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On complex symmetric weighted Toeplitz operators

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Abstract In this paper, we provide a characterization of a complex symmetric weighted Toeplitz operator T_ϕ on $H^2(\beta)$ with respect to the conjugation $C_{\mu,\lambda}$. Moreover, for any conjugation C and the symbol ϕ of the form $\phi(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$, we give necessary and sufficient conditions for the weighted Toeplitz operator T_ϕ to be complex symmetric. We also discuss complex symmetric weighted Toeplitz operator T_ϕ with non-harmonic symbols on $H^2(\beta)$. Finally, we study the Toeplitz graphs of some complex symmetric weighted Toeplitz operators with finite symbols.

1 Introduction

Let $B(H)$ denote the set of all bounded linear operators on a separable complex Hilbert space H.

Definition 1.1. A conjugation on H is an anti-linear operator $C : H \to H$ such that $\langle Cf, Cg \rangle = \langle g, f \rangle$ for all $f, g \in H$ and $C^2 = I$.

Given any conjugation C, we can find an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for H satisfying $Ce_n = e_n$ for all $n \in \mathbb{N} \cup \{0\}$ [4].

Definition 1.2. An operator $T \in B(H)$ is complex symmetric if there exists a conjugation C on H such that $T = CT^*C$.

Complex symmetric operators on Hilbert spaces are natural generalizations of complex symmetric matrices. The study of complex symmetric operators was initiated by Garcia and Putinar [4] in the year 2006. An operator is complex symmetric if and only if it can be written as a symmetric matrix relative to some orthonormal basis $\{e_n\}$ for H. Many well-known operators such as normal operators, Hankel operators, truncated Toeplitz operators, etc., are included in the class of complex symmetric operators. In 2016, Ko and Lee [7] gave a characterization of complex symmetric Toeplitz operators on the Hardy space. Recently, many authors have studied complex symmetric Toeplitz on various spaces [2, 6, 7, 8, 9, 11]. In 2005, Lauric studied the weighted Toeplitz operators on $H^2(\beta)$. Toeplitz graphs have been introduced by G. Sierksma and the hamiltonian properties of such graphs have been discussed by Van Dal et al. [12]. In this paper, we concentrate on characterizing the complex symmetric weighted Toeplitz operators on $H^2(\beta)$ with symbols from $L^\infty(\beta)$. Finally, we draw the Toeplitz graphs of some complex symmetric weighted Toeplitz operators with finite symbols.

Definition 1.3. A Toeplitz graph is an undirected graph associated with a symmetric Toeplitz adjacency matrix, say A, of order n. Let $0, 1, 2, \ldots, n-1$ denote the distinct n diagonals of A. The main diagonal 0 contains only zeroes. Let $d_1, d_2, \ldots, d_k$ be the diagonals which contain ones such that $0 \le d_1 \le d_2 \le \cdots \le d_k$. The vertex set is $\{1, 2, \ldots, n\}$, and the two vertices u

and v are connected by an edge if and only if $|u-v| \in \{d_1, d_2, \ldots, d_k\}$. The Toeplitz graph will be denoted by $T_n\langle d_1, d_2, \ldots, d_k\rangle$. If $n = \infty$, then the Toeplitz graph is infinite.

Let $\beta = (\beta_m)_{m\in\mathbb{Z}}$ be a sequence of positive real numbers with $\beta_0 = 1$ such that for all $m \in \mathbb{N}$,

$$M_1 \leq \frac{\beta_m}{\beta_{m+1}} \leq M_2$$

for some $0 < M_1 \leq M_2 < \infty$. Then the space $L^2(\beta)$ defined by

$$L^2(\beta) = \left\{ f(z) = \sum_{m=-\infty}^{\infty} a_m z^m \mid a_m \in \mathbb{C}, \|f\|_\beta^2 := \sum_{m=-\infty}^{\infty} |a_m|^2 \beta_m^2 < \infty \right\}$$

is a Hilbert space [10] with the inner product given by

$$\left\langle \sum_{m=-\infty}^{\infty} a_m z^m, \sum_{m=-\infty}^{\infty} b_m z^m \right\rangle = \sum_{m=-\infty}^{\infty} a_m \bar{b}_m \beta_m^2.$$

