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APPROXIMATION BY GBS OF BIVARIATE RIEMANN-LIOUVILLE TYPE FRACTIONAL α-BERNSTEIN-KANTOROVICH OPERATORS

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Abstract In this article, we construct the GBS operators of the operators defined by Kaur et al.[29]. We analyze both its convergence and error of approximation properties using the conventional tools of approximation theory. Furthermore, we give some graphical examples on which GBS operators can be applied.

1 Introduction

The most renowned fundamental conclusion about the convergence of linear positive operators is attributed to Weierstrass, who presented a significant theorem known as Weierstrass approximation theorem. In 1912, Bernstein created the eminent algebraic polynomials in approximation theory to provide a constructive proof of Weierstrass theorem. Bernstein's [11] most remarkable contribution is the invention of Bernstein's polynomials, a series of polynomial approximations for continuous functions. The Bernstein polynomial of degree m associated with the continuous function f on the interval [0,1] is defined as:

$$B_m(f) = \sum_{i=0}^{m} \binom{m}{i} y^i (1-y)^{m-i} f\left(\frac{i}{m}\right) \qquad m \in \mathbb{N}, \qquad f \in C[0,1]. \tag{1.1}$$

These polynomials possess the unique property of uniformly approximating the function f on the interval [0,1]. The approximation improves as m increases.

In [18], Chen et al. constructed α-Bernstein operators which is generalization of the Bernstein operators.

$$B_m^{\alpha}(f;y) = \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y) f\left(\frac{i}{m}\right), \quad y \in [0,1], \tag{1.2}$$

By incorporating a parameter $\alpha \in [0,1]$ that permits different approximation weightings and for $i = 0, 1, ..., m$, the α-Bernstein polynomial $p_{m,i}^{(\alpha)}(y)$ of degree m is defined by $p_{1,0}^{(\alpha)}(y) = 1 - y, p_{1,1}^{(\alpha)}(y) = y$ and

$$p_{m,i}^{(\alpha)}(y) = \left[\binom{m-2}{i}(1-\alpha)y + \binom{m-2}{i-2}(1-\alpha)(1-y) + \binom{m}{i}\alpha y(1-y)\right] y^{i-1}(1-y)^{m-i-1}.$$

where $m \geq 2, y \in [0,1]$.

Mohiuddine et al.[33] constructed a new family of α-Bernstein-Kantorovich operators. The Kantorovich modifications of sequences of linear positive operators can be utilized to approximate Lebesgue integrable functions. The α-Bernstein-Kantorovich operators are defined as:

$$K_{m,\alpha}(f;y) = (m+1)\sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y)\int_{i/(m+1)}^{(i+1)/(m+1)} f(t)dt. \tag{1.3}$$

If we choose α=1, then the operators (1.3) converted into the classical Bernstein-Kantorovich operators [28]. The operators $K_{m,\alpha}$ can be rewrite as:

$$\begin{aligned} K_{m,\alpha}(f;y) &= \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y)\int_0^1 f\left(\frac{i+t}{m+1}\right)dt, & (1.4)\\ &= \sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y)({}_{0^+}I_1 f)\left(\frac{i+.}{m+1}\right). & (1.5) \end{aligned}$$

Berwal et al.[12] extended the concept of operators (1.3) to the Riemann-Liouville type fractional α-Bernstein-Kantorovich operators. These operators are defined for order of $\beta > 0$,

$$K_m^{\beta}(f;y) = \Gamma(\beta+1)\sum_{i=0}^{m} p_{m,i}^{(\alpha)}(y)\int_0^1 \frac{(1-t)^{\beta-1}}{\Gamma(\beta)} f\left(\frac{i+t}{m+1}\right)dt, \quad (y \in [0,1], m \in \mathbb{N}). \tag{1.6}$$

For additional information regarding Bernstein-Kantorovich operators, Grüss-Voronovskaya type theorems and fractional type operators, readers may consult prior studies [7, 8, 13, 14, 20, 21, 22, 24, 25, 26, 28, 30, 31, 32, 34, 39, 43, 44] and references therein.

