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ADVANCES IN BALANCING NUMBERS USING A MATRIX APPROACH

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Abstract Balancing numbers and their generalizations have been the focus of considerable research, with previous studies often generating these numbers through linear or nonlinear recurrences. Another approach involves generating balancing numbers via specific matrices. In this paper, we employ a matrix of order 4 to derive new identities for balancing numbers. Additionally, we establish other relations involving co-balancing numbers, expanding the framework for analyzing these sequences.

1 Introduction

The concept of balancing numbers was first introduced by Behera and Panda [1] in 1999 in connection with a Diophantine equation. It involves finding a natural number n such that:

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some natural number r. They called n a balancing number and r the balancer corresponding to n.

An important result about balancing numbers is that B_n is a balancing number if and only if B_n^2 is a triangular number; that is, $8B_n^2 + 1$ is a perfect square. These numbers can be generated by the linear recurrence:

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geqslant 1, \; B_0 = 0, \; B_1 = 1. \tag{1.1}$$

or by the nonlinear recurrence

$$B_n^2 = B_{n+1}B_{n-1} + 1, \quad n \geqslant 1. \tag{1.2}$$

For each balancing number B_n, the number $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number [8]. The sequence of balancing numbers has been extensively studied and generalized in various ways [12, 2, 7, 3, 5, 10, 6].

In [11], Ray introduced a second-order balancing matrix Q_B, whose entries are the first three balancing numbers: 0, 1, and 6. He also showed that the n^{th} power of the balancing matrix $(n \in \mathbb{N})$ is given by:

$$Q_B = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

This matrix representation provides an elegant method for deriving relationships between balancing and Lucas-balancing numbers. In this paper, we pose an equivalent problem using a matrix of order 4, with $n \in \mathbb{Z}$, whose entries are balancing numbers and Lucas-balancing numbers.

2 Balancing matrices

Matrices can be used to represent balancing numbers and can be extended to related sequences.

2.1 Balancing Q-Matrices

The concept of Q-matrices was first studied by Charles King [4] in 1960. These matrices play an important role in the study of Fibonacci numbers. More recently, Ray [11] introduced a *balancing Q-matrix* whose entries are the first three balancing numbers: 0, 1, and 6. Motivated by this, we introduce a new balancing Q-matrix, defined by

$$Q_{BC} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

and prove the following main theorem.

Theorem 2.1. *Let Q_{BC} be the balancing Q-matrix given in [4]. Then for every integer $n \in \mathbb{Z}$,*

$$Q_{BC}^n = \begin{bmatrix} -B_{n-1} & -B_n & 0 & 0 \\ B_n & B_{n+1} & 0 & 0 \\ \frac{1}{4}(-B_n + B_{n-1} + 1) & \frac{1}{4}(-5B_n + B_{n-1} + 1) & 1 & 0 \\ \frac{1}{2}(B_n + B_{n-1} + 1) & \frac{1}{2}(7B_n - B_{n-1} - 1) & 0 & 1 \end{bmatrix}. \tag{2.1}$$

Proof. We consider the two cases $n \geq 0$ and $n \leq 0$.
Case when $n \geq 0$: First, we prove the identity

$$\sum_{k=0}^{n} B_k = \frac{1}{4}(5B_n - B_{n-1} - 1), \quad n \geq 0. \tag{2.2}$$

Indeed, from the recurrence relation (1.1), it follows that

$$\begin{aligned} \sum_{k=0}^{n} B_k &= B_0 + B_1 + \sum_{k=1}^{n-1} B_{k+1} \\ &= B_0 + B_1 + \sum_{k=1}^{n-1} (6B_k - B_{k-1}) \\ &= B_0 + B_1 + 6\sum_{k=1}^{n-1} B_k - \sum_{k=0}^{n-2} B_k \\ &= B_0 + B_1 + 6\left(\sum_{k=0}^{n} B_k - B_0 - B_n\right) - \left(\sum_{k=0}^{n} B_k - B_{n-1} - B_n\right) \\ &= 5\sum_{k=0}^{n} B_k + 1 + B_{n-1} - 5B_n \end{aligned}$$

from which identity (2.2) follows.
Next, we prove the theorem by induction on n. Since $B_0 = 0$, $B_1 = 1$ and $B_2 = 6$, the theorem holds for $n = 1$,

$$Q_{BC}^1 = \begin{bmatrix} B_0 & -B_1 & 0 & 0 \\ B_1 & B_2 & 0 & 0 \\ B_0 & -B_1 & 1 & 0 \\ B_1 & 3B_1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}.$$

