leq_{pe} M$.

An R-module M has the pure intersection (pure finite intersection) property if the intersection of any family (finite family) of a pure submodule of M is again pure ([5], Definition 2.16). It is denoted by PIP (and we use the symbol PFIP for the pure finite intersection property).

Theorem 2.3. ([5], Theorem 4.4)

For submodules K and N of M with $K \leq N \leq M$, *the statements below are satisfied:*

a. If $K \leq_{pe} M$ *then* $N \leq_{pe} M$.

b. If $N \leq_{pu} M$ *and* $K \leq_{pe} M$, *then* $N \leq_{pe} M$.

c. Let M has PFIP and $N \leq_{pu} M$, *then* $K \leq_{pe} M$ *if and only if* $K \leq_{pe} N \& N \leq_{pe} M$.

Proposition 2.4. ([5], Corollary 4.5)

Assume that M has PFIP. If H is pure in M then $H \cap K \leq_{pe} M$ if and only if $H \leq_{pe} M$ and $K \leq_{pe} M$ for any $K \leq M$.

Remember that if T and K are submodules of M with $K \leq_{pu} M$, then K is called a pure relative complement of T in M if K is the maximal submodule with the property $K \cap T = 0$, [5].

Remark 2.5. ([5], Proposition 4.14 and Proposition 4.15)

1. Every submodule of module M has a pure relative complement in M.
2. Given the ideals A and B of a ring R. If B is a pure relative complement of A in R, then $A \oplus B \leq_{pe} R$.

3 More properties of P-essential submodules

This section introduces other characteristics of P-essential submodules as analogues of those known in the essential submodules. Firstly, we will provide another characterization of the definition of P-essential.

Proposition 3.1. *Let A be a submodule of a module M. Then $A \leq_{pe} M$ if and only if $A \cap L \neq 0$ for each $0 \neq L \leq_{pu} M$.*

Proof.

$\Rightarrow$) Let $0 \neq L \leq_{pu} M$, to prove $A \cap L \neq 0$. If $A \cap L = 0$ then $L = 0$ since $A \leq_{pe} M$. But $L \neq 0$, so we get a contradiction.

$\Leftarrow$) Let $L \leq_{pu} M$ with $A \cap L = 0$, to prove $L = 0$. Suppose that $L \neq 0$, by assumption $A \cap L \neq 0$, which is a contradiction. Thus, $A \leq_{pe} M$.

Proposition 3.2. *Let M have the PIP and B_1, B_2 are submodules of M, with $A_1 \leq B_1 \leq M$, $A_2 \leq B_2 \leq M$. Assume that $A_1 \leq_{pu} M$ and $A_2 \leq_{pu} M$. If A_1 and A_2 are P-essential in B_1 and B_2, respectively, then $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.*

Proof. Let $0 \neq N \leq B_1 \cap B_2 \leq_{pu} M$. To show that $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$, we must verify that $(A_1 \cap A_2) \cap N \neq 0$. Since A_1 and A_2 are P-essential in B_1 and B_2, respectively, then $A_1 \cap N \neq 0$ and $A_2 \cap N \neq 0$. On the other hand, $A_1 \leq_{pu} M$ and $N \leq_{pu} M$, as well as M has PIP therefore, $A_1 \cap N \leq_{pu} M$. This implies that $A_1 \cap N \leq_{pu} B_2$, [22]. But $A_2 \leq_{pe} B_2$ then $(A_1 \cap N) \cap A_2 \neq 0$, hence $(A_1 \cap A_2) \cap N \neq 0$. Thus, $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.

When $B_1 = B_2 = M$, we deduce the following.

Corollary 3.3. *Let M have the PIP. If the pure submodules A_1 and A_2 are P-essential in M, then $A_1 \cap A_2 \leq_{pe} M$.*

Recall that a ring R is called regular if, for each $r \in R$ there exists $x \in R$ such that $r = rxr$, [15].