The set $\left\{ e_m(z) = \frac{z^m}{\beta_m} \mid m \in \mathbb{Z} \right\}$ forms an orthonormal basis for $L^2(\beta)$. Also, the set $H^2(\beta)$ defined by

$$H^2(\beta) = \left\{ f(z) = \sum_{m=0}^{\infty} a_m z^m \mid a_m \in \mathbb{C}, \|f\|_\beta^2 := \sum_{m=0}^{\infty} |a_m|^2 \beta_m^2 < \infty \right\}$$

forms a subspace of $L^2(\beta)$. If $\beta_m = 1$, $\beta_m = \frac{1}{\sqrt{m+1}}$, $\beta_m = \sqrt{m+1}$ or $\beta_m = \sqrt{\frac{(m+\nu)(m+\xi)}{\nu\xi}}$ where $\nu,\xi \in \mathbb{N}$, then $H^2(\beta)$ can be viewed as a Hilbert Hardy space, Bergman space, Dirichlet space or generalized derivative Hardy space respectively [5, 9]. Again, the space $L^\infty(\beta)$ defined by

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{m=-\infty}^{\infty} a_m z^m \mid \exists\, c \in \mathbb{R}, \phi L^2(\beta) \subseteq L^2(\beta), \|\phi f\|_\beta \leq c\|f\|_\beta \;\forall\, f \in L^2(\beta) \right\}$$

is a Banach space [1] with respect to the norm defined by

$$\|\phi\|_\beta = \inf\{c \mid \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for all } f \in L^2(\beta)\}.$$

Moreover, if $\beta_m = 1$ for each $m \in \mathbb{Z}$ and the functions considered are complex valued measurable functions defined over the *unit circle* $\mathbb{T}$, then the spaces $L^2(\beta)$, $H^2(\beta)$ and $L^\infty(\beta)$ coincide with the classical spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ respectively.

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$.

Definition 1.4. For $\phi \in L^\infty(\beta)$, the *weighted Toeplitz operator*, T_ϕ, on $H^2(\beta)$ is defined by

$$T_\phi f := P(\phi f) \text{ for all } f \in H^2(\beta).$$

Now, we prove the following lemmas.

Lemma 1.5. *For $m, n \in \mathbb{N} \cup \{0\}$,*

$$\langle z^n, z^m \rangle = \begin{cases} \beta_m^2 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. The result follows immediately from the definition of inner product defined on $L^2(\beta)$. □

Lemma 1.6. *For* $m, n \in \mathbb{N} \cup \{0\}$,

$$P(\overline{z}^n z^m) = \begin{cases} \dfrac{\beta_m^2}{\beta_{m-n}^2} z^{m-n} & \textit{if } m \geq n, \\ 0 & \textit{if } m < n. \end{cases}$$

Proof. Let s be a non-negative integer. If $s \neq m - n$, then

$$\langle P(\overline{z}^n z^m), z^s \rangle = 0.$$

Let $s = m - n$. If $m < n$, then

$$\langle P(\overline{z}^n z^m), z^s \rangle = 0.$$

If $m \geq n$, then

$$\left\langle P(\overline{z}^n z^m), z^{m-n} \right\rangle = \left\langle \overline{z}^n z^m, z^{m-n} \right\rangle = \|z^m\|_\beta^2 = \beta_m^2.$$

Therefore, we have

$$P(\overline{z}^n z^m) = \begin{cases} \dfrac{\beta_m^2}{\beta_{m-n}^2} z^{m-n} & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

□

Theorem 1.7. *Let C be a conjugation on $H^2(\beta)$. Then the following statements hold.*

(i) The Parseval's identity holds, i.e., $\sum_{m=0}^{\infty} |\langle f, Ce_m \rangle|^2 = \|f\|_\beta^2$ *for every* $f \in H^2(\beta)$.

(ii) The set $\left\{ Ce_m(z) := \dfrac{Cz^m}{\beta_m} \right\}$ *forms an orthonormal basis for* $H^2(\beta)$.

Proof. Since C is a conjugation on $H^2(\beta)$, we have

$$\langle Ce_n, Ce_m \rangle = \langle e_m, e_n \rangle = \frac{1}{\beta_m \beta_n} \langle z^m, z^n \rangle = \delta_{mn}.$$

We first show that Parseval's identity holds. Let $Cf(z) = \sum_{k=0}^{\infty} \tilde{a}_k z^k$. Now, for any $f \in H^2(\beta)$,

$$\|f\|_\beta^2 = \|Cf\|_\beta^2 = \sum_{k=0}^{\infty} {\beta_k}^2 |\tilde{a}_k|^2$$

and

$$\begin{aligned} \langle f(z), Ce_m(z) \rangle &= \langle e_m(z), Cf(z) \rangle \\ &= \left\langle \frac{z^m}{\beta_m}, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right\rangle \\ &= \frac{1}{\beta_m} \sum_{k=0}^{\infty} \overline{\tilde{a}}_k \langle z^m, z^k \rangle \\ &= \frac{1}{\beta_m} \overline{\tilde{a}}_m \langle z^m, z^m \rangle \\ &= \beta_m \overline{\tilde{a}}_m. \end{aligned}$$