Assume that the property holds for order n.
Then, from the recurrence relation (1.1), it follows that

$$\begin{aligned} Q_{BC}^{n+1} &= Q_{BC}^{n} \times Q_{BC}^{1} \\ &= \begin{bmatrix} -B_n & -B_{n+1} & 0 & 0 \\ B_{n+1} & B_{n+2} & 0 & 0 \\ \frac{1}{4}\left(-5B_n + B_{n-1} + 1\right) & \frac{1}{4}\left(-29B_n + 5B_{n-1} - 1\right) & 1 & 0 \\ B_n + \frac{1}{2}\left(5B_n - B_{n-1} + 1\right) & \frac{1}{2}\left(41B_n - 7B_{n-1} - 1\right) & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -B_n & -B_{n+1} & 0 & 0 \\ B_{n+1} & B_{n+2} & 0 & 0 \\ \frac{1}{4}\left(-B_{n+1} + B_n + 1\right) & \frac{1}{4}\left(-5B_{n+1} + B_n + 1\right) & 1 & 0 \\ \frac{1}{2}\left(B_{n+1} + B_n + 1\right) & \frac{1}{2}\left(7B_{n+1} - B_n - 1\right) & 0 & 1 \end{bmatrix}, \end{aligned}$$

it results that the induction step holds.
Case when $n \leq 0$: Assume that the theorem holds for n. Since

$$Q_{BC}^{-1} = \begin{bmatrix} 6 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

then, from the recurrence relation (1.1) it follows that

$$\begin{aligned} Q_{BC}^{-(n+1)} &= Q_{BC}^{-n} \times Q_{BC}^{-1} \\ &= \begin{bmatrix} 6B_{n+1} - B_n & B_{n+1} & 0 & 0 \\ -6B_n + B_{n-1} & -B_n & 0 & 0 \\ \frac{1}{4}\left(-30B_n + 5B_{n-1} + B_n + 1\right) & \frac{1}{4}\left(-5B_n + B_{n-1} + 1\right) & 1 & 0 \\ \frac{1}{2}\left(-42B_n + 7B_{n-1} + B_n + 1\right) & \frac{1}{2}\left(-7B_n + B_{n-1} + 1\right) - 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} B_{n+2} & B_{n+1} & 0 & 0 \\ -B_{n+1} & -B_n & 0 & 0 \\ \frac{1}{4}\left(-5B_{n+1} + B_n + 1\right) & \frac{1}{4}\left(-B_{n+1} + B_n + 1\right) & 1 & 0 \\ \frac{1}{2}\left(-7B_{n+1} + B_n + 1\right) & \frac{1}{2}\left(-B_{n+1} - B_n - 1\right) & 0 & 1 \end{bmatrix}, \end{aligned}$$

which completes the induction step. □

Corollary 2.2. *For every integer $n \in \mathbb{Z}$, the following identity holds:*

$$B_n^2 - B_{n-1}B_{n+1} = 1.$$

Proof. Let $|Q_{BC}|$ denote the determinant of the matrix Q_{BC}. Then

$$|Q_{BC}|^n = 1 = |Q_{BC}^n| = B_n^2 - B_{n-1}B_{n+1},$$

which proves the identity.

□

Corollary 2.3. *For every natural number n, the following identity holds:*

$$\sum_{k=0}^{n} B_k = \frac{B_n B_{n+1}}{B_{n+1} - B_n + 1}. \tag{2.3}$$

Proof. By computing the product Q_{BC}^{n+1} with its inverse $Q_{BC}^{-(n+1)}$, we obtain:

$$Q_{BC}^{n+1} \times Q_{BC}^{-(n+1)} = I_d.$$

More precisely, by calculating the product of the $(3, 2)$-elements, we find

$$-B_{n+1}\sum_{k=0}^{n} B_k + B_n\sum_{k=0}^{n+1} B_k - \sum_{k=0}^{n} B_k = 0.$$

This simplifies to:

$$(-B_{n+1} - 1 + B_n)\sum_{k=0}^{n} B_k + B_nB_{n+1} = 0,$$

from which the relation (2.3) follows. □

Corollary 2.4. *For every natural number n, the following identity holds:*

$$\sum_{k=0}^{n} B_k = \frac{B_{n+1}(B_{n+1} - 1)}{B_{n+2} - B_{n+1} + 1}. \tag{2.4}$$