Corollary 3.4. *Let M be a module over a regular ring R, and let B_1 and B_2 be submodules of M with $A_1 \leq B_1 \leq M$, $A_2 \leq B_2 \leq M$. If A_1 and A_2 are P-essential in B_1 and B_2, respectively, then $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.*

Proof. Since R is a regular ring, then M is F- regular ([22], Remark 1.2(2), P.29). This means every submodule of M is pure; in particular, both A_1 and A_2 pure. Hence, the result follows from proposition 3.2.

Remember that M is a multiplication module if each $T \leq M$ can be written as T=IM, [8, 20]. It is known that every multiplication module has the PIP, so we have the following.

Corollary 3.5. *Let M be a multiplication module, B_1 and B_2 are submodules of M with $A_1 \leq B_1 \leq M$, $A_2 \leq B_2 \leq M$. Assume that $A_1 \leq_{pu} M$ and $A_2 \leq_{pu} M$. If A_1 and A_2 are P-essential in B_1 and B_2, respectively, then $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.*

Recall that any monomorphism $f : N \to M$ is P-essential monomorphism whenever $f(N) \leq_{pe} M$, [5]. The proof of the remark below is obvious, so it is omitted.

Remark 3.6. $N \leq_{pe} M$ if and only if the inclusion $i : N \to M$ is a P-essential monomorphism.

Remark 3.7. If $A \leq_{pe} B$ and $R \leq_{pe} S$ then in general $A + R \nleq_{pe} B + S$ as shown in the example below.

Example 3.8. Consider the Z-module $M = Z \oplus (Z/2Z)$, assume that $A = R = Z(2, \overline{0})$, $B = Z(1, \overline{0})$ and $S = Z(1, \overline{1})$. Note that $A \leq_{pe} B$ and $R \leq_{pe} S$, while A+R is not P-essential in B+S. Take $Z(0, \overline{1})$. One can show that the submodule $Z(0, \overline{1})$ is pure in B+S. Since $A + R = A$, then $A \cap Z(0, \overline{1}) = 0$. Thus $A + R \nleq_{pe} B + S$.

Proposition 3.9. *Given two P-essential submodules S and T of M. If $S + T \leq_{pu} M$, then $S + T \leq_{pe} M$.*

Proof. We have $S \leq S+T \leq M$. Since $S+T \leq_{pu} M$ then by Theorem 2.3(b), $S+T \leq_{pe} M$.

A module M has "pure sum property" (for short, PSP) if the sum of any two pure submodules of M is again pure, [4]. This property is applicable as follows.

Proposition 3.10. *If a module M has PSP, then the sum of any two P-essential submodules of M is also P-essential.*

To generalize Proposition 3.10, we introduce the following.

Definition 3.11. A module M has a “generalized pure sum property” (simply, GPSP) if the sum of any number (finite or infinite) of pure submodules of M is again pure.

Proposition 3.12. *Take a module M that satisfies GPSP. The sum of any number (finite or infinite) of P-essential submodules of M is P-essential.*

Proof. Take a family of P-essential submodules $\{N\}_{i \in I}$ of M, where I is the index set of numbers. Since M satisfies GPSP, $\sum_{i \in I} N_i$ is pure. So, we have $N_i \leq \sum_{i \in I} N_i \leq M$. By Theorem 2.3(b), $\sum_{i \in I} N_i \leq_{pe} M$.

We need the following lemma.

Lemma 3.13. ([19], Remark 1.4, P.37)

Let $K \leq T \leq M$ with $T \leq_{pu} M$, then T/K is pure in M/K.

Proposition 3.14. *Let M be an R-module, $K \leq M$, with $\pi : M \to M/K$ a natural epimorphism. If $T/K \leq_{pe} M/K$, then $T \leq_{pe} M \quad \forall\, T \leq M$.*

Proof. Take $L \leq_{pu} M$ with $T \cap L = 0$, to prove that $L = 0$. We can easily show that $\pi(T \cap L) = (T \cap L)/K = T/K \cap L/K$. But $T \cap L = 0$, thus $T/K \cap L/K = 0$. On the other hand, $L \leq_{pu} M$, so by Lemma 3.8, L/K is pure in M/K. Since $T/K \leq_{pe} M/K$, then $L/K = 0$, hence $L = 0$. That is $T \leq_{pe} M$.