Therefore,

$$\sum_{m=0}^{\infty} |\langle f, Ce_m \rangle|^2 = \sum_{m=0}^{\infty} {\beta_m}^2 |\overline{\tilde{a}}_m|^2 = \|f\|_\beta^2 \ \forall\ f \in H^2(\beta).$$

Thus, the Parseval's identity holds. So, $f = \sum_{m=0}^{\infty} \langle f, Ce_m \rangle Ce_m$ for every $f \in H^2(\beta)$. Hence, $\{Ce_m\}$ forms an orthonormal basis for $H^2(\beta)$. □

Remark 1.8. If $\beta_m = \sqrt{\dfrac{(m+\nu)(m+\xi)}{\nu\xi}}$, then lemma 1.6 and theorem 1.7 become lemma 2.1 and theorem 2.3 respectively of [9].

In 2016, the authors in [7] gave the conjugation $C_{\mu,\lambda}$ on the Hardy space. Now, we prove the following lemma.

Lemma 1.9. *For every μ and λ with $|\mu| = |\lambda| = 1$, the map $C_{\mu,\lambda} : H^2(\beta) \to H^2(\beta)$ defined by*

$$C_{\mu,\lambda}\, f(z) = \mu\overline{f(\lambda\overline{z})}$$

is a conjugation on $H^2(\beta)$.

Proof. It is clear that $C_{\mu,\lambda}$ is an antilinear involutive operator on $H^2(\beta)$. Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \text{ and } g(z) = \sum_{m=0}^{\infty} b_m z^m$$

be two elements of $H^2(\beta)$. Now,

$$\begin{aligned}
\langle C_{\mu,\lambda}\, f(z), C_{\mu,\lambda}\, g(z)\rangle &= \left\langle \mu\overline{f(\lambda\overline{z})}, \mu\overline{g(\lambda\overline{z})}\right\rangle \\
&= \left\langle \sum_{m=0}^{\infty} \bar{a}_m \overline{\lambda}^m z^m, \sum_{m=0}^{\infty} \bar{b}_m \overline{\lambda}^m z^m \right\rangle \\
&= \sum_{m=0}^{\infty} \bar{a}_m b_m \beta_m^2 \\
&= \langle g(z), f(z)\rangle.
\end{aligned}$$

Hence, $C_{\mu,\lambda}$ is a conjugation on $H^2(\beta)$. □

2 Complex symmetric weighted Toeplitz operators

In this section, we characterize complex symmetric weighted Toeplitz operator T_ϕ on $H^2(\beta)$ with respect to the above conjugation $C_{\mu,\lambda}$. Also, for any conjugation C and the symbol ϕ of the form $\phi(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$, we give the necessary and sufficient conditions for the weighted Toeplitz operator T_ϕ to be complex symmetric.

Lemma 2.1. *If P denotes the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$, then the operators $C_{\mu,\lambda}$ and P commute.*

Proof. If $n \in \mathbb{Z}$, then

$$Pe_n(z) = \begin{cases} e_n(z) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

and

$$C_{\mu,\lambda}\, e_n(z) = \mu\overline{e_n(\lambda\overline{z})} = \frac{\mu\overline{\lambda}^n z^n}{\beta_n} = \mu\overline{\lambda}^n e_n(z).$$

For $n \geq 0$, we have

$$\begin{aligned}
PC_{\mu,\lambda}\, e_n(z) &= P\left(\mu\overline{\lambda}^n e_n(z)\right) \\
&= \mu\overline{\lambda}^n e_n(z) \\
&= C_{\mu,\lambda}\, e_n(z) \\
&= C_{\mu,\lambda}\, Pe_n(z).
\end{aligned}$$

For $n < 0$, we have

$$PC_{\mu,\lambda}\, e_n(z) = P\left(\mu\overline{\lambda}^n e_n(z)\right) = 0 = C_{\mu,\lambda} 0 = C_{\mu,\lambda}\, Pe_n(z).$$

Hence, the operators $C_{\mu,\lambda}$ and P commute. □

Theorem 2.2. *If $\phi \in L^\infty(\beta)$, then the weighted Toeplitz operator T_ϕ on $H^2(\beta)$ is complex symmetric with conjugation $C_{\mu,\lambda}$ if and only if $\phi(z) = \phi(\lambda\overline{z})$.*