Proof. By calculating the product of the third row of the matrix Q_{BC}^{n+1} with the first column of the matrix $Q_{BC}^{-(n+1)}$, we obtain:

$$-B_{n+2}\sum_{k=0}^{n} B_k + B_{n+1}\sum_{k=0}^{n+1} B_k - \sum_{k=0}^{n+1} B_k = 0.$$

This simplifies to:

$$(B_{n+1} - 1 - B_{n+2})\sum_{k=0}^{n} B_k - B_{n+1} + B_{n+1}^2 = 0,$$

from which the relation (2.4) follows. □

Corollary 2.5. *For any non-negative integers m, n, the following identity holds:*

$$\sum_{k=0}^{m+n} B_k = \frac{B_{n+1}\left(5B_m - B_{m-1}\right) - B_n\left(B_m - B_{m-1}\right) - 1}{4}.$$

Proof. By comparing the $(3, 2)$-elements in the matrix equality $Q_{BC}^{m+n} = Q_{BC}^{m} \times Q_{BC}^{n}$, we obtain:

$$\begin{aligned}\sum_{k=0}^{m+n} B_k &= B_{n+1}\sum_{k=0}^{m} B_k - B_n\sum_{k=0}^{m-1} B_k + \sum_{k=0}^{n} B_k \\ &= (B_{n+1} - B_n)\sum_{k=0}^{m-1} B_k + B_{n+1}B_m + \sum_{k=0}^{n} B_k.\end{aligned}$$

Using the identity from [2]:

$$\sum_{k=0}^{n} B_k = \frac{-1 - B_n + B_{n+1}}{4}$$

and similarly for $\sum_{k=0}^{m-1} B_k$, we can simplify the expression and derive the required result. □

2.2 Some results obtained by using the characteristic equation of Q_{BC}^n

The roots $\lambda_1 = 3 - 2\sqrt{2}$ and $\lambda_2 = 3 + 2\sqrt{2}$ and $\lambda_3 = 1$ are eigenvalues of the matrix Q_{BC}. It is well known that if λ be an eigenvalue of a matrix A, λ^n is also an eigenvalue of A^n.
Therefore, 1, λ_1^n, λ_2^n are eigenvalues of the matrix Q_{BC}^n. To make this result more insightful, we state the following theorem.

Theorem 2.6. *We have:*

$$C_n - \sqrt{C_n^2 - 1} = \left(3 - 2\sqrt{2}\right)^n,$$
$$C_n + \sqrt{C_n^2 - 1} = \left(3 + 2\sqrt{2}\right)^n,$$

or equivalently,

$$B_{n+1} - B_{n-1} = \frac{1}{4}\left(\left(3 - 2\sqrt{2}\right)^n + \left(3 + 2\sqrt{2}\right)^n\right).$$

Proof. Let I denote the identity matrix of the same order as Q_{BC}^n. The characteristic equation of Q_{BC}^n is given by:

$$\left|Q_{BC}^n - \lambda I\right| = (1-\lambda)^2\left(\lambda^2 - \lambda\left(B_{n+1} - B_{n-1}\right) + 1\right),$$

where $B_n^2 - B_{n+1}B_{n-1} = 1$ (see [1]) and $B_{n+1} - B_{n-1} = 2C_n$ (see [7]).
Solving the characteristic equation and using Binet's formulas for B_n and C_n (see [8]), we obtain:

$$\left(3 - 2\sqrt{2}\right)^n = \lambda_1^n = C_n - \sqrt{C_n^2 - 1},$$
$$\left(3 + 2\sqrt{2}\right)^n = \lambda_2^n = C_n + \sqrt{C_n^2 - 1}.$$

Each of these identities yields the explicit expression of C_n stated in the theorem. □

Proposition 2.7. *For any integer n, the following identities hold:*

$$\left(B_{n-1} - 2B_{2n-1} + B_{3n-1}\right)\left(B_{n+1} - B_{n-1} + 1\right) = B_{4n-1} - B_{3n-1} - B_{n-1} - 1,$$
$$\left(2B_{2n} - B_n - B_{3n}\right)\left(B_{n+1} - B_{n-1} + 1\right) = B_n + B_{3n} - B_{4n},$$
$$\left(B_{n+1} - 2B_{2n+1} + B_{3n+1}\right)\left(B_{n+1} - B_{n-1} + 1\right) = 1 - B_{n+1} - B_{3n+1} + B_{4n+1}.$$