Proposition 3.15. *Let T and V be non-zero submodules of M with a pure submodule V. If V is a pure relative complement of T in M, then $(T \oplus V)/V \leq_{pe} M/V$.*

Proof. Consider the natural epimorphism $f : M \to M/V$ and a pure relative complement V of T in M. Take a non-zero $A/V \leq_{pu} M/V$ such that $((T \oplus V)/V) \cap A/V = 0$. So $(T \oplus V) \cap A = V$, and by modular Law, $(T \cap A) + V = V$. This implies that $T \cap A \leq V$ as well as $T \cap A \leq T$, hence:

$$T \cap A \leq T \cap V \ldots\ldots (*)$$

In contrast, V is a pure relative complement of T; thus, $T \cap V = 0$, so by (*), $T \cap A = 0$. But V is the maximal pure submodule with $T \cap V = 0$, therefore $A = V$, hence $A/V = 0$. That is $(T \oplus V)/V \leq_{pe} M/V$.

Proposition 3.16. *Given an R-module and $0 \neq V \leq M$. If V' is a pure relative complement of V in M, then $V \oplus V' \leq_{pe} M$.*

Proof. Take the projection homomorphism $\pi : M \to M/V'$. Since V' is a pure relative complement of V in M, then by Proposition 3.15, $(V \oplus V')/V' \leq_{pe} M/V'$, and by Proposition 3.14, $V \oplus V' \leq_{pe} M$.

Proposition 3.17. *Let M be an R-module, $V \leq M$. If $\ker f \cap V \neq 0$ for each homomorphism f from M to any R-module M', then $V \leq_{pe} M$.*

Proof. Given $0 \neq P \leq_{pu} M$, and take the projection homomorphism $\pi : M \to M/P$. By hypothesis, $\ker \pi \cap V \neq 0$, hence $P \cap V \neq 0$. Thus, $V \leq_{pe} M$.

Proposition 3.18. *For any isomorphism $f : M \to M'$. If $V \leq_{pe} M$ then $f(V) \leq_{pe} M'$.*

Proof. Take $0 \neq P \leq_{pu} M'$. f is an epimorphism implies $0 \neq f^{-1}(P) \leq_{pu} M$, [14]. Since $V \leq_{pe} M$, then $V \cap f^{-1}(P) \neq 0$. But f is a monomorphism, therefore $f(V) \cap P \neq 0$. That is $f(V) \leq_{pe} M'$.

Next, another characterization of the P-essential submodule is introduced by using elements.

Proposition 3.19. *A submodule U of an R-module M is P-essential if and only if, for all $V \leq_{pu} M$, $(V \neq 0)$, $\exists\, v \in V$ and $r \in R$ with $0 \neq rv \in U$.*

Proof. Assume that $U \leq_{pe} M$, so $U \cap V \neq 0$, $\forall V \leq_{pu} M$. This implies $0 \neq v \in U \cap V$, which means $v \in U$ and $v \in V$. Therefore, we found $r = 1 \in R$ with $0 \neq 1.v = v \in U$. Conversely, suppose V is a non-zero pure submodule and there exists $x \in V$ and $r \in R$ such that $0 \neq rx \in U$. Since r and x belong to R and V, respectively, $rx \in V$. This implies that $0 \neq rx \in U \cap V$. Hence $U \cap V \neq 0$, thus, $U \leq_{pe} M$.

Compare the following theorem with ([11], Theorem 2.13).

Theorem 3.20. *Let M be a finitely generated, faithful, and multiplication module. A submodule $N \leq_{pe} M$ if and only if there is a P-essential ideal I of R with $N = IM$.*

