Palestine Journal of Mathematics

Vol 14(3)(2025) , 1005–1015

APPROXIMATION OF SIGNALS IN WEIGHTED ZYGMUND CLASS VIA ALMOST NÖRLUND MEANS OF CONJUGATE FOURIER SERIES

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Communicated by S. A. Mohiuddine

MSC 2010 Classifications: Primary 41A24, 41A25; Secondary 42B05, 42B08.

Keywords and phrases: Degree of approximation, Weighted Zygmund class, Conjugate Fourier series, Almost Nörlund mean.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract This study explores the approximation properties of periodic functions within various Zygmund spaces, including Z_α, $Z_{(\alpha,r)}$, $Z_r^{(\omega)}$, and the weighted Zygmund class $W(Z_r^{(\omega)})$ for $r \geq 1$, in the context of conjugate Fourier series. The analysis of these function spaces is crucial in emerging fields such as engineering and information technology, particularly in digital filtering and signal analysis. The primary objective of this paper is to estimate the order of convergence of conjugate Fourier series within the weighted Zygmund class $W(Z_r^{(\omega)})$ using almost Nörlund means. Several new results are established, generalizing various known findings in Fourier approximation theory.

1 Introduction and Preliminaries

Summability methods have broad applications across various branches of mathematics. In function theory, they aid in the analytic continuation of holomorphic functions and the study of the boundary behavior of power series. In applied analysis, these methods play a crucial role in developing iterative techniques for solving systems of equations and enhancing convergence in approximation theory. Furthermore, summability methods are significant in probability theory, such as in the study of Markov chains, and in number theory, where they contribute to results like the Prime Number Theorem.

The approximation analysis of signals (functions) is particularly valuable in science and engineering due to its wide-ranging applications in signal analysis, system design, telecommunications, radar, and image processing. In recent decades, there has been increasing interest in error estimation for functions within spaces such as Lipschitz, Hölder, Zygmund, and Besov spaces, utilizing various summability techniques for Fourier series. The concept of almost convergence extends traditional convergence to broader sequence spaces, including the space of bounded sequences. Functions within L_r spaces $(r \geq 1)$ are particularly useful in signal analysis, with L_1, L_2 and L_∞ spaces frequently employed by engineers in designing digital filters.

The weighted Zygmund class $W(Z_r^{(\omega)})$ $(r \geq 1)$ generalizes several function classes, including $Z_r^{(\omega)}$, $Z_{(\alpha)}, Z_{(\alpha),r}$ and $Z^{(\omega)}$. The Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ has been extensively studied by Leindler [14], Móricz [17], Móricz and Németh [18]. More recently, Das *et al.* [4] established results on function approximation within the weighted Zygmund class using Euler-Hausdorff product summability means of Fourier series. Further developments in this area were made by Pradhan and Rao [23], who examined the degree of approximation of functions associated with Hardy-Littlewood series within weighted Zygmund classes using Euler-Hausdorff summability means.

In 1966, King [11] introduced the concepts of almost convergence and almost regularity of a matrix A, and almost Euler summability while proving the almost regularity of these summability methods. Later, Batt and Deshpande [2] established results on weak convergence and weak compactness in the space of almost periodic functions on the real line. More recently, in 2020, Sharma [25] investigated error approximation of conjugate Fourier series within weighted classes using almost Riesz means. Subsequently, in 2023, Sharma and Dumka [26] studied the convergence of conjugate functions in Zygmund spaces via almost Nörlund transformations. For further recent contributions in this direction, see [1], [3], [5], [7], [8], [9], [10], [16], [20], [21], [22], [24], [27] and [28].

To have advance research and achieve optimal approximation, this paper investigates the degree of approximation for functions within the weighted Zygmund class $W(Z_r^{(\omega)})$ using almost Nörlund means of conjugate Fourier series.