Kaur et al.[29] build a novel set of positive linear operators. They defined the bivariate Riemann-Liouville type fractional α-Bernstein-Kantorovich operators and investigate its approximation properties. Let $C([0,1]\times[0,1])$ be the space of continuous bivariate functions on $[0,1]\times[0,1]$. The operators is defined as follows:

$$\begin{aligned} K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(f;x,y) &= \Gamma(\beta_1+1)\Gamma(\beta_2+1)\sum_{i_1=0}^{n} p_{n,i_1}^{(\alpha_1)}(x)\sum_{i_2=0}^{m} p_{m,i_2}^{(\alpha_2)}(y)\\ &\times \int_0^1\int_0^1 \frac{(1-t)^{\beta_1-1}}{\Gamma(\beta_1)}\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} f\left(\frac{i_1+t}{n+1}, \frac{i_2+s}{m+1})\right)dtds, \forall\ (x,y)\in[0,1]\times[0,1]. \end{aligned} \tag{1.7}$$

When $\beta_1 = \beta_2 = 1$, we get back the bivariate α-Bernstein-Kantorovich operators. Similar for the case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, the operators reduced back to bivariate Bernstein-Kantorovich operators.

The motivation for the investigation and application of Bernstein polynomials is in their capacity to deliver efficient and refined answers to challenges related to approximation and interpolation. Bernstein polynomials are fundamental to Bézier curves, which are extensively utilized in computer graphics, animation and computer-aided design (CAD). Bernstein fundamental functions were employed by French engineer Bézier in the 1960s to create the contours and surfaces of automobiles based on their geometric configurations. These functions are referred as Bézier basis functions. Bernstein polynomials are a computationally efficient tool for modeling and manipulating curves in a variety of contexts due to their relationship with Bézier curves. Their widespread usage in animation and graphic design is due in large part to their user-friendliness, local controllability and smoothness. Multiple mathematical fields have made use of Bernstein's excellent polynomials, including numerical solutions to partial differential equations, statistical modeling, machine learning, signal processing, geometric modeling, computer-aided geometric design (CAGD), computer graphics and many more. Readers are directed to the literature[5, 15, 16, 17, 35, 42] for further information.

In this research paper, we extend our concept of operators (1.7) to the GBS operators of the bivariate Riemann-Liouville type fractional α-Bernstein-Kantorovich operators. We began by computing the moments for the operators (1.7) and utilized them in the results of the GBS operators, since the moments are crucial for determining the direct results of positive linear operators.

Lemma 1.1. *[12] For the test functions $e_i(t) = t^i$, i=0-4, we have*

(i) $K_m^{\beta}(e_0; y) = 1$;

(ii) $K_m^{\beta}(e_1; y) = \frac{m}{m+1}y + \frac{1}{(\beta+1)(m+1)}$;

(iii) $K_m^{\beta}(e_2; y) = \frac{m^2}{(m+1)^2}\left(y^2 + \frac{m+2(1-\alpha)}{m^2}y(1-y)\right) + \frac{2my}{(\beta+1)(m+1)^2} + \frac{2}{(\beta+1)(\beta+2)(m+1)^2}$;

(iv)
$$K_m^{\beta}(e_3; y) = \frac{m^3}{(1+m)^3}\left(y^3 + \frac{3(1-y)y^2(m+2(1-\alpha))}{m^2} + \frac{(1-2y)(1-y)y(m+6(1-\alpha))}{m^3}\right) + \frac{3m^2}{(1+m)^3(1+\beta)}\left(y^2 + \frac{(1-y)y(m+2(1-\alpha))}{m^2}\right) + \frac{6my}{(1+m)^3(1+\beta)(2+\beta)} + \frac{6}{(1+m)^3(1+\beta)(2+\beta)(3+\beta)};$$

(v)
$$K_m^{\beta}(e_4; y) = \frac{m^4}{(1+m)^4}\left(y^4 + \frac{(1-y)y(m+14(1-\alpha))) + (1-y)y(3(-2+m)m+12(-6+m)(1-\alpha))}{m^4} + \frac{6(1-y)y^3(m+2(1-\alpha))}{m^2}\right) + \frac{m^4}{(1+m)^4}\left(\frac{4(1-2y)(1-y)y^2(m+6(1-\alpha))}{m^3}\right) + \frac{4m^3}{(1+m)^4(1+\beta)}\left(y^3 + \frac{3(1-y)y^2(m+2(1-\alpha))}{m^2} + \frac{(1-2y)(1-y)y(m+6(1-\alpha))}{m^3}\right) + \frac{12m^2}{(1+m)^4(1+\beta)(2+\beta)}\left(y^2 + \frac{(1-y)y(m+2(1-\alpha))}{m^2}\right) + \frac{24my}{(1+m)^4(1+\beta)(2+\beta)(3+\beta)} + \frac{24}{(1+m)^4(1+\beta)(2+\beta)(3+\beta)(4+\beta)}.$$