Proof. Let T_ϕ be any weighted Toeplitz operator on $H^2(\beta)$. It can be easily seen that

$$C_{\mu,\lambda} = \mu I C_{1,\lambda} = C_{1,\lambda}\, \overline{\mu} I.$$

So, we have

$$C_{\mu,\lambda}\, T_\phi C_{\mu,\lambda} = \mu I C_{1,\lambda}\, T_\phi\, \mu I C_{1,\lambda} = \mu I C_{1,\lambda}\, \mu I T_\phi C_{1,\lambda} = C_{1,\lambda}\, \overline{\mu} I \mu I T_\phi C_{1,\lambda} = C_{1,\lambda}\, T_\phi C_{1,\lambda}.$$

From this, we see that T_ϕ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if T_ϕ is complex symmetric with the conjugation $C_{1,\lambda}$, i.e.,

$$T_{\phi(z)} = C_{1,\lambda}\, T^*_{\phi(z)}\, C_{1,\lambda} = C_{1,\lambda}\, T_{\overline{\phi(z)}}\, C_{1,\lambda}.$$

By lemma 2.1, we have

$$\begin{aligned}
C_{1,\lambda} T_{\overline{\phi(z)}} C_{1,\lambda} f(z) &= C_{1,\lambda} T_{\overline{\phi(z)}}\, \overline{f(\lambda\overline{z})} \\
&= C_{1,\lambda} P\left(\overline{\phi(z)}\,\overline{f(\lambda\overline{z})}\right) \\
&= P C_{1,\lambda}\left(\overline{\phi(z)}\,\overline{f(\lambda\overline{z})}\right) \\
&= P\left(\phi(\lambda\overline{z}) f(z)\right) \\
&= T_{\phi(\lambda\overline{z})} f(z).
\end{aligned}$$

This is true for all $f \in H^2(\beta)$. So,

$$C_{1,\lambda}\, T_{\overline{\phi(z)}}\, C_{1,\lambda} = T_{\phi(\lambda\overline{z})},$$

and hence $T_{\phi(z)} = T_{\phi(\lambda\overline{z})}$, i.e., $\phi(z) = \phi(\lambda\overline{z})$. □

Corollary 2.3. ([6], Theorem 2.2) *Let $\beta_n = 1$ for each $n \in \mathbb{Z}$ and $\phi \in L^\infty(\mathbb{T})$, then the weighted Toeplitz operator T_ϕ on $H^2(\mathbb{T})$ is complex symmetric with conjugation $C_{\mu,\lambda}$ if and only if $\phi(z) = \phi(\lambda\overline{z})$ on $|z| = 1$.*

Theorem 2.4. *Let $\phi \in L^\infty(\beta)$ such that $\phi(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$. If T_ϕ is a Toeplitz operator on $H^2(\beta)$, then the following statements are equivalent.*

(i) T_ϕ is complex symmetric with the conjugation $C_{\mu,\lambda}$.

(ii) $\phi(z) = \phi(\lambda\overline{z})$.

(iii) $\overline{\hat{\phi}(-n)} = \lambda^n \hat{\phi}(n)$ for all $n \in \mathbb{N} \cup \{0\}$ with $|\lambda| = 1$.

Proof. The equivalence of (1) and (2) follows from the theorem 2.2. Now, we prove the equivalence of (2) and (3). If $\phi(z) = \phi(\lambda\overline{z})$, then

$$\sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{\lambda}^n z^n + \sum_{n=0}^{\infty} \hat{\phi}(n) \lambda^n\, \overline{z}^n.$$

Therefore, $\overline{\hat{\phi}(-n)} = \lambda^n \hat{\phi}(n)$ for all $n \in \mathbb{N} \cup \{0\}$ with $|\lambda| = 1$.

Conversely, if $\overline{\hat{\phi}(-n)} = \lambda^n \hat{\phi}(n)$ for all $n \in \mathbb{N} \cup \{0\}$, then

$$\phi(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n = \sum_{n=1}^{\infty} \lambda^n \hat{\phi}(n) \overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$$

and

$$\phi(\lambda\overline{z}) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{\lambda}^n z^n + \sum_{n=0}^{\infty} \hat{\phi}(n) \lambda^n\, \overline{z}^n = \sum_{n=1}^{\infty} \hat{\phi}(n) z^n + \sum_{n=0}^{\infty} \lambda^n \hat{\phi}(n) \overline{z}^n.$$