Proof. For the first direction, suppose that $N \leq_{pe} M$, so there exists $I \leq R$ such that $N = IM$. We must show that $I \leq_{pe} R$. Assume that $I \cap B = 0$ for some $B \leq_{pu} R$. Now, by ([11], Theorem 1.6):

$$0 = (I \cap B)M = IM \cap BM = N \cap BM$$

Hence $N \cap BM = 0$. Now, $B \leq_{pu} R$ implice $BM \leq_{pu} M$ ([6], Theorem 1.4(2), P. 67). On the other hand, $N \leq_{pe} M$, therefore $BM = 0$. But M is faithful, thus $B = 0$. That is, $I \leq_{pe} R$. Conversely, assume there is $E \leq_{pe} R$ with $N = EM$, and take a pure submodule K of M such that $EM \cap K = 0$. M is multiplication, which implies $K = CM$ for some ideal C of R. By ([11], Theorem 1.6):

$$0 = EM \cap K = EM \cap CM = (E \cap C)M$$

Hence $(E \cap C)M = 0$. Because M is faithful, then $E \cap C = 0$. In contrast, K is a pure submodule and M is faithful; therefore, C is pure [6]. Since $E \leq_{pe} R$, then $C = 0$, hence $K = 0$. Thus, $EM \leq_{pe} M$. Put $EM \equiv N$, so $N \leq_{pe} M$.

Proposition 3.21. *Given a non-zero multiplication module M having only one maximal submodule T. If $T \neq 0$, then $T \leq_{pe} M$.*

Proof. Assume that $P \leq_{pu} M$ with $P \cap T = 0$. If $P = M$, then $M \cap T = 0$, hence $T = 0$ which is a contradiction. Otherwise, P is a proper submodule. M is a non-zero multiplication module implies that P is contained in some maximal submodule of M ([11], Theorem 2.5). But M has only one maximal submodule T, so $P \subseteq T$; thus, $P = 0$.

Proposition 3.22. *Take a finitely generated module M having only one non-zero maximal submodule T, then $T \leq_{pe} M$.*

Proof. Since M is finitely generated, then as in the same proof of Proposition 3.16, using ([21], Proposition 1.6, P. 7) instead of ([11], Theorem 2.5), we obtain $T \leq_{pe} M$.

An R-module M satisfies the ascending chain condition (simply, ACC) if all ascending chains of submodules of M are stationary. For a P-essential submodule, we introduce the following:

Definition 3.23. An R-module M satisfies the ascending chain condition on P-essential submodules if any ascending chain of P-essential submodules:

$$V_1 \leq V_2 \leq V_3 \leq \cdots V_n \leq \cdots$$

is stationary.

Proposition 3.24. *Given a module M satisfying GPSP. If each P-essential submodule of M is finitely generated, then M satisfies ACC on P-essential submodules.*

Proof. Suppose that:

$$V_1 \leq V_2 \leq V_3 \leq \cdots V_n \leq \cdots$$

It is an ascending chain of P-essential submodules in M. Put $\sum_i V_i = V$. Since M satisfies GPSP, then by Proposition 3.12, $V \leq_{pe} M$, and by assumption, V is finitely generated, so there is a finite set I_0 of I with $V = \sum_{i \in I_0} V_i$. Hence, the chain is stationary.

Any module is Noetherian if all its submodules are finitely generated, equivalently, any module is Noetherian if it satisfies ACC [16].

Proposition 3.25. *Let M be an R-module satisfying GPSP. Every P-essential R-submodule of M is finitely generated if and only if M is Noetherian.*

Proof. The necessity follows from Proposition 3.24. For the converse, since M is a Noetherian module, then every submodule of M is finitely generated, hence every P-essential submodule is finitely generated.

4 Direct sum of P-essential submodules

This section investigates and discusses the direct sum of P-essential submodules.

Proposition 4.1. *Consider $M = M_1 \oplus M_2$, where M_1 and M_2 are submodules of M. Take the submodules K_1 and K_2 of M_1 and M_2 respectively. If $K_1 \oplus K_2 \leq_{pe} M_1 \oplus M_2$ then $K_1(K_2) \leq_{pe} M_1(M_2)$, provided that each pure submodule of $M_1(M_2)$ is also pure in M.*

Proof. Take $P_1 \leq_{pu} M_1$ with $K_1 \cap P_1 = 0$. One can easily show that $(K_1 \oplus K_2) \cap P_1 = 0$. Since $P_1 \leq_{pu} M_1$, then by assumption, $P_1 \leq_{pu} M$. Besides that $K_1 \oplus K_2 \leq_{pe} M$, Thus, $P_1 = 0$. This means $K_1 \leq_{pe} M_1$. Use a similar proof for K_2.