Lemma 1.2. *[29] Let $e_{rs}(x,y) = x^r y^s, (r,s)$ in $\mathbb{N}\times\mathbb{N}$, with $r+s \le 4$ be the bivariate test functions. Then the moments for bivariate Riemann-Liouville type fractional α-Bernstein-Kantorovich operators $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(f; x, y)$ are as follows:*

(i) $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{00}; x, y) = 1$;

(ii) $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{10}; x, y) = \frac{n}{n+1}x + \frac{1}{(\beta_1+1)(n+1)}$;

(iii) $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{01}; x, y) = \frac{m}{m+1}y + \frac{1}{(\beta_2+1)(m+1)}$;

(iv) $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{20}; x, y) = \frac{n^2}{(n+1)^2}\left(x^2 + \frac{n+2(1-\alpha_1)}{n^2}x(1-x)\right) + \frac{2nx}{(\beta_1+1)(n+1)^2} + \frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2}$;

(v) $K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{02}; x, y) = \frac{m^2}{(m+1)^2}\left(y^2 + \frac{m+2(1-\alpha_2)}{m^2}y(1-y)\right) + \frac{2my}{(\beta_2+1)(m+1)^2} + \frac{2}{(\beta_2+1)(\beta_2+2)(m+1)^2}$;

(vi)
$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{30}; x, y) = \frac{n^3}{(1+n)^3}\left(x^3 + \frac{3(1-x)x^2(n+2(1-\alpha_1))}{n^2} + \frac{(1-2n)(1-x)x(n+6(1-\alpha_1))}{n^3}\right) + \frac{3n^2}{(1+n)^3(1+\beta_1)}\left(x^2 + \frac{(1-x)x(n+2(1-\alpha_1))}{n^2}\right) + \frac{6mx}{(1+n)^3(1+\beta_1)(2+\beta_1)} + \frac{6}{(1+n)^3(1+\beta_1)(2+\beta_1)(3+\beta_1)};$$

(vii)
$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{03}; x, y) = \frac{m^3}{(1+m)^3}\left(y^3 + \frac{3(1-y)y^2(m+2(1-\alpha_2))}{m^2} + \frac{(1-2y)(1-y)y(m+6(1-\alpha_2))}{m^3}\right) + \frac{3m^2}{(1+m)^3(1+\beta)}\left(y^2 + \frac{(1-y)y(m+2(1-\alpha_2))}{m^2}\right) + \frac{6my}{(1+m)^3(1+\beta_2)(2+\beta_2)} + \frac{6}{(1+m)^3(1+\beta_2)(2+\beta_2)(3+\beta_2)};$$

(viii)
$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(e_{40}; x, y) = \frac{n^4}{(1+n)^4}\left(x^4 + \frac{(1-x)x(n+14(1-\alpha_1)) + (1-x)x(3(-2+n)n+12(-6+n)(1-\alpha_1)}{n^4} + \frac{6(1-x)y^3(n+2(1-\alpha_1))}{n^2}\right) + \frac{n^4}{(1+n)^4}(\frac{4(1-2x)(1-x)x^2(n+6(1-\alpha_1))}{n^3}) + \frac{4n^3}{(1+n)^4(1+\beta_1)}\left(x^3 + \frac{3(1-x)x^2(n+2(1-\alpha_1))}{n^2} + \frac{(1-2x)(1-x)x(n+6(1-\alpha_1))}{n^3}\right) + \frac{12n^2}{(1+n)^4(1+\beta_1)(2+\beta_1)}\left(x^2 + \frac{(1-x)x(n+2(1-\alpha_1))}{n^2}\right) + \frac{24nx}{(1+n)^4(1+\beta_1)(2+\beta_1)(3+\beta_1)} + \frac{24}{(1+n)^4(1+\beta_1)(2+\beta_1)(3+\beta_1)(4+\beta_1)};$$