Therefore, $\phi(z) = \phi(\lambda\overline{z})$. □

Theorem 2.5. *Let $\phi \in L^\infty(\beta)$ such that $\phi(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{z}^n + \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$ and C be a conjugation on $H^2(\beta)$. Then the operator T_ϕ on $H^2(\beta)$ is complex symmetric with conjugation C if and only if $\hat{\phi}(-k) = C\hat{\phi}(k)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. Let $f(z) = \sum_{j=0}^{\infty} a_j z^n$ and $Cf(z) = \sum_{j=0}^{\infty} \tilde{a}_j z^j$. Let $\phi_+(z) = \sum_{n=0}^{\infty} \hat{\phi}(n) z^n$ and $\phi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\phi}(-n)}\,\overline{z}^n$. By lemma 1.6, we have

$$\phi_+ f = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \hat{\phi}(k) a_n z^{n+k},$$

$$P(\phi_- f) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\beta_n^2}{\beta_{n-k}^2} \overline{\hat{\phi}(-k)} a_n z^{n-k},$$

$$P(\overline{\phi_+} Cf) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{\beta_n^2}{\beta_{n-k}^2} \overline{\hat{\phi}(k)} \tilde{a}_n z^{n-k} \text{ and}$$

$$\overline{\phi_-} Cf = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \hat{\phi}(-k) \tilde{a}_n z^{n+k}.$$

Let C be a conjugation on $H^2(\beta)$ and T_ϕ be a complex symmetric operator with conjugation C. Now, we have

$$\phi_+ f + P(\phi_- f) = CP(\overline{\phi_+} Cf) + C(\overline{\phi_-} Cf).$$

This equation implies that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \hat{\phi}(k) a_n z^{n+k} + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\beta_n^2}{\beta_{n-k}^2} \overline{\hat{\phi}(-k)} a_n z^{n-k}$$
$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{\beta_n^2}{\beta_{n-k}^2} \widetilde{\overline{\hat{\phi}(k)}} a_n z^{n-k} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \widetilde{\hat{\phi}(-k)} a_n z^{n+k}.$$

Comparing the constant terms, we have

$$\beta_k^2\, \overline{\hat{\phi}(-k)}\, a_k = \beta_k^2\, \widetilde{\overline{\hat{\phi}(k)}}\, a_k.$$

Since a_k is arbitrary, we get $\hat{\phi}(-k) = C\hat{\phi}(k)$ for all $k \in \mathbb{N} \cup \{0\}$. Conversely, if $\hat{\phi}(-k) = C\hat{\phi}(k)$ for all $k \in \mathbb{N} \cup \{0\}$, then we find that T_ϕ is complex symmetric with the conjugation C. □

3 Complex symmetric weighted Toeplitz operators with non-harmonic symbols

In this section, we discuss complex symmetric weighted Toeplitz operators with non-harmonic symbols. The condition $\bar{z}^n z^m = z^{m-n}$ is true in $H^2(\mathbb{T})$. However, this is not true in $H^2(\beta)$. The following theorem gives necessary and sufficient conditions for weighted Toeplitz operators to be complex symmetric with conjugation on non-harmonic symbols.

Theorem 3.1. *Let $\phi \in L^\infty(\beta)$ such that $\phi(z) = \sum_{i=0}^{\infty}(a_i \overline{z}^{m_i} z^{n_i} + b_i \overline{z}^{s_i} z^{t_i})$ for $a_i, b_i \in \mathbb{C}$ and let $m_i - n_i = t_i - s_i$ hold. Then the weighted Toeplitz operator T_ϕ on $H^2(\beta)$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if ϕ is of the form*

$$\phi(z) = \sum_{i=0}^{\infty} \left(a_i|z|^{2n_i} + b_i|z|^{2t_i}\right)$$

or

$$\phi(z) = \sum_{i=0}^{\infty} a_i(\overline{z}^{m_i} z^{n_i} + \lambda^{n_i - m_i} \overline{z}^{n_i} z^{m_i})$$

for $a_i, b_i \in \mathbb{C}$.

Proof. Let us assume that $m_i > n_i$ for $i \in \mathbb{N}$ and that T_ϕ is complex symmetric with the conjugtaion $C_{\mu,\lambda}$. If $k \geq \max_{i\in\mathbb{N}}\{m_i - n_i\}$, then

$$\begin{aligned} C_{\mu,\lambda} T_\phi z^k &= C_{\mu,\lambda} P\left(\sum_{i=0}^{\infty}(a_i \overline{z}^{m_i} z^{n_i+k} + b_i \overline{z}^{s_i} z^{t_i+k})\right) \\ &= C_{\mu,\lambda}\left(\sum_{i=0}^{\infty}\left(\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}} a_i z^{n_i+k-m_i} + \frac{\beta^2_{t_i+k}}{\beta^2_{t_i+k-s_i}} b_i z^{t_i+k-s_i}\right)\right) \\ &= \sum_{i=0}^{\infty}\left(\mu\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}}\overline{a}_i\overline{\lambda}^{n_i+k-m_i} z^{n_i+k-m_i} + \mu\frac{\beta^2_{t_i+k}}{\beta^2_{t_i+k-s_i}}\overline{b}_i\,\overline{\lambda}^{t_i+k-s_i} z^{t_i+k-s_i}\right) \end{aligned}$$