Recall that a non-zero module M is fully P-essential if every P-essential submodule is essential [10].

Proposition 4.2. *Consider the fully P-essential module $M = M_1 \oplus M_2$, where M_1 and M_2 are submodules of M. Suppose that $T_1 \leq M_1$ and $T_2 \leq M_2$. If $T_1 \oplus T_2 \leq_{pe} M_1 \oplus M_2$ then $T_1 \leq_{pe} M_1$ and $T_2 \leq_{pe} M_2$.*

Proof. M is a fully P-essential module implying $T_1 \oplus T_2 \leq_e M_1 \oplus M_2$. By ([7], Proposition 5.16, P.74), $T_1 \leq_e M_1$ and $T_2 \leq_e M_2$. But every essential submodule is P-essential. Therefore, the desired is achieved.

Proposition 4.3. *Given $M = M_1 \oplus M_2$, where M_1 and M_2 are fully P-essential submodules of M. Assume that $T_1 \leq M_1$ and $T_2 \leq M_2$. If $T_1 \leq_{pe} M_1$ and $T_2 \leq_{pe} M_2$ then $T_1 \oplus T_2 \leq_{pe} M_1 \oplus M_2$.*

Proof. Because M_1 and M_2 are fully P-essential modules, then both T_1 and T_2 are essential in M_1 and M_2, respectively. Thus, $T_1 \oplus T_2 \leq_e M_1 \oplus M_2$, ([7], Proposition 5.20(2), P.75). But each essential is P-essential, so $T_1 \oplus T_2 \leq_{pe} M_1 \oplus M_2$.

Corollary 4.4. *Suppose that $M = M_1 \oplus M_2$, where all submodules of M are fully P-essential. Given $T_1 \leq M_1$ and $T_2 \leq M_2$. Then $T_1 \oplus T_2 \leq_{pe} M_1 \oplus M_2$ if and only if $T_1 \leq_{pe} M_1$ and $T_2 \leq_{pe} M_2$.*

Proof. M is a submodule of itself, so it is fully P-essential, and the necessity follows directly from Proposition 4.2. For sufficiency, since M_1 and M_2 are submodules of M, then they are entirely essential, and the result obtained from Proposition 4.3.

5 P-Socle of modules

This section studies in detail the intersection of all P-essential submodules.

Definition 5.1. Let M be an R-module. We call the set $\bigcap\{K \setminus K \leq_{pe} M\}$ P-socle of M. It is denoted by $Soc_p(M)$.

Remarks and Examples 5.2.

1. $Soc_p(M) \neq \phi$ and it is a submodule of M.
2. $Soc_p(Z) = 0$, $Soc_p(Z_{10}) = Z_{10}$, $Soc_p(Z_{12}) = 2Z_{12}$.
3. $Soc_p(Z_{p^\infty}) = (\frac{1}{p} + Z)$.
4. $Soc_p(M) \subseteq Soc(M)$.
5. $L \subseteq M$ implies $Soc_p(L) \subseteq Soc_p(M)$.
6. If R is a regular ring, then $Soc_p(M) = Soc(M)$.

A module M is pure split if every pure submodule of M is a direct summand [5].

Proposition 5.3. *For any R-module M the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold:*

1. *$Soc_p(M) = M$.*
2. *M has no proper P-essential submodule.*
3. *M is pure split.*

Also, $(3) \Rightarrow (1)$ when R is a regular ring.

Proof.

$(1) \Rightarrow (2)$. Take $L \leq_{pe} M$. Since $Soc_p(M) = \bigcap\{K \setminus K \leq_{pe} M\}$, then $Soc_p(M) \subseteq L$. On the other hand, $Soc_p(M) = M$, therefore, $M = L$.

$(2) \Rightarrow (3)$. Given $N \leq M$. Let N' be a pure relative complement of N; by Proposition 3.16, $N \oplus N' \leq_{pe} M$. But if M has no proper P-essential submodule, then $N \oplus N' = M$. Thus, N is a direct summand of M.

$(3) \Rightarrow (1)$. Suppose that R is a regular ring, so there is no difference between a pure and any other submodule. By (3), every submodule of M is a direct summand. This implies that $Soc(M) = M$, ([15], P.26). Since R is regular, so by Remark 5.2(6), $Soc_p(M) = Soc(M)$. Hence $Soc_p(M) = M$.