$$\text{(ix)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(e_{04};x,y) = \frac{m^4}{(1+m)^4}\Bigg(y^4 + \frac{(1-y)y(m+14(1-\alpha_2)) + (1-y)y(3(-2+m)m + 12(-6+m)(1-\alpha_2))}{m^4}$$
$$+ \frac{6(1-y)y^3(m+2(1-\alpha_2))}{m^2}\Bigg) + \frac{m^4}{(1+m)^4}(\frac{4(1-2y)(1-y)y^2(m+6(1-\alpha_2))}{m^3})$$
$$+ \frac{4m^3}{(1+m)^4(1+\beta_2)}\Bigg(y^3 + \frac{3(1-y)y^2(m+2(1-\alpha_2))}{m^2} + \frac{(1-2y)(1-y)y(m+6(1-\alpha_2))}{m^3}\Bigg)$$
$$+ \frac{12m^2}{(1+m)^4(1+\beta_2)(2+\beta_2)}\Bigg(y^2 + \frac{(1-y)y(m+2(1-\alpha_2))}{m^2}\Bigg) + \frac{24my}{(1+m)^4(1+\beta_2)(2+\beta_2)(3+\beta_2)}$$
$$+ \frac{24}{(1+m)^4(1+\beta_2)(2+\beta_2)(3+\beta_2)(4+\beta_2)}.$$

Corollary 1.3. *Using the previous lemma as a foundation, we identify the following properties.*

$$\text{(i)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x);x,y) = \frac{n}{n+1}x + \frac{1}{(\beta_1+1)(n+1)} - x,$$
$$= \frac{-x}{n+1} + \frac{1}{(\beta_1+1)(n+1)};$$

$$\text{(ii)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y);x,y) = \frac{m}{m+1}y + \frac{1}{(\beta_2+1)(m+1)} - y,$$
$$= \frac{-y}{m+1} + \frac{1}{(\beta_2+1)(m+1)};$$

$$\text{(iii)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)^2;x,y) = \frac{n^2}{(n+1)^2}\Bigg(x^2 + \frac{n+2(1-\alpha_1)}{n^2}x(1-x)\Bigg) + \frac{2nx}{(\beta_1+1)(n+1)^2} + \frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2}\Bigg)$$
$$- 2x\Bigg(\frac{n}{n+1}x + \frac{1}{(\beta_1+1)(n+1)}\Bigg) + x^2,$$
$$= \frac{x(1-x)[2(1-\alpha_1)+n-1]}{(n+1)^2} + \frac{x(\beta_1-1)}{(\beta_1+1)(n+1)^2} + \frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2};$$

$$\text{(iv)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(s-y)^2;x,y) = \frac{m^2}{(m+1)^2}\Bigg(y^2 + \frac{m+2(1-\alpha_2)}{m^2}y(1-y)\Bigg) + \frac{2my}{(\beta_2+1)(m+1)^2} + \frac{2}{(\beta_2+1)(\beta_2+2)(m+1)^2}\Bigg)$$
$$- 2y\Bigg(\frac{m}{m+1}y + \frac{1}{(\beta_2+1)(m+1)}\Bigg) + y^2,$$
$$= \frac{y(1-y)[2(1-\alpha_2)+m-1]}{(m+1)^2} + \frac{y(\beta_2-1)}{(\beta_2+1)(m+1)^2} + \frac{2}{(\beta_2+1)(\beta_2+2)(m+1)^2};$$