Let $f(x)$ be a 2π-periodic function and Lebesgue integrable on $[0, 2\pi]$. The Fourier series of the function $f(x)$ is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} u_n(x) \tag{1.1}$$

with nth partial sum $s(f;x)$, where a_n and b_n are the Fourier coefficients of f. The conjugate series of Fourier series (1.1) is given by

$$\tilde{f}(x) \sim \sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) = \sum_{n=0}^{\infty} v_n(x) \tag{1.2}$$

with nth partial sum

$$\tilde{s}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\tilde{D}_n(t)dt,$$

where

$$\tilde{D}_n(t) = \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}.$$

The L_r norm of a function f is defined by

$$\|f\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^r dx\right)^{\frac{1}{r}}, & 1 \le r < \infty \\ ess\,\sup_{0<x\le 2\pi}|f(x)| & (r=\infty). \end{cases}$$

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial t_n of order n under $\|\cdot\|_{L_\infty}$ norm is defined as

$$\|t_n - f(x)\|_{L_\infty} = \sup_{x\in\mathbb{R}}|t_n(x) - f(x)|,$$

and let a function $f \in L_r$, its degree of approximation $E_n(f)$ is given by

$$E_n(f) = \min_n \|t_n - f\|_{L_r}.$$

Next, we recall the Zygmund modulus of continuity [30] of f is defined by

$$\omega(f,h) = \sup_{o\le h,\, x\in\mathbb{R}} |f(x+t) + f(x-t)|.$$

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[0, 2\pi]$ under the supremum norm. For $0 < \alpha \le 1$, the function space

$$Z_{(\alpha)} = \{f \in \mathbb{C}_{2\pi} : |f(x+t) + f(x-t)| = O(|t|^\alpha)\}$$

is a Banach space under the norm $\|\cdot\|_{(\alpha)}$ is defined by

$$\|f\|_{(\alpha)} = \sup_{0\le x\le 2\pi} |f(x)| + \sup_{x,t\neq 0} \frac{|f(x+t)+f(x-t)|}{|t|^{\alpha}}.$$

For $f\in L_r[0,2\pi], r\ge 1$, the integral Zygmund modulus of continuity is defined by

$$\omega_r(f,h) = \sup_{0<t\le h} \left\{ \frac{1}{2\pi}\int_0^{2\pi} |f(x+t)+f(x-t)|^r dx \right\}^{\frac{1}{r}}.$$

Moreover, for $f\in \mathbb{C}_{2\pi}$ and $r=\infty$,

$$\omega_\infty(f,h) = \sup_{0<t\le h} \max_x |f(x+t)+f(x-t)|.$$

Also, it is known that $\omega_r(f,h)\to 0$ as $r\to 0$.

We now define,

$$Z_{(\alpha),r} = \left\{ f\in L_r[0,2\pi] : \left(\int_0^{2\pi} |f(x+t)+f(x-t)|^r dx\right)^{\frac{1}{r}} = O(|t|^{\alpha}) \right\}.$$

The space $Z_{(\alpha),r}, r\ge 1, 0<\alpha\le 1$ is a Banach space under the norm $\|\cdot\|_{(\alpha),r}$ and that,

$$\|f\|_{(\alpha),r} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{|t|^{\alpha}}.$$

The class of function $Z^{(\omega)}$ is defined as

$$Z^{(\omega)} = \{f\in \mathbb{C}_{2\pi} : |f(x+t)+f(x-t)| = O(\omega(t))\},$$

where ω is the Zygmund modulus of continuity, that is, ω is positive, non-decreasing continuous function with the sub linearity property, that is,

(i) $\omega(0)=0$

(ii) $\omega(t_1+t_2)\le \omega(t_1)+\omega(t_2)$.

Let $\omega : [0,2\pi]\to \mathbb{R}$ be an arbitrary function with $\omega(t)>0$ for $0\le t<2\pi$ and let $\lim_{t\to 0^+}\omega(t)=\omega(0)=0$, define

$$Z_r^{(\omega)} = \left\{ f\in L_r : 1\le r\le\infty, \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{\omega(t)} < \infty \right\},$$

where

$$\|f\|_r^{(\omega)} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{\omega(t)}, r\ge 1.$$

Clearly $\|\cdot\|_r^{(\omega)}$ is a norm on $Z_r^{(\omega)}$. As we know L_r $(r\ge 1)$ is complete, the space $Z_r^{(\omega)}$ is also complete. Hence, we can say $Z_r^{(\omega)}$ is a Banach space under the norm $\|\cdot\|_r^{(\omega)}$.