An R-module M is called finitely cogenerated if, for every set $\{A_i \setminus i \in I\}$ of submodules A_i of M with $\bigcap_{i \in I} A_i = 0$, there is a finite subset $\{A_i \setminus i \in I_0\}$ with $\bigcap_{i \in I_0} A_i = 0$ (i.e. $I_0 \subset I$ and I_0 is finite), ([16], Definition 2.3.14, P.29). This motivates us to introduce the concept below.

Definition 5.4. A module M is finitely pure cogenerated if for every set $\{A_i \setminus i \in I\}$ of a pure submodules A_i of M with $\bigcap_{i \in I} A_i = 0$, there is a finite subset $\{A_i \setminus i \in I_0\}$ of a pure submodules with $\bigcap_{i \in I_0} A_i = 0$.

Proposition 5.5. *Let M be a module that has PFIP. If $Soc_p(M)$ is a finitely pure cogenerated and P-essential submodule of M, then M is a finitely pure cogenerated.*

Proof. Assume that $Soc_p(M)$ is a finitely cogenerated submodule and $Soc_p(M) \leq_{pe} M$. Let $T = \{A_i \setminus i \in I\}$ be a set of submodules of M with $\bigcap_{i\in I} A_i = 0$, then $\bigcap_{i\in I}\{A_i \cap Soc_p(M) \setminus i \in I\} = 0$. Now, $A_i \cap Soc_p(M) \subset Soc_p(M)$ and $Soc_p(M)$ is finitely cogenerated, so there exists a finite set $I_0 \subset I$ with $\bigcap_{i\in I_0}\{A_i \cap Soc_p(M)\} = 0$. Hence $(\bigcap_{i\in I_0} A_i) \cap Soc_p(M) = 0$. Because A_i is pure $\forall i \in I_0$ and M has PIFP, then $\bigcap_{i\in I_0} A_i$ is a pure submodule of M. Moreover, $Soc_p(M) \leq_{pe} M$, therefore $\bigcap_{i\in I_0} A_i = 0$. Thus, M is a finitely pure cogenerated.

It is known that the intersection of two P-essential submodules may not be P-essential. For the following result, we need to provide the following.

Condition 5.6. The intersection of any number of P-essential submodules is again P-essential.

Proposition 5.7. *Given a module that satisfies condition 5.6. If M is a finitely pure cogenerated, then $Soc_p(M)$ is a finitely pure cogenerated and $Soc_p(M) \leq_{pe} M$.*

Proof. Assume that M is a finitely pure cogenerated. It is clear that any submodule (especially, $Soc_p(M)$) of a finitely pure cogenerated module is also finitely pure cogenerated. To prove that $Soc_p(M) \leq_{pe} M$, take a pure submodule K of M with $Soc_p(M) \cap K = 0$. By definition of $Soc_p(M)$, $\exists\, U_1, U_2, \ldots\ldots\ldots, U_n \leq_{pe} M$ such that $(U_1 \cap U_2, \ldots\ldots\ldots, \cap U_n) \cap K = 0$. Since M satisfies condition 5.6, then $U_1 \cap U_2, \ldots\ldots\ldots, \cap U_n \leq_{pe} M$, hence $K = 0$.

The following is obtained from Proposition 5.5 and Proposition 5.7.

Theorem 5.8. *Take module M, which has PFIP and satisfies condition 5.6. Then $Soc_p(M) \leq_{pe} M$ and a finitely pure cogenerated in M if and only if M is a finitely pure cogenerated.*

6 P-essential and semi-essential submodules

This section discusses the relationship between P-essential and semi-essential submodules, where a submodule N of M is semi-essential if $N \cap L \neq 0$ for each prime submodule L$of M$ [18]. An R-module M is called torsion-free if $T(M) = 0$, where $T(M) = \{m \in M | rm = 0$, for some non-zero $r \in R\}$, [15]. And module M is prime if $ann_R(M) = ann_R(N)$ for every non-zero submodule N of M, [3].

Firstly, we need the following lemma.