Proof. It is known that

$$P(\lambda) = \left|Q_{BC}^n - \lambda I\right| = (1-\lambda)^2\left(\lambda^2 - \lambda\left(B_{n+1} - B_{n-1}\right) + 1\right)$$

and since

$$P\left(Q_{BC}^n\right) = 0,$$

we have the matrix equation

$$\left(Q_{BC}^n - I\right)^2\left(Q_{BC}^{2n} - \left(B_{n+1} - B_{n-1}\right)Q_{BC}^n + I\right) = 0.$$

Expanding the expression, we obtain:

$$I - Q_{BC}^n\left(B_{n+1} - B_{n-1} + 2\right) + 2Q_{BC}^{2n}\left(B_{n+1} - B_{n-1} + 1\right)$$
$$-Q_{BC}^{3n}\left(B_{n+1} - B_{n-1} + 2\right) + Q_{BC}^{4n} = 0,$$

This simplifies to:

$$\left(Q_{BC}^n - 2Q_{BC}^{2n} + Q_{BC}^{3n}\right)\left(B_{n+1} - B_{n-1} + 1\right) + Q_{BC}^n + Q_{BC}^{3n} - Q_{BC}^{4n} = I.$$

By substituting the expressions for Q_{BC}^n, Q_{BC}^{2n}, Q_{BC}^{3n} and Q_{BC}^{4n} from Theorem 2.1, we obtain the desired result. □

For example the number $B_n + B_{3n} - B_{4n}$ can be factored as

$$\left(2B_{2n} - B_n - B_{3n}\right)\left(B_{n+1} - B_{n-1} + 1\right),$$

which shows that this number is never prime.

2.3 Consequences of the identity $Q_{BC}^{m+n} = Q_{BC}^{m} \times Q_{BC}^{n}$

Proposition 2.8. *For any integers n and m, we have*

$$B_{m+n+1} = B_{m+1}B_{n+1} - B_m B_n. \tag{2.5}$$

This identity leads to the following relations:

$$B_{m+n+1} - B_{m+n} = B_{m+1}(B_{n+1} - B_n) + B_m(B_{n-1} - B_n)$$
$$B_{m+n+1} - B_{m+n-1} = B_{m+1}B_{n+1} - 2B_m B_n + B_{m-1}B_{n-1}$$

Proof. The identity $Q_{BC}^{m+n} = Q_{BC}^{m} \times Q_{BC}^{n}$ gives the following matrix:

$$\begin{bmatrix} -B_{m+n-1} & -B_{m+n} & 0 & 0 \\ B_{m+n} & B_{m+n+1} & 0 & 0 \\ \frac{1}{4}\left(-B_{m+n} + B_{m+n-1} + 1\right) & \frac{1}{4}\left(-5B_{m+n} + B_{m+n-1} + 1\right) & 1 & 0 \\ \frac{1}{2}\left(B_{m+n} + B_{m+n-1} + 1\right) & \frac{1}{2}\left(7B_{m+n} - B_{m+n-1} - 1\right) & 0 & 1 \end{bmatrix}.$$

By comparing the likes coefficients from the same sides, we obtain the desired result. □

Proposition 2.9. *For any integer m, we have*

$$B_m \mid B_{2m} \quad \textit{and} \quad B_{2m-1} = B_m^2 - B_{m-1}^2.$$

Proof. From Proposition 2.8 we know that

$$B_{m+n-1} = B_m B_n - B_{m-1}B_{n-1}.$$

Applying this identity to the cases $n = m + 1$ and $n = m$, we obtain:

$$B_{2m} = B_m(B_{m+1} - B_{m-1}) \quad \text{and} \quad B_{2m-1} = B_m^2 - B_{m-1}^2.$$

This proves the result. □

Proposition 2.10. *For any integer n, we have*

$$B_{2n+1} = B_{n+1}^2 - B_n^2 = (B_{n+1} + B_n)(B_{n+1} - B_n)$$

and

$$B_n^2 - B_{n+1}B_{n-1} = 1.$$

Proof. In Proposition 2.8, setting $m = n$ and $m = -n$ yields the desired results. □