$$\text{(v)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)^3;x,y) = x^3\Bigg(-1 + \frac{n^3}{(1+n)^3} - \frac{3n^2}{(1+n)^2} + \frac{3n}{1+n} - \frac{3n(n+2(1-\alpha_1))}{(1+n)^3} + \frac{3(n+2(1-\alpha_1)}{(1+n)^2}$$
$$+ \frac{2(n+6(1-\alpha_1))}{(1+n)^3}\Bigg) + x^2\Bigg(\frac{3n(n+2(1-\alpha_1))}{(1+n)^3} - \frac{3(n+2(1-\alpha_1))}{(1+n)^2}$$
$$- \frac{3(n+6(1-\alpha_1)}{(1+n)^3} + \frac{3n^2}{(1+n)^3(1+\beta_1)} - \frac{6n}{(1+n)^2(1+\beta_1)}$$
$$+ \frac{3}{(1+n)(1+\beta_1)} - \frac{3(n+2(1-\alpha_1))}{(1+n)^3(1+\beta_1)}\Bigg) + x\Bigg(\frac{n+6(1-\alpha_1))}{(1+n)^3} + \frac{3(n+2(1-\alpha_1))}{(1+n)^3(1+\beta_1)}$$
$$+ \frac{6n}{(1+n)^3(1+\beta_1)(2+\beta_1)} \frac{6}{(1+n)^2(1+\beta_1)(2+\beta_1)}\Bigg) + \frac{6}{(1+n)^3(1+\beta_1)(2+\beta_1)(3+\beta_1)};$$

$$\text{(vi)}\ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y)^3;x,y) = y^3\Bigg(-1 + \frac{m^3}{(1+m)^3} - \frac{3m^2}{(1+m)^2} + \frac{3m}{1+m} - \frac{3m(m+2(1-\alpha_2))}{(1+m)^3} + \frac{3(m+2(1-\alpha_2)}{(1+m)^2}$$
$$+ \frac{2(m+6(1-\alpha_2))}{(1+m)^3}\Bigg) + y^2\Bigg(\frac{3m(m+2(1-\alpha_2))}{(1+m)^3} - \frac{3(m+2(1-\alpha_2))}{(1+m)^2}$$
$$- \frac{3(m+6(1-\alpha_2))}{(1+m)^3} + \frac{3m^2}{(1+m)^3(1+\beta_2)} - \frac{6m}{(1+m)^2(1+\beta_2)}$$
$$+ \frac{3}{(1+m)(1+\beta_2)} - \frac{3(m+2(1-\alpha_2))}{(1+m)^3(1+\beta_2)}\Bigg) + y\Bigg(\frac{m+6(1-\alpha_2))}{(1+m)^3} + \frac{3(m+2(1-\alpha_2)}{(1+m)^3(1+\beta_2)}$$
$$+ \frac{6m}{(1+m)^3(1+\beta_2)(2+\beta_2)} - \frac{6}{(1+m)^2(1+\beta_2)(2+\beta_2)}\Bigg) + \frac{6}{(1+m)^3(1+\beta_2)(2+\beta_2)(3+\beta_2)};$$

(vii)
$$
\begin{aligned}
K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^4;x,y) &= x^4\Big(1+\frac{n^4}{(1+n)^4}-\frac{4n^3}{(1+n)^3}+\frac{6n^2}{(1+n)^2}-\frac{4n}{1+n}-\frac{6n^2(n+2(1-\alpha_1))}{(1+n)^4}\\
&+\frac{12n(n+2(1-\alpha_1))}{(1+n)^3}-\frac{6(n+2(1-\alpha_1))}{(1+n)^2}+\frac{8n(n+6(1-\alpha_1))}{(1+n)^4}-\frac{8(n+6(1-\alpha_1))}{(1+n)^3}\Big)\\
&+x^3\Big(\frac{6n^2(n+2(1-\alpha_1))}{(1+n)^4}-\frac{12n(n+2(1-\alpha_1))}{(1+n)^3}+\frac{6(n+2(1-\alpha_1))}{(1+n)^2}\\
&-\frac{12n(n+6(1-\alpha_1))}{(1+n)^4}+\frac{12(n+6(1-\alpha_1))}{(1+n)^3}+\frac{4n^3}{(1+n)^4(1+\beta_1)}\\
&-\frac{12n^2}{(1+n)^3(1+\beta_1)}\Big)+x^3\Big(\frac{12n}{(1+n)^2(1+\beta_1)}-\frac{4}{(1+n)(1+\beta_1)}-\frac{12n(n+2(1-\alpha_1))}{(1+n)^4(1+\beta_1)}\\
&+\frac{12(n+2(1-\alpha_1))}{(1+n)^3(1+\beta_1)}+\frac{8(n+6(1-\alpha_1))}{(1+n)^4(1+\beta_1)}\Big)+x^2\Big(\frac{4n(n+6(1-\alpha_1))}{(1+n)^4}-\frac{4(n+6(1-\alpha_1))}{(1+n)^3}\\
&-\frac{n+14(1-\alpha_1))}{(1+n)^4}-\frac{3(-2+n)n+12(-6+n)(1-\alpha_1))}{(1+n)^4}+\frac{12n(n+2(1-\alpha_1))}{(1+n)^4(1+\beta_1)}\\
&-\frac{12(n+2(1-\alpha_1))}{(1+n)^3(1+\beta_1)}-\frac{12(n+6(1-\alpha_1))}{(1+n)^4(1+\beta_1)}+\frac{12n^2}{(1+n)^4(1+\beta_1)(2+\beta_1)}\\
&-\frac{24n}{(1+n)^3(1+\beta_1)(2+\beta_1)}+\frac{12}{(1+n)^2(1+\beta_1)(2+\beta_1)}-\frac{12(n+2(1-\alpha_1))}{(1+n)^4(1+\beta_1)(2+\beta_1)}\Big)\\
&+x\Big(\frac{n+14(1-\alpha_1))}{(1+n)^4}+\frac{3(-2+n)n+12(-6+n)(1-\alpha_1))}{(1+n)^4}+\frac{4(n+6(1-\alpha_1))}{(1+n)^4(1+\beta_1)}\\
&+\frac{12(n+2(1-\alpha_1))}{(1+n)^4(1+\beta_1)(2+\beta_1)}+\frac{24n}{(1+n)^4(1+\beta_1)(2+\beta_1)(3+\beta_1)}\\
&-\frac{24}{(1+n)^3(1+\beta_1)(2+\beta_1)(3+\beta_1)}\Big)+\frac{24}{(1+n)^4(1+\beta_1)(2+\beta_1)(3+\beta_1)(4+\beta_1)};
\end{aligned}
$$