Lemma 6.1.

1. *Given a torsion-free module M, a proper submodule T of M is pure if and only if it is prime and $(T\colon {}_R M) = 0$ [17].*
2. *Given a prime R-module M, a proper submodule T of M is pure if and only if it is a prime and $(T\colon {}_R M) = ann_R(M)$ [3].*

Proposition 6.2. *Let M be a non-zero torsion-free R-module. A submodule T of M is P-essential if and only if T is a semi-essential submodule, provided that $(T\colon {}_R M) = 0$.*

Proof. Suppose that $T \leq_{pe} M$, so $T \cap L \neq 0\ \forall L \leq_{pu} M$. Since M is torsion-free as well as $(T\colon {}_R M) = 0$, by Lemma 6.1(1), every pure submodule T of M is prime. Thus, $T \cap L \neq 0$ for each prime submodule L of M, and T is a semi-essential submodule. In the same way, the converse can be done.

Corollary 6.3. *Take a faithful and multiplication module M. Any submodule T of M is P-essential if and only if T is semi-essential, provided that $(T\colon {}_R M) = 0$.*

Proof. M is a faithful and multiplication module implies that M is torsion-free ([11], Lemma 4.1, P. 773), and the result follows from Proposition 6.2.

Proposition 6.4. *Let M be a non-zero prime R-module. A submodule T of M is P-essential if and only if T is a semi-essential submodule, provided that $(T\colon {}_R M) = ann_R(M)$.*

Proof. Assume that $T \leq_{pe} M$, this means $T \cap L \neq 0\ \forall\ L \leq_{pu} M$. Because M is a prime module and $(T\colon {}_R M) = ann_R(M)$, then every pure submodule is prime by Lemma 6.1(2). Therefore, $T \cap L \neq 0$ for each prime $L \leq M$.

7 P-uniform Modules

As mentioned earlier, a P-uniform module is a non-zero module in which all non-zero submodules are P-essential. This section aims to develop this category of modules.

Proposition 7.1. *If M/T is a P-uniform module for any R-module M, then M is P-uniform.*

Proof. Take $0 \neq L \leq M$. L/T is a non-zero submodule of M/T. Because M/T is P-uniform, so $L/T \leq_{pe} M/T$. By Proposition 3.14, $L \leq_{pe} M$; therefore, M is a P-uniform module.

It is known that any submodule of a uniform module is uniform. As an analogue to this idea, we introduce the following.

Proposition 7.2. *Every pure submodule of a P-uniform module is P-uniform.*

Proof. Given a pure submodule N of M, assume that $0 \neq K \leq N$. Since $K \leq M$ and M is P-uniform, then $K \leq_{pe} M$. But $N \leq_{pu} M$, therefore $K \leq_{pe} N$, [5]. Thus, N is P-uniform.

In the following, we give another characterization of the P-uniform module.

Theorem 7.3. *An R-module M is P-uniform if and only if any non-zero submodule N of M satisfies $N \cap L \neq 0$ for any $0 \neq L \leq_{pu} M$.*

Proof. Let M be a P-uniform module and $0 \neq N \leq M$. Take $0 \neq L \leq_{pu} M$ with $N \cap L \neq 0$. If $N \cap L = 0$, then $N \nleq_{pe} M$, hence M is not P-uniform. But this is not true; thus, $N \cap L \neq 0$. For the converse, suppose that $0 \neq N \leq M$ and take $L \leq_{pu} M$ with $N \cap L = 0$. If $L \neq 0$ then by assumption; $N \cap L \neq 0$. But this contradicts our assumption; therefore, $L = 0$. This implies $N \leq_{pe} M$, hence M is P-uniform.

As a consequence of Theorem 7.3, we have the following.

Corollary 7.4. *Let M be an R-module having PIP, and N is a pure and essential submodule of M. If N is a P-uniform submodule, then M is a P-uniform module.*

Proof. Given non-zero submodules L and K of M with $L \leq_{pu} M$. Since N is essential, $N \cap K \neq 0$ and $N \cap L \neq 0$. Since N is pure and M has PIP, then $N \cap L \leq_{pu} N$. On the other hand, N is P-uniform and both $N \cap K$, $N \cap L$ belong to N, so by Theorem 7.3 $(N \cap K) \cap (N \cap L) \neq 0$, hence $N \cap (K \cap L) \neq 0$. But $N \leq_e M$ then $K \cap L \neq 0$. Again, by Theorem 7.3, we obtain that M is a P-uniform module.