Proposition 2.11. *For any non-negative integers m and n, the following identity holds:*

$$\sum_{k=0}^{m+n} kB_k = B_{m+1}B_{n+1}\left(\frac{B_n}{B_{n+1} - B_n + 1} + \frac{B_m}{B_{m+1} - B_m + 1}\right) + B_m B_n. \tag{2.6}$$

Proof. From Proposition 2.8, we have

$$B_{i+j+1} = B_{i+1}B_{j+1} - B_i B_j.$$

Then, summing over $i = 0, \ldots, m$ and $j = 0, \ldots, n$, we obtain:

$$\sum_{i,j=0}^{m,n} B_{i+j+1} = \left(\sum_{i=0}^{m} B_{i+1}\right)\left(\sum_{j=0}^{n} B_{j+1}\right) - \left(\sum_{i=0}^{m} B_i\right)\left(\sum_{j=0}^{n} B_j\right).$$

Equivalently, this can be rewritten as:

$$\sum_{k=0}^{m+n} kB_k + B_{m+n+1} = \left(\sum_{i=0}^{m} B_i + B_{m+1}\right)\left(\sum_{j=0}^{n} B_j + B_{n+1}\right) - \left(\sum_{i=0}^{m} B_i\right)\left(\sum_{j=0}^{n} B_j\right),$$

which simplifies to:

$$\sum_{k=0}^{m+n} kB_k = B_{m+1}\sum_{j=0}^{n} B_j + B_{n+1}\sum_{i=0}^{m} B_i + B_{m+1}B_{n+1} - B_{m+n+1}, \quad m, n \geq 0,$$

Using identity (2.5), we find

$$\sum_{k=0}^{m+n} kB_k = B_{m+1}\sum_{j=0}^{n} B_j + B_{n+1}\sum_{i=0}^{m} B_i + B_m B_n, \quad m, n \geq 0,$$

Now, applying identity (2.3), the desired result (2.6) follows. □

Corollary 2.12. *For any positive integer* n, *we have*

$$\sum_{k=0}^{2n+1} kB_k = \frac{2B_n B_{n+1}^2}{B_{n+1} - B_n + 1} + B_n^2.$$

Proof. This follows by setting $n = m$ in the identity (2.6). □

3 Other results derived from the generating function

Behera and Panda established the generating function for balancing numbers in [1] as

$$g(s) := \sum_{k\geq 0} B_k s^k = \frac{s}{1 - 6s + s^2}. \tag{3.1}$$

This result can also be derived using the matrix method. Consider the matrix

$$I - sQ_{BC} = \begin{bmatrix} 1 & s & 0 & 0 \\ -s & 1-6s & 0 & 0 \\ 0 & s & 1-s & 0 \\ -s & -3s & 0 & 1-s \end{bmatrix}, \tag{3.2}$$

where I denotes the identity matrix of order 4. The determinant of the matrix in (3.2) is

$$(1-s)^2\left(s^2 - 6s + 1\right),$$

and hence, its inverse is given by

$$(I - sQ_{BC})^{-1} = \begin{bmatrix} \frac{1-6s}{s^2-6s+1} & \frac{-s}{s^2-6s+1} & 0 & 0 \\ \frac{s}{s^2-6s+1} & \frac{1}{s^2-6s+1} & 0 & 0 \\ \frac{-s^2}{(1-s)(s^2-6s+1)} & \frac{-s}{(1-s)(s^2-6s+1)} & \frac{1}{1-s} & 0 \\ \frac{3s^2-s}{(1-s)(s^2-6s+1)} & \frac{3s-s^2}{(1-s)(s^2-6s+1)} & 0 & \frac{1}{1-s} \end{bmatrix}. \tag{3.3}$$

Now, define

$$G(s) := \sum_{k\geq 0} s^k Q_{BC}^k = (I - sQ_{BC})^{-1}.$$

Using expression (3.3), we obtain

$$G(s) = \begin{bmatrix} \frac{1-6s}{s^2-6s+1} & \frac{-s}{s^2-6s+1} & 0 & 0 \\ \frac{s}{s^2-6s+1} & \frac{1}{s^2-6s+1} & 0 & 0 \\ \frac{-s^2}{(1-s)(s^2-6s+1)} & \frac{-s}{(1-s)(s^2-6s+1)} & \frac{1}{1-s} & 0 \\ \frac{3s^2-s}{(1-s)(s^2-6s+1)} & \frac{3s-s^2}{(1-s)(s^2-6s+1)} & 0 & \frac{1}{1-s} \end{bmatrix}.$$

By comparing the corresponding coefficients from both sides, we retrieve the generating function for the balancing numbers.