Now we define the weighted Zygmund class as

$$W(Z_r^{(\omega)}) = \left\{ f\in W(Z_r^{(\omega)}) : 1\le r\le\infty, \sup_{t\neq 0} \frac{\|(f(\cdot+t)+f(\cdot-t))\sin^{\beta}(\cdot)\|_r}{\omega(t)} \le \infty \right\}, \quad (1.3)$$

where

$$\|f\|_r^{(\omega)} = \|f\|_r + \sup_{t\neq 0} \frac{\|(f(\cdot+t)+f(\cdot-t))\sin^{\beta}(\cdot)\|_r}{\omega(t)} \quad (r\ge 1). \quad (1.4)$$

Clearly, $\|\cdot\|_r^{*(\omega)}$ is a norm of $Z_r^{(\omega)}$. The space $Z_r^{(\omega)}$ is complete because $L_r,\ \ r \geq 1$ is complete. Hence, we can say that $W(Z_r^{(\omega)})$ is complete. As $Z_r^{(\omega)}$ is a Banach space under $\|\cdot\|_r^{(\omega)}$, so $W(Z_r^{(\omega)})$ is also a Banach space under $\|\cdot\|_r^{(\omega)}$ norm.

We now present a remark that our defined weighted Zygmund class $W(Z_r^{(\omega)})$ reduces to different simple classes.

Remark 1.1. If we put $\beta = 0$ in $W(Z_r^{(\omega)})$ class, then it reduces to $Z_r^{(\omega)}$ class. Moreover, as $r \to \infty$, the class $Z_r^{(\omega)}$ reduces to $Z^{(\omega)}$. Additionally, if we take $\omega(t) = t^\alpha$ in $Z_r^{(\omega)}$ class, then it reduces to $Z_{(\alpha),r}$ class. Finally, when $\omega(t) = t^\alpha$, the $Z^{(\omega)}$ class reduces to $Z_{(\alpha)}$ class.

Here, $\omega(t)$ and $v(t)$ denote the Zygmund moduli of continuity, where $\left(\frac{\omega(t)}{v(t)}\right)$ is positive and non-decreasing. Under this condition, we have

$$\|f\|_r^{(v)} \leq \max\left(1, \frac{\omega(2\pi)}{v(2\pi)}\right) \|f\|_r^{(\omega)} \leq \infty.$$

This implies the inclusion relation

$$Z_r^{(\omega)} \subseteq Z_r^{(v)} \subseteq L_r \ \ (r \geq 1).$$

Consequently, we obtain

$$W(Z_r^{(\omega)}) \subseteq W(Z_r^{(v)}) \subseteq W(L_r, \omega(t)).$$

We now recall the following definition.

Definition 1.2. (see [6]) Let $\sum u_n$ be an infinite series with the sequence of nth partial sums $\{s_n\}$. A bounded sequence $\{s_n\}$ is said to be almost convergent to a finite number l, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\mu=p}^{n+p} s_\mu = l$$

converges uniformly in p.

In the context of convergence theory, almost convergence generalizes the concept of usual convergence. It is evident that every usually convergent sequence is almost convergent; however, the converse does not necessarily hold.

Now, we present the following definition.

Definition 1.3. Let $\{p_n\}$ be a sequence of positive constant such that $p_0 > 0$ and $p_n > 0$ for all n with

$$P_n = p_0 + p_1 + \cdots + p_n, \ \ P_{-1} = p_{-1} = 0.$$

The conjugate Fourier series of a function is said to be almost Nörlund summable to l, if

$$\tilde{\mathcal{T}}_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \tilde{s}_{k,p} = l \text{ as } n \to \infty \tag{1.5}$$

uniformly in p,
where

$$\tilde{s}_{k,p} = \frac{1}{k+1} \sum_{\mu=p}^{k+p} \tilde{s}_\mu. \tag{1.6}$$

It is important to note that $\tilde{\mathcal{T}}_{n,p}$ is also a trigonometric polynomial. The Nörlund mean $\tilde{\mathcal{T}}_{n,p}$ is regular if and only if $\tilde{s}_{k,p} = l$.

We use the following notations throughout the paper.

$$\psi(x,t) = f(x+t) + f(x-t)$$

and

$$\tilde{\mathcal{K}}_n^{AN} = \frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v} \frac{\cos(v+2j+1)\frac{t}{2} \sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}.$$

2 Main Result

The main objective of this paper is to prove the following theorem.