and

$$\begin{aligned} T^*_\phi C_{\mu,\lambda} z^k &= T_{\bar{\phi}}\mu\overline{\lambda}^k z^k \\ &= \mu\overline{\lambda}^k P\left(\sum_{i=0}^{\infty}(\overline{a}_i z^{m_i+k}\,\overline{z}^{n_i} + \overline{b}_i z^{s_i+k}\,\overline{z}^{t_i})\right) \\ &= \sum_{i=0}^{\infty}\left(\mu\overline{\lambda}^k\frac{\beta^2_{m_i+k}}{\beta^2_{m_i+k-n_i}}\overline{a}_i z^{m_i+k-n_i} + \mu\overline{\lambda}^k\frac{\beta^2_{s_i+k}}{\beta^2_{s_i+k-t_i}}\overline{b}_i z^{s_i+k-t_i}\right). \end{aligned}$$

Since T_ϕ is complex symmetric with conjugation $C_{\mu,\lambda}$, we have

$$m_i = n_i \text{ and } s_i = t_i$$

for all $i \geq 0$. That is, ϕ is of the form

$$\phi(z) = \sum_{i=0}^{\infty} a_i|z|^{2n_i} + b_i|z|^{2t_i}$$

or

$$\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}}\overline{a}_i\overline{\lambda}^{m_i-n_i} = \frac{\beta^2_{s_i+k}}{s_i+k-t_i}\overline{b}_i \tag{3.1}$$

and

$$\frac{\beta^2_{t_i+k}}{\beta^2_{t_i+k-s_i}}\overline{b}_i\overline{\lambda}^{t_i-s_i} = \frac{\beta^2_{m_i+k}}{m_i+k-n_i}\overline{a}_i \tag{3.2}$$

for all $i \in \mathbb{N}$. By equations (3.1) and (3.2), we get

$$s_i = n_i,\ t_i = m_i \text{ and } a_i = b_i\lambda^{m_i-n_i} \text{ for all } i \geq 0$$

and so, ϕ is of the form

$$\phi(z) = \sum_{i=0}^{\infty} a_i(\overline{z}^{m_i} z^{n_i} + \lambda^{n_i-m_i}\,\overline{z}^{n_i} z^{m_i}).$$

Conversely, if ϕ is of the form

$$\phi(z) = \sum_{i=0}^{\infty} a_i(\overline{z}^{m_i} z^{n_i} + \lambda^{n_i-m_i}\, \overline{z}^{n_i} z^{m_i}),$$

then

$$\begin{aligned}
& C_{\mu,\lambda}\, T_\phi \sum_{k=0}^{\infty} \alpha_k z^k \\
&= C_{\mu,\lambda}\, P\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(a_i\alpha_k \overline{z}^{m_i} + a_i\alpha_k\lambda^{n_i-m_i}\,\overline{z}^{n_i} z^{m_i+k})\right) \\
&= C_{\mu,\lambda}\left(\sum_{i=0}^{\infty}\left(\sum_{k=m_i-n_i}^{\infty}\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}} a_i\alpha_k z^{n_i+k-m_i} + \sum_{k=0}^{\infty}\frac{\beta^2_{m_i+k}}{\beta^2_{m_i+k-n_i}} a_i\alpha_k\lambda^{n_i-m_i} z^{m_i+k-n_i}\right)\right) \\
&= \sum_{i=0}^{\infty}\left(\sum_{k=m_i-n_i}^{\infty}\mu\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}}\overline{a}_i\overline{\alpha}_k\overline{\lambda}^{n_i+k-m_i} z^{n_i+k-m_i} + \sum_{k=0}^{\infty}\mu\frac{\beta^2_{m_i+k}}{\beta^2_{m_i+k-n_i}}\overline{a}_i\overline{\alpha}_k\overline{\lambda}^{k} z^{m_i+k-n_i}\right)
\end{aligned}$$

and

$$\begin{aligned}
& T^* C_{\mu,\lambda}\sum_{k=0}^{\infty}\alpha_k z^k \\
&= T_{\overline{\phi}}\sum_{k=0}^{\infty}\mu\overline{\alpha}_k\overline{\lambda}^k z^k \\
&= P\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\mu\overline{\alpha}_k\overline{\lambda}^k\left(\overline{a}_i\overline{z}^{n_i} z^{m_i+k} + \overline{a}_i\overline{\lambda}^{n_i-m_i}\,\overline{z}^{m_i} z^{n_i+k}\right)\right) \\
&= \sum_{i=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu\overline{\alpha}_k\overline{\lambda}^k\frac{\beta^2_{m_i+k}}{\beta^2_{m_i+k-n_i}}\overline{a}_i z^{m_i+k-n_i} + \sum_{k=m_i-n_i}^{\infty}\mu\overline{\alpha}_k\overline{\lambda}^k\frac{\beta^2_{n_i+k}}{\beta^2_{n_i+k-m_i}}\overline{a}_i\overline{\lambda}^{n_i-m_i} z^{n_i+k-m_i}\right).
\end{aligned}$$