Proposition 3.1. *We have*

$$B_n = \frac{1}{4\sqrt{2}}\left(\left(3+2\sqrt{2}\right)^n - \left(3-2\sqrt{2}\right)^n\right) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \times 8^k \times 3^{n-1-2k},$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof. From the generating function (3.1), we have

$$\begin{aligned} g(s) &= \frac{s}{1-6s+s^2} \\ &= \frac{1}{4\sqrt{2}}\left(\frac{1}{1-\left(3+2\sqrt{2}\right)s} - \frac{1}{1-\left(3-2\sqrt{2}\right)s}\right) \\ &= \sum_{n\geq 0}\frac{1}{4\sqrt{2}}\left(\left(3+2\sqrt{2}\right)^n - \left(3-2\sqrt{2}\right)^n\right)s^n. \end{aligned}$$

□

Corollary 3.2. *We havee*

$$B_{2n} \equiv 3^{2(n-1)} \times 6n \pmod{24}, \quad n \geq 1,$$
$$B_{2n+1} \equiv 9^n \times (2n+1) + 8^n \pmod{72}, \quad n \geq 3.$$

Moreover, and if $n = p \notin \{2,3\}$ is a prime number, then

$$B_p \equiv \pm 1 \pmod{p} \quad \textit{and} \quad B_{p-1} \equiv -6 \textit{ or } 0 \pmod{p}.$$

Proof. From Proposition 3.1, we have

$$\begin{aligned} B_{2n} &= 3^{2n-1} \times 2n + \sum_{k=1}^{n-1}\binom{2n}{2k+1} \times 8^k \times 3^{2(n-k)-1} \\ &= 3^{2n-1} \times 2n + 24\sum_{k=1}^{n-1}\binom{2n}{2k+1} \times 8^{k-1} \times 3^{2(n-k-1)} \\ &\equiv 3^{2n-1} \times 2n \pmod{24}, \quad n \geq 2. \end{aligned}$$

Similarly, for B_{2n+1}, we have

$$\begin{aligned} B_{2n+1} &= 9^n \times (2n+1) + 8^n + 72 \times \sum_{k=1}^{n-1}\binom{2n+1}{2k+1} \times 8^{k-1} \times 3^{2(n-k-1)} \\ &\equiv 9^n \times (2n+1) + 8^n \pmod{72}, \quad n \geq 3. \end{aligned}$$

Moreover, using the following known congruences,

$$\binom{p}{k} \equiv 0 \pmod{p}, \; 1 \leq k \leq p-1,$$
$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}, \; 0 \leq k \leq p,$$
$$k^p \equiv k \pmod{p}, \quad k \text{ is integer},$$

we obtain

$$B_p = \sum_{k=0}^{\frac{p-1}{2}}\binom{p}{2k+1} \times 8^k \times 3^{p-1-2k} \equiv 8^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}.$$

$$B_{p-1} \equiv -3^{p-2}\sum_{k=0}^{(p-1)/2-1}\left(\frac{8}{9}\right)^k \equiv 3 \times 8^{(p-1)/2} - 3^p \equiv 3(\pm 1 - 1) \pmod{p}.$$

□

4 Connection with the associated co-balancing numbers

Let b_n denote the co-balancing number k that satisfies the identity

$$1+2+\cdots+k=(k+1)+(k+2)+\cdots+(k+r),$$

for some natural number r. It is known that

$$b_0=b_1=0,\ b_2=2 \text{ and } b_{n+1}=6b_n-b_{n-1}+2,\ n\geq 2.$$

Proposition 4.1. *For any positive integer n, we have*

$$B_nB_{n+1}=b_{n+1}+b_{n+1}^2,$$

and

$$\sqrt{1+4B_nB_{n+1}}=1+2b_{n+1}.$$

Proof. We aim to show that $B_nB_{n+1}=b_{n+1}+b_{n+1}^2$. Indeed, observe that

$$\sqrt{1+4B_nB_{n+1}}=1+2b_{n+1}\Leftrightarrow B_nB_{n+1}=b_{n+1}+b_{n+1}^2.$$

Using the identities from [7]