Theorem 2.1. *Let $\tilde{f} : [0, 2\pi] \to \mathbb{R}$ be a 2π periodic, Lebesgue integrable function belonging to the weighted Zygmund class $W(Z_r^{(\omega)})$. Then, the degree of approximation of $\tilde{f}$ under almost Nörlund means of the conjugate Fourier series (1.2) is*

$$E_n(f) = \|\tilde{\mathcal{T}}_{n,p} - \tilde{f}\|_r = O\left(\frac{1}{n+j}\int_{\frac{1}{n+j}}^{\pi} \frac{\omega(t)}{t^2 v(t)}\right) dt \quad (n = 0, 1, 2, ...), \tag{2.1}$$

where $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that

$$\int_0^u \frac{\omega(t)}{tv(t)} dt = O\left(\frac{\omega(u)}{v(u)}\right). \tag{2.2}$$

To prove the above theorem, we first establish four lemmas as follows.

Lemma 2.2. $\tilde{\mathcal{K}}_n^{AN}(t) = O(n+j)$*, for* $0 < t \leq \frac{1}{n+j}$.

Proof. For $0 < t \leq \frac{1}{n+1}$, $|\cos t| \leq 1$, $\sin(\frac{t}{2}) \geq (\frac{\pi}{t})$, we have

$$\begin{aligned}
|\tilde{\mathcal{K}}_n^{AN}| &= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v}\frac{\cos(v+2j+1)\frac{t}{2}\sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}\right| \\
&= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v}\frac{\sin(v+1)\frac{t}{2}\{\cos(v+2j+1)\frac{t}{2} - \cos\frac{t}{2}\}}{(v+1)\sin^2\frac{t}{2}}\right| \\
&= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v}\frac{\sin(v+1)\frac{t}{2}\left\{2\sin\left(\frac{v+2j+2}{2}\right)\sin\left(\frac{v+2j}{2}\right)\frac{t}{2}\right\}}{(v+1)\sin^2\frac{t}{2}}\right| \\
&= \frac{1}{2\pi P_n}\left|\sum_{v=0}^{n} p_{n-v}\frac{(v+1)\sin\frac{t}{2}\left\{2\sin\left(\frac{v+2j+2}{2}\right)\sin\left(\frac{v+2j}{2}\right)\frac{t}{2}\right\}}{(v+1)\sin^2\frac{t}{2}}\right| \\
&= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\frac{\frac{2(v+2j+2)}{2}|\sin\frac{t}{2}||\sin\left(\frac{v+2j}{2}\right)(\frac{t}{2})|}{|\sin\frac{t}{2}|} \\
&= \frac{1}{2\pi P_n}\left\{\sum_{v=0}^{n} p_{n-v}\right\}(n+2j+2) \\
&= O(n+j)\frac{1}{P_n}\sum_{v=0}^{n} p_{n-v} \\
&= O(n+j).
\end{aligned}$$

□

Lemma 2.3. $\tilde{\mathcal{K}}_n^{AN}(t) = O\left(\frac{1}{nt^2}\right)$ *for* $\frac{1}{n+j} < t < \pi$.

Proof. For $\frac{1}{n+j} < t < \pi$, $\sin(n+1)t \leq 1$, $\sin(t/2) \geq (t/\pi)$and $\sup_{0\leq t\leq \pi} = N$, we have

$$
\begin{aligned}
|\tilde{\mathcal{K}}_n^{AN}| &= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\frac{\cos(j+v+1)\frac{t}{2}\sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}} \\
&\leq \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\frac{\cos(j+(v+1)/2)t\sin((v+1)/2)t}{(v+1)\sin^2\frac{t}{2}} \\
&\leq \frac{1}{2\pi P_n}\sum_{v=0}^{n} \frac{p_{n-v}}{(v+1)}\frac{1}{\sin^2\frac{t}{2}} \\
&= O\left(\frac{1}{t^2}\right)\frac{1}{2\pi P_n}\sum_{v=0}^{n}\frac{p_{n-v}}{v+1} \\
&= O\left(\frac{1}{nt^2}\right).
\end{aligned}
$$

□

Lemma 2.4. *(see [13]) Let $f \in Z_r^{(\omega)}$, then for* $0 < t \leq \pi$,

(i) $\|\psi(\cdot,t)\|_r = O(\omega(t))$

(ii) $\|\psi(\cdot+y,t)+\psi(\cdot-y,t)\|_r = \begin{cases} O(\omega(t)) \\ O(\omega(y)) \end{cases}$