Therefore, we obtain

$$C_{\mu,\lambda}\, T_\phi \sum_{k=0}^{\infty}\alpha_k z^k = T^*_\phi C_{\mu,\lambda}\sum_{k=0}^{\infty}\alpha_k z^k.$$

Hence, T_ϕ is complex symmetric with conjugation $C_{\mu,\lambda}$. Similarly, T_ϕ is complex symmetric with conjugation $C_{\mu,\lambda}$ if ϕ is of the form $\phi(z) = \sum_{i=0}^{\infty}(a_i|z|^{2n_i} + b_i|z|^{2t_i})$. This completes the proof. □

4 Toeplitz graph

A Toeplitz graph is an undirected graph with a symmetric Toeplitz adjacency matrix. In this section, we provide the matrix characterization of complex symmetric weighted Toeplitz operators with a particular class of symbols. We draw the graph of a complex symmetric Toeplitz operator on Hardy space by considering a specific symbol ϕ from $L^\infty(\mathbb{T})$. Again, we draw the graph of a complex symmetric weighted Toeplitz operator by finding the indicator binary matrix (see [3]).

Let us consider a particular class of weighted Toeplitz operators T_ϕ with symbols in the set

$$S = \left\{\phi \mid \phi(z) = \sum_{n=1}^{\infty} a_n\overline{z}^n + \sum_{n=0}^{\infty} a_n z^n\right\} \subseteq L^\infty(\beta).$$

We have seen that the operator T_ϕ is complex symmetric with respect to the conjugation $C_{\mu,1}$. Clearly,

$$\begin{aligned} T_\phi e_j &= P(\phi e_j) \\ &= P\left(\left(\sum_{n=1}^{\infty} a_n \overline{z}^n + \sum_{n=0}^{\infty} a_n z^n\right)\frac{z^j}{\beta_j}\right) \\ &= \sum_{n=1}^{j} a_n \frac{\beta_j}{\beta_{j-n}^2} z^{j-n} + \sum_{n=0}^{\infty} \frac{a_n}{\beta_j} z^{n+j} \\ &= \sum_{n=0}^{j-1} a_{j-n}\frac{\beta_j}{\beta_n} e_n + \sum_{n=j}^{\infty} a_{n-j}\frac{\beta_n}{\beta_j} e_n. \end{aligned}$$

The matrix of this weighted Toeplitz operator, denoted by $(\lambda_{ij})_{i,j=0}^{\infty}$, is defined by

$$\begin{aligned} \lambda_{ij} &= \langle T_\phi e_j, e_i\rangle \\ &= \left\langle \sum_{n=0}^{j-1} a_{j-n}\frac{\beta_j}{\beta_n} e_n + \sum_{n=j}^{\infty} a_{n-j}\frac{\beta_n}{\beta_j} e_n, e_i \right\rangle \\ &= \begin{cases} \dfrac{\beta_j}{\beta_i} a_{j-i} & \text{if } i < j, \\ \dfrac{\beta_i}{\beta_j} a_{i-j} & \text{if } i \geq j. \end{cases} \end{aligned}$$

Now, the matrix representation of T_ϕ is given below.

$$[T_\phi] = \begin{pmatrix} a_0 & \frac{\beta_1}{\beta_0}a_1 & \frac{\beta_2}{\beta_0}a_2 & \frac{\beta_3}{\beta_0}a_3 & \cdots \\ \frac{\beta_1}{\beta_0}a_1 & a_0 & \frac{\beta_2}{\beta_1}a_1 & \frac{\beta_3}{\beta_1}a_2 & \cdots \\ \frac{\beta_2}{\beta_0}a_2 & \frac{\beta_2}{\beta_1}a_1 & a_0 & \frac{\beta_3}{\beta_2}a_1 & \cdots \\ \frac{\beta_3}{\beta_0}a_3 & \frac{\beta_3}{\beta_1}a_2 & \frac{\beta_3}{\beta_2}a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $\dfrac{\beta_{n+1}}{\beta_n} = k$ for some $k \neq 0$ and for all $n \in \mathbb{N} \cup \{0\}$, then the matrix representation of T_ϕ becomes a Toeplitz matrix given by

$$[T_\phi] = \begin{pmatrix} a_0 & ka_1 & k^2a_2 & k^3a_3 & \cdots \\ ka_1 & a_0 & ka_1 & k^2a_2 & \cdots \\ k^2a_2 & ka_1 & a_0 & ka_1 & \cdots \\ k^3a_3 & k^2a_2 & ka_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we can now draw the Toeplitz graph of the complex symmetric weighted Toeplitz operator T_ϕ with symbols from the set S.