$$B_n=\frac{b_{n+1}-b_n}{2},\quad b_{n+1}=6b_n-b_{n-1}+2,\quad (b_n-1)^2=1+b_{n-1}b_{n+1},\quad n\geq 2, \tag{4.1}$$

along with identity (1.1), we compute:

$$\begin{aligned} B_nB_{n+1} &= \left(\frac{b_{n+1}}{2}-\frac{b_n}{2}\right)(6B_n-B_{n-1}) \\ &= \left(\frac{b_{n+1}-b_n}{2}\right)\left(3b_{n+1}-3b_n-\frac{b_n-b_{n-1}}{2}\right) \\ &= \left(\frac{b_{n+1}-b_n}{2}\right)\left(\frac{6b_{n+1}-6b_n-b_n+b_{n-1}}{2}\right) \\ &= \left(\frac{b_{n+1}-b_n}{2}\right)\left(\frac{6b_{n+1}+2-b_{n+1}-b_n}{2}\right) \\ &= \frac{5b_{n+1}^2-6b_nb_{n+1}+b_n^2-2b_n+2b_{n+1}}{4}. \end{aligned}$$

Using the identity $(b_n-1)^2=1+b_{n-1}b_{n+1}$, we substitute $b_n^2-2b_n=b_{n-1}b_{n+1}$ to obtain:

$$\begin{aligned} B_nB_{n+1} &= \frac{5b_{n+1}^2-6b_nb_{n+1}+b_{n-1}b_{n+1}+2b_{n+1}}{4} \\ &= \frac{b_{n+1}(5b_{n+1}-6b_n+2+b_{n-1})}{4}. \end{aligned}$$

Now, using the recurrence relation $b_{n+1}=6b_n-b_{n-1}+2$, we simplify:

$$\begin{aligned} 5b_{n+1}-6b_n+2+b_{n-1} &= 5(6b_n-b_{n-1}+2)-6b_n+2+b_{n-1} \\ &= 30b_n-5b_{n-1}+10-6b_n+2+b_{n-1} \\ &= 24b_n-4b_{n-1}+12. \end{aligned}$$

Similarly, we compute:

$$\begin{aligned} b_{n+1}+1 &= 6b_n-b_{n-1}+2+1=6b_n-b_{n-1}+3, \\ 4(b_{n+1}+1) &= 24b_n-4b_{n-1}+12. \end{aligned}$$

Therefore, we conclude:

$$B_nB_{n+1}=\frac{b_{n+1}\cdot 4(b_{n+1}+1)}{4}=b_{n+1}(b_{n+1}+1),$$

as claimed. □

Proposition 4.2. *For any integers $k \geq 1$ and $n \geq 0$, the following identity holds:*

$$B_{n+k} = B_{n+k-1} + (1 + 2b_k)B_{n+1} - (1 + 2b_{k-1})\,B_n.$$

Proof. Using identity (2.3) and the first two identities of (4.1) along with (1.1), we obtain:

$$\begin{aligned}
B_{n+k} &= B_k B_{n+1} - B_{k-1} B_n \\
&= \left(\frac{b_{k+1} - b_k}{2}\right) B_{n+1} - \left(\frac{b_k - b_{k-1}}{2}\right) B_n \\
&= \left(\frac{6b_k - b_{k-1} + 2 - b_k}{2}\right) B_{n+1} - \left(\frac{6b_{k-1} - b_{k-2} + 2 - b_{k-1}}{2}\right) B_n \\
&= (1 + 2b_k)\, B_{n+1} + \left(\frac{b_k - b_{k-1}}{2}\right) B_{n+1} - (1 + 2b_{k-1})\, B_n + \left(\frac{b_{k-1} - b_{k-2}}{2}\right) B_n \\
&= (1 + 2b_k)\, B_{n+1} - (1 + 2b_{k-1})\, B_n + B_{k-1} B_{n+1} + B_{k-2} B_n \\
&= (1 + 2b_k)\, B_{n+1} - (1 + 2b_{k-1})\, B_n + B_{n+k-1}.
\end{aligned}$$

□

For example, setting $k = 2$ yields the known recurrence relation (1.1).

Conclusion

This paper presented a matrix-based approach to studying balancing numbers, leading to new identities, recurrence relations, and congruences. The use of the balancing Q-matrix provided a unified method for deriving and generalizing results, including connections with co-balancing numbers. These findings enhance our understanding of balancing numbers and suggest directions for future research, such as extending this method to related sequences.

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