(viii)
$$
\begin{aligned}
K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((s-y)^4;x,y) &= y^4\Big(1+\frac{m^4}{(1+m)^4}-\frac{4m^3}{(1+m)^3}+\frac{6m^2}{(1+m)^2}-\frac{4m}{1+m}-\frac{6m^2(m+2(1-\alpha_2))}{(1+m)^4}\\
&+\frac{12m(m+2(1-\alpha_2))}{(1+m)^3}-\frac{6(m+2(1-\alpha_2))}{(1+m)^2}+\frac{8m(m+6(1-\alpha_2))}{(1+m)^4}-\frac{8(m+6(1-\alpha_2)}{(1+m)^3}\Big)\\
&+y^3\Big(\frac{6m^2(m+2(1-\alpha_2))}{(1+m)^4}-\frac{12m(m+2(1-\alpha_2)}{(1+m)^3}+\frac{6(m+2(1-\alpha_2))}{(1+m)^2}\\
&-\frac{12m(m+6(1-\alpha_2))}{(1+m)^4}+\frac{12(m+6(1-\alpha_2))}{(1+m)^3}+\frac{4m^3}{(1+m)^4(1+\beta_2)}\\
&-\frac{12m^2}{(1+m)^3(1+\beta_2)}+\frac{12m}{(1+m)^2(1+\beta_2)}-\frac{4}{(1+m)(1+\beta_2)}-\frac{12m(m+2(1-\alpha_2))}{(1+m)^4(1+\beta_2)}\\
&+\frac{12(m+2(1-\alpha_2))}{(1+m)^3(1+\beta_2)}+\frac{8(m+6(1-\alpha_2)}{(1+m)^4(1+\beta_2)}\Big)+y^2\Big(\frac{4m(m+6(1-\alpha_2))}{(1+m)^4}-\frac{4(m+6(1-\alpha_2))}{(1+m)^3}\\
&-\frac{m+14(1-\alpha_2))}{(1+m)^4}-\frac{3(-2+m)m+12(-6+m)(1-\alpha_2))}{(1+m)^4}+\frac{12m(m+2(1-\alpha_2))}{(1+m)^4(1+\beta_2)}\\
&-\frac{12(m+2(1-\alpha_2))}{(1+m)^3(1+\beta_2)}-\frac{12(m+6(1-\alpha_2))}{(1+m)^4(1+\beta_2)}+\frac{12m^2}{(1+m)^4(1+\beta_2)(2+\beta_2)}\\
&-\frac{24m}{(1+m)^3(1+\beta_2)(2+\beta_2)}+\frac{12}{(1+m)^2(1+\beta_2)(2+\beta_2)}-\frac{12(m+2(1-\alpha_2))}{(1+m)^4(1+\beta_2)(2+\beta_2)}\Big)\\
&+y\Big(\frac{m+14(1-\alpha_2))}{(1+m)^4}+\frac{3(-2+m)m+12(-6+m)(1-\alpha_2))}{(1+m)^4}+\frac{4(m+6(1-\alpha_2))}{(1+m)^4(1+\beta_2)}\\
&+\frac{12(m+2(1-\alpha_2))}{(1+m)^4(1+\beta_2)(2+\beta_2)}+\frac{24m}{(1+m)^4(1+\beta_2)(2+\beta_2)(3+\beta_2)}\\
&-\frac{24}{(1+m)^3(1+\beta_2)(2+\beta_2)(3+\beta_2)}\Big)+\frac{24}{(1+m)^4(1+\beta_2)(2+\beta_2)(3+\beta_2)(4+\beta_2)}.
\end{aligned}
$$