Example 4.1. We consider the sequence $\beta_n = 1$ for all n. Let $\phi(z) = z^{-3} + z^{-1} + z + z^3 \in L^\infty(\mathbb{T})$. Then the weighted Toeplitz operator T_ϕ is a complex symmetric operator with conjugation $C_{\mu,1}$ (by corollary 2.3). The matrix representation of T_ϕ with respect to the orthonormal basis $\mathcal{B} = \{e_n\}$ is a symmetric Toeplitz adjacency matrix given by

$$[T_\phi]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Toeplitz graph $T_\infty\langle 1, 3\rangle$ of the weighted Toeplitz operator T_ϕ is given in figure 1. It may be observed that the vertex 1 is of degree 2, the vertices 2 and 3 are of degree 3, and all the remaining vertices are of degree 4.

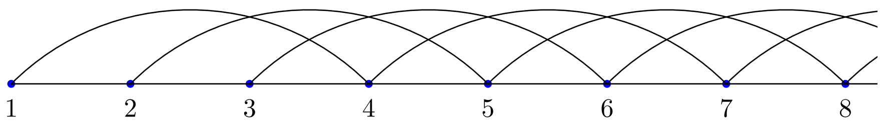

Figure 1. The graph $T_\infty\langle 1, 3\rangle$.

Example 4.2. Let $\phi(z) = 3\overline{z}^4 - \overline{z}^3 + 2\overline{z}^2 + 2z^2 - z^3 + 3z^4 \in S$ and $\dfrac{\beta_{n+1}}{\beta_n} = k$ for some $k \neq 0$ and for all $n \in \mathbb{N} \cup \{0\}$. Then the weighted Toeplitz operator T_ϕ is a complex symmetric operator with conjugation $C_{\mu,1}$. Its matrix representation is given by

$$[T_\phi] = \begin{pmatrix} 0 & 0 & 2k^2 & -k^3 & 3k^4 & 0 & \cdots \\ 0 & 0 & 0 & 2k^2 & -k^3 & 3k^4 & \cdots \\ 2k^2 & 0 & 0 & 0 & 2k^2 & -k^3 & \cdots \\ -k^3 & 2k^2 & 0 & 0 & 0 & 2k^2 & \cdots \\ 3k^4 & -k^3 & 2k^2 & 0 & 0 & 0 & \cdots \\ 0 & 3k^4 & -k^3 & 2k^2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The indicator binary matrix A of T_ϕ is given below.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 1 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The indicator binary matrix is symmetric and the Toeplitz graph of the adjacency matrix A of T_ϕ is given in figure 2. It may be observed that the vertices 1 and 2 are of degree 3, the vertices 3 and 4 are of degree 4 and 5 respectively, and all the remaining vertices are of degree 6.

5 Conclusion

Several authors have characterized complex symmetric Toeplitz operators on various spaces such as Hardy Hilbert space, Bergman space, Dirichlet space, Weighted Hardy space, etc., with respect to a particular conjugation or arbitrary conjugation. In this paper, we have given a characterization of complex symmetric weighted Toeplitz operator over $H^2(\beta)$ and symbols from

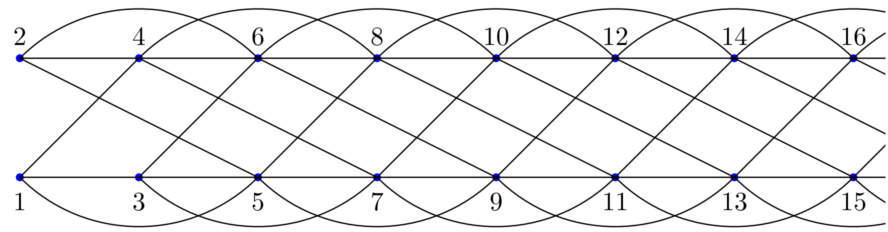

Figure 2. Graph $T_\infty\langle 2,3,4\rangle$ of the adjacency matrix A of T_ϕ.

$L^\infty(\beta)$. In the last section, we have drawn the graph of a Toeplitz operator on Hardy space with $\beta_m = 1$. The Toeplitz graph of the adjacency matrix of a complex symmetric weighted Toeplitz operator is also produced.

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