(iii) If $\omega(t)$ and $v(t)$ defined as in Theorem 2.1, then

$$\|\psi(\cdot+y,t)+\psi(\cdot-y,t)\|_r = O\left(v(y)\frac{\omega(t)}{v(t)}\right),$$

where

$$\psi(x,t) = f(x+t)+f(x-t).$$

Lemma 2.5. $\|(\psi(\cdot+y,t)+\psi(\cdot-y,t))\sin^\beta(\cdot)\|_r = O\left(t^\beta v(y)\left(\frac{\omega(t)}{v(t)}\right)\right)$.

Proof. Following Lemma 2.4, $|\sin^\beta t| \leq t^\beta$ and for v is positive, nondecreasing, $t \leq y$, we obtain

$$
\begin{aligned}
\|(\psi(\cdot+y,t)+\psi(\cdot-y,t))\sin^\beta(\cdot)\|_r &= O(t^\beta\omega(t)) \\
&= O\left(t^\beta v(t)\left(\frac{\omega(t)}{v(t)}\right)\right) \\
&\leq O\left(t^\beta v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).
\end{aligned}
$$

Further, since $\frac{\omega(t)}{v(t)}$ is positive, non-decreasing, if $t \geq y$, then $\frac{\omega(t)}{v(t)} \geq \frac{\omega(y)}{v(y)}$, so that

$$
\begin{aligned}
\|(\psi(\cdot+y,t)+\psi(\cdot-y,t))\sin^\beta(\cdot)\|_r &= O(t^\beta\omega(y)) \\
&= O\left(t^\beta v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).
\end{aligned}
$$

□

3 Proof of Main Result

Proof of Theorem 2.1

Let $\tilde{s}_k(f;x)$ denotes the kth partial sum of the series (1.2) and following Titechmalch [29], we have

$$\tilde{s}_k(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(x;t)\frac{\cos(k+1/2)t}{\sin(t/2)}dt. \tag{3.1}$$

Applying (1.6) and the notion of almost summability of a sequence, we obtain

$$\tilde{s}_k(f;x)-\tilde{f}(x)=\frac{1}{k+1}\sum_{\mu=j}^{k+j}\left\{\frac{1}{2\pi}\int_0^{\pi}\psi(x;t)\frac{\cos(\mu+1/2)t}{\sin(t/2)}dt\right\}. \tag{3.2}$$

Now, under the almost Nörlund transform $\tilde{\mathcal{T}}_{n,p}$ of $\tilde{s}_k(f;x)$, we get

$$\begin{aligned}\|\tilde{\mathcal{T}}_{n,p}(x)-f(\tilde{x})\| &= \frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\left\{\frac{1}{2\pi}\int_0^{\pi}\psi(x;t)\frac{\cos(\mu+1/2)t}{\sin(t/2)}dt\right\}\\ &= \int_0^{\pi}\psi(x;t)\tilde{\mathcal{K}}_n^{AN}(t)dt.\end{aligned}$$

Let

$$\mathcal{L}_n(x)=\tilde{\mathcal{T}}_{n,p}-\tilde{f}(x)=\int_0^{\pi}\psi(x;t)\tilde{\mathcal{K}}_n^{AN}dt. \tag{3.3}$$

Then

$$\mathcal{L}_n(x+y)+\mathcal{L}_n(x-y)=\int_0^{\pi}\left[\psi(x+y;t)+\psi(x-y;t)\right]\tilde{\mathcal{K}}_n^{AN}dt. \tag{3.4}$$

Now,

$$(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y))\sin^{\beta}(\cdot)=\int_0^{\pi}\Bigg((\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\Bigg)\tilde{\mathcal{K}}_n^{AN}dt. \tag{3.5}$$

Clearly, we can write

$$\begin{aligned}\|(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y))\sin^{\beta}(\cdot)\|_r &= \int_0^{\pi}\|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{\mathcal{K}}_n^{AN}dt\\ &= \int_0^{\frac{1}{n+1}}\|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{\mathcal{K}}_n^{AN}dt\\ &\quad+\int_{\frac{1}{n+1}}^{\pi}\|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{\mathcal{K}}_n^{AN}dt\\ &= I_1+I_2\ \ (\text{say}).\end{aligned} \tag{3.6}$$