Corollary 1.4. *From Corollary (1.3), we get*

$$
K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^2;x,y) < \frac{1}{n+1}\Big(x(1-x)+\frac{x}{(n+1)}+\frac{2}{(n+1)}\Big), \tag{1.8}
$$

$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((s-y)^2;x,y) < \frac{1}{m+1}\left(y(1-y)+\frac{y}{(m+1)}+\frac{2}{(m+1)}\right). \tag{1.9}$$

Proof. For all $x \in [0,1]$, we have

$$\begin{aligned}
K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^2;x,y) &= \frac{x(1-x)[2(1-\alpha_1)+n-1]}{(n+1)^2}+\frac{x(\beta_1-1)}{(\beta_1+1)(n+1)^2}+\frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2}\\
&< \frac{x(1-x)}{(n+1)}+\frac{x(\beta_1-1)}{(\beta_1+1)(n+1)^2}+\frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2}\\
&< \frac{x(1-x)}{(n+1)}+\frac{x}{(n+1)^2}+\frac{2}{(\beta_1+1)(\beta_1+2)(n+1)^2}\\
&< \frac{1}{n+1}\left(x(1-x)+\frac{x}{(n+1)}+\frac{2}{(n+1)}\right).
\end{aligned}$$

The demonstration of inequality (1.9) is similar to the demonstration of inequality (1.8). As a result, the details are omitted. □

2 Construction of GBS operators of Generalized Bivariate Riemann-Liouville type Fractional α-Bernstein-Kantorovich Type

An interesting concept called GBS operators arises in the study of bivariate linear positive operators. Recently, function theory and approximation theory have taken an interest in studying generalized Boolean sum (GBS) operators of certain linear positive operators. We will talk about GBS operators that work with bivariate fractional Riemann-Liouville α-Bernstein-Kantorovich operators and look at some of its smoothness properties in this part. Bögel [13, 14] introduced the concepts of B-continuous and B-differentiable functions. The Korovkin theorem for B-continuous functions, which is widely recognized in approximation theory, is developed by Badea et al.[7, 8]. In [22], Dobrescu and Matei proved that for any B-continuous function defined on a bounded interval, it is possible to get a uniform approximation by using a boolean sum of bivariate Bernstein polynomials. The researchers Agrawal et al. [3] conducted a study to investigate the degree of approximation for bivariate Lupas-Durrmeyer type operators. These operators were based on the Pòlya distribution and related with GBS operators. Recently, a large number of researchers have made important contributions to the discussion on this subject. We direct the reader to a few relevant publications[1, 2, 4, 10, 23, 27, 36, 37, 38, 40].