Next, the relationships between P-uniform and the two concepts of uniform and semi-uniform modules are given in the following two propositions.

Proposition 7.5.

1. *Any module M is a P-uniform module and fully P-essential if and only if M is a uniform module.*
2. *Any module M is a P-uniform and F-regular if and only if M is a uniform module.*

Proof.

1. For the sufficiency, take $0 \neq T \leq M$. Because M is a P-uniform module, then $T \leq_{pe} M$. But M is fully P-essential; therefore, $T \leq_e M$. That is, M is uniform. The necessity is obvious.
2. It is similar to point (1), except replacing the concept fully P-essential with F-regular and using Remark 2.1(3).

Proposition 7.6. *Let M be a non-zero torsion-free R-module and $(T\colon {}_RM) = 0$ for each submodule T of M. Then M is a P-uniform module if and only if M is a semi-uniform module.*

Proof. Given $0 \neq T \leq M$. Because M is a P-uniform module, then $T \leq_{pe} M$. But M is a torsion-free R-module and $(T\colon {}_RM) = 0$, so by Proposition 6.2, $T \leq_{sem} M$, hence M is a semi-uniform module. The other direction is done similarly.

It is known that any faithful and multiplication module is torsion-free. This fact helps us to obtain the following.

Corollary 7.7. *Any faithful and multiplication module M that satisfies* $(T\colon {}_RM) = 0\ \forall\, T \leq M$, *is a semi-uniform module if and only if M is a P-uniform module.*

Proposition 7.8. *Given a non-zero prime module M and* $(T\colon {}_RM) = ann_R(M)\ \ \forall\, T \leq M$. *Then M is a semi-uniform module if only if M is a P-uniform module.*

Proof. For the necessity, take $0 \neq T \leq M$. Because M is semi-uniform, then $T \leq_{sem} M$. But M is a prime R-module and $(T\colon {}_RM) = ann_R(M)$, so by Proposition 6.4, $T \leq_{pe} M$. Hence, M is a P-uniform module. The sufficiency is proved similarly.

8 Results and Discussion

Proposition 3.2 determined the condition under which the intersection of two P-essential submodules is P-essential. Proposition 3.16 presents the important result that the direct sum of any non-zero submodule with its pure relative complement is a P-essential submodule. Besides, it has illustrated in Propositions 3.19 and 7.3 other characterizations of the definitions of P-essential submodules and P-uniform modules, respectively. In the same context, we found the connection between P-essential submodules and P-essential ideals if they satisfy certain conditions, and these conditions exist in Theorem 3.20. Based on Proposition 4.1, Propositions 4.2 and 4.3, we explain how the finite direct sum of the P-essential submodules is P-essential and vice versa if they have specific criteria. Moreover, we introduced P-socal of modules and show in Theorem 5.8, that if a module M has PFIP and satisfies condition 5.6, then Soc_p is a finitely pure cogenerated and P-essential in M if and only if M is a finitely pure cogenerated. Also, we found the link between the P-uniform module and both uniform and semi-uniform modules, where they were equivalent under a certain condition, which is clarified in Propositions 7.5, 7.6, and Proposition 7.8.

9 Conclusions

Nada M. Al-Thani first proposed P-essential submodules in 1997, while P-uniform modules were first proposed by Maria M. Baher and Muna A. Ahmed in their work in 2023. In this paper, these important notions were studied in more detail. The reason that motivated us to study these kinds of modules is their important role in module theory, as we mentioned at the beginning of this paper. This study answered several questions related to the characteristics of P-essential submodules and P-uniform modules. Many results, analogous to those in essential submodules and uniform modules, were discussed. P-Socle of any R-module was given and established. Other characterizations of these concepts were explored. In addition, the relationships of P-essential and P-uniform with semi-essential submodules and semi-uniform modules, respectively, were considered.

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