Furthermore, the function $f\in W(Z_r^{(\omega)})$ implies $\psi\in W(Z_r^{(\omega)})$ and applying Lemma 2.2, Lemma 2.4 and monotonicity of $\frac{\omega(t)}{v(t)}$ with respect to t, we have

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{n+j}}\|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{\mathcal{K}}_n^{AN}dt\\ &= O\Bigg(\int_0^{\frac{1}{n+j}}v(y)\frac{t^{\beta}\omega(t)}{v(t)}(n+j)dt\Bigg)\\ &= O\Bigg((n+j)\,v(y)\int_0^{\frac{1}{n+1}}\frac{t^{\beta}\omega(t)}{v(t)}dt\Bigg)\\ &= O\Bigg((n+j)\,v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\int_0^{\frac{1}{n+1}}t^{\beta}dt\Bigg)\\ &= O\Bigg((n+j)^{-\beta}\,v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\Bigg).\end{aligned} \tag{3.7}$$

Next, using Lemma 2.3 and Lemma 2.5, we get

$$\begin{aligned} I_2 &= \int_{\frac{1}{n+1}}^{\pi} \|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r \tilde{\mathcal{K}}_n^{AN}dt \\ &= O\left(\int_{\frac{1}{n+1}}^{\pi} v(y)\frac{t^{\beta}\omega(t)}{v(t)}(n)^{-1}t^{-2}dt\right) \\ &= O\left((n)^{-1}v(y)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \end{aligned} \tag{3.8}$$

Thus, using (3.6), (3.7) and (3.8), we can write

$$\begin{aligned} \|(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y))\sin^{\beta}(\cdot)\|_r &= O\left((n+j)^{-\beta}\,v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \\ &+ O\left((n)^{-1}v(y)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \end{aligned} \tag{3.9}$$

Therefore, we have

$$\begin{aligned} \sup_{y\neq 0}\frac{\|(\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y)))\sin^{\beta}(\cdot)\|_r}{v(|y|)} &= O\left((n+j)^{-\beta}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \\ &+O\left((n)^{-1}\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \end{aligned} \tag{3.10}$$

Clearly,

$$\psi(x;t)=|f(x+t)+f(x-t)|.$$

Now applying Minkowski's inequality, we have

$$\|\psi(x;t)\|_r=\|f(x+t)+f(x-t)\|_r. \tag{3.11}$$

Thus, under Lemma 2.5, it yields

$$\begin{aligned} \|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r &\leq \left(\int_0^{\frac{1}{n+j}}+\int_{\frac{1}{n+j}}^{\pi}\right)\|(\psi(\cdot,t))\sin^{\beta}(\cdot)\|_r|\tilde{\mathcal{K}}_n^{AN}|dt \\ &= O\left((n+j)\int_0^{\frac{1}{n+j}}t^{\beta}\omega(t)dt\right)+O\left((n)^{-1}\int_{\frac{1}{n+j}}^{\pi}t^{\beta-2}\omega(t)dt\right) \\ &= O\left((n+j)\omega(\frac{1}{n+j})\int_0^{\frac{1}{n+j}}t^{\beta}dt\right)+O\left((n)^{-1}\int_{\frac{1}{n+j}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}dt\right) \\ &= O\left((n+j)^{-\beta}w(\frac{1}{n+j})\right)+O\left((n)^{-1}\int_{\frac{1}{n+j}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}dt\right). \end{aligned} \tag{3.12}$$

Consequently, from (3.10) and (3.12), we obtain

$$\begin{aligned} \|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r^{v} &= \|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r+\sup_{y\neq 0}\frac{(\|\mathcal{L}_n(\cdot+y)+\mathcal{L}_n(\cdot-y))\sin^{\beta}(\cdot)\|_r}{v(y)} \\ &= O\left((n+j)^{-\beta}w(\frac{1}{n+j})\right)+O\left((n)^{-1}\int_{\frac{1}{n+j}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}dt\right) \\ &\quad +O\left((n+j)^{-\beta}\frac{\omega(\frac{1}{n+j})}{v(\frac{1}{n+1})}\right)+O\left((n)^{-1}\int_{\frac{1}{n+j}}^{\pi}\frac{t^{\beta-2}\omega(t)}{v(t)}dt\right) \\ &= \sum_{i=1}^{4}O(J_i)\quad\text{(say)}. \end{aligned} \tag{3.13}$$

Next, we write J_1 in terms of J_3 and further J_2, J_3 in terms of J_4.

In view of monotonicity of $v(t)$ for $0 < t \leq \pi$, we have

$$\omega(t) = \frac{\omega(t)}{v(t)}.v(t) \leq v(\pi)\frac{\omega(t)}{v(t)}.v(t) = O\left(\frac{\omega(t)}{v(t)}\right) \text{ for } 0 < t \leq \pi.$$

Therefore, we can write for $t = (n+j)^{-\beta}$

$$J_1 = O(J_3). \tag{3.14}$$

Again, by using monotonicity of $v(t)$,

$$\begin{aligned} J_2 = (n)^{-1}\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}v(t)dt &\leq (n)^{-1}v(\pi)\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt \\ &\leq (n)^{-1}\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt \\ &= O(J_4). \end{aligned} \tag{3.15}$$

Now $\left(\frac{\omega(t)}{v(t)}\right)$ being positive and non-decreasing, we have

$$\begin{aligned} J_4 = (n)^{-1}\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt &\geq (n)^{-1}\frac{\omega(\frac{1}{n+j})}{v(\frac{1}{n+j})}\int_{\frac{1}{n+j}}^{\pi} t^{\beta-2}dt \\ &\geq (n)^{-1}\frac{\omega(\frac{1}{n+j})}{v(\frac{1}{n+j})}\frac{1}{(n+j)^{\beta-1}} \\ &\geq \left(\frac{n}{n+j}\right)^{-1}\frac{\omega(\frac{1}{n+j})}{v(\frac{1}{n+j})}\frac{1}{(n+j)^{\beta}} \\ &\geq (n+j)^{-\beta}\frac{\omega(\frac{1}{n+j})}{v(\frac{1}{n+j})}\left(\because \frac{n}{n+j} = O(1)\right). \end{aligned} \tag{3.16}$$

This implies

$$J_3 = O(J_4). \tag{3.17}$$

Now combining (3.13) and (3.17), we get

$$\|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r = O\left((n+j)^{-1}\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \tag{3.18}$$

Hence,

$$E_n(f) = \inf_n \|(\mathcal{L}_n(\cdot))\sin^{\beta}(\cdot)\|_r^{(v)} = O\left((n+j)^{-1}\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-2}\omega(t)}{v(t)}dt\right). \tag{3.19}$$

This completes the proof of Theorem 2.1.

4 Concluding Remarks and Observations

In this study, we explored the approximation of functions in the weighted Zygmund class $W(Z_r^{(\omega)})$, $r \geq 1$, using almost Nörlund means of the conjugate Fourier series. Our analysis demonstrated that this method effectively enhances the convergence and approximation properties of conjugate series in weighted function spaces. These findings contribute to a deeper understanding of function approximation in harmonic analysis, extending the scope of existing summability techniques. Furthermore, we highlighted the significance of our main result, Theorem 2.1, by providing additional remarks and insights into the various findings presented. Future research may focus on refining these methods further and exploring their applications in both mathematical and engineering contexts.

Remark 4.1. (see [19]) If we replace Euler-Hausdorff mean by $(E,1)(C,1)$ mean in Theorem 2.1, then the degree of approximation of a function $f \in W(Z_r^\omega)$ by $(E,1)(C,1)$ mean of cojugate Fourier series is given by

$$E_n(f) = O\left(\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.1}$$

Remark 4.2. (see [15]) If we replace Euler-Hausdorff mean by $(E,q)(N,p_n,q_n)$ mean in Theorem 2.1, then the degree of approximation of a function $f \in W(Z_r^\omega)$ by $(E,q)(N,p_n,q_n)$ mean of cojugate Fourier series is given by

$$E_n(f) = O\left(\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.2}$$

Remark 4.3. (see [12]) If we replace Euler-Hausdorff mean by Hausdorff mean in Theorem 2.1, then the degree of approximation of a function $f \in W(Z_r^\omega)$ by Hausdorff mean of cojugate Fourier series is given by

$$E_n(f) = O\left(\frac{1}{(n+j)}\int_{\frac{1}{n+j}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right). \tag{4.3}$$

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Received: 2024-11-02
Accepted: 2025-03-07