Let X and Y be compact real intervals and let $\Delta_{(x,y)}f[t,s;x,y]$ be the mixed difference of f defined for every $(x,y)\in X\times Y$. The function $f: X\times Y\to\mathbb{R}$ is B-bounded on $X\times Y$ if there exists a constant $N>0$ such that $|\Delta_{(x,y)}f[t,s;x,y]|\le N$ for every $(x,y),(t,s)\in X\times Y$. Observe that if $X\times Y$ is a compact subset of $\mathbb{R}^2$, then any B-continuous function from $X\times Y$ to $\mathbb{R}$ is also a B-bounded function. In this article, $B_b(X\times Y)$ refers to all B-bounded functions on $X\times Y$. The collection of all B-continuous functions on $X\times Y$ is denoted by $C_b(X\times Y)$. The space of all bounded functions and the space of all continuous functions on $X\times Y$ endowed with the sup-norm $\|\cdot\|_\infty$ are denoted by $B(X\times Y)$ and $C(X\times Y)$, respectively and a subset of $C_b(X\times Y)$ is conformable.

When approximating a function f, the GBS operators achieve an accuracy comparable to or better than operators. A broader range of functions will be covered by the validity of approximation properties and the Korovkin type theorem. For $I=[0,1]$, $I^2=[0,1]\times[0,1]$, $C_b(I^2)$ denotes the space of all B-continuous functions on I^2. For any $f\in C(I^2)$ and $m,n\in\mathbb{N}$, we construct the GBS operators of generalized Riemann-Liouville type fractional α-Bernstein-Kantorovich operators as follows:

$$S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f(t,s);x,y) := K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f(t,y) - f(t,s) + f(x,s);x,y), \quad \forall\ (x,y) \in I^2.$$

More precisely, the generalized Riemann-Liouville fractional α-Bernstein-Kantorovich type GBS operators are well defined as follows:

$$S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) = \Gamma(\beta_1+1)\Gamma(\beta_2+1)\sum_{i_1=0}^{n} p^{(\alpha_1)}_{n,i_1}(x)\sum_{i_2=0}^{m} p^{(\alpha_2)}_{m,i_2}(y)$$
$$\times \int_0^1\int_0^1 \frac{(1-t)^{\beta_1-1}}{\Gamma(\beta_1)}\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}\left(f\left(\frac{i_1+t}{n+1},y\right) - f\left(\frac{i_1+t}{n+1},\frac{i_2+s}{m+1}\right) + f\left(x,\frac{i_2+s}{m+1}\right)\right)dtds, \tag{2.1}$$

where $(x,y) \in I^2$ and $m_i, n_i \in \mathbb{N}$, where the operators $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ are well defined on the space $C_b(I^2)$ into $C_b(I^2)$.

The GBS operators $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ to a function f has an approximation degree at least as good as operators $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$. The approximation characteristics and the Korovkin type theorem will hold in a larger space of functions since all continuous functions are Bögel continuous functions. The motive behind defining GBS operators is to extend univariate approximation operators to bivariate functions in a way that accounts for mixed partial derivatives. This leads to better approximation accuracy, improved convergence rates and practical applications in computational mathematics, image processing and numerical analysis.

3 Degree of approximation by $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$

The mixed modulus of smoothness of $f \in C_b(I^2)$ is defined by :

$$\omega_{mixed}(f;\delta_1,\delta_2) := \left\{\sup|\Delta_{(x,y)}f[t,s;x,y]| : |x-t| < \delta_1, |y-s| < \delta_2\right\}, \tag{3.1}$$

for all $(x,y),(t,s) \in I^2$ and for any $(\delta_1,\delta_2) \in (0,\infty)\times(0,\infty)$ with $\omega_{mixed} : [0,\infty)\times[0,\infty) \to \mathbb{R}$.

The basic characteristics of ω_{mixed}, which are similar to the conventional modulus of continuity, were established by Badea et al. [7, 9]. The mixed modulus of smoothness will be used to assess the convergence rate of the operators (2.1) sequences to $f \in C_b(I^2)$.

Theorem 3.1. *For every $f \in C_b(I^2)$, at each point $(x,y) \in I^2$, the operators satisfies the following inequality*

$$|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)| \le 8\,\omega_{mixed}(f;(n+1)^{\frac{-1}{2}},(m+1)^{\frac{-1}{2}}).$$

Proof. According to the definition of $\omega_{mixed}(f;\delta_1,\delta_2)$ and using the inequality

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(f;\delta_1,\delta_2);\lambda_1,\lambda_2 > 0,$$

we can write

$$|\Delta_{(x,y)}f[t,s;x,y]| \le \omega_{mixed}(f;|t-x|,|s-y|)$$
$$\le \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_{mixed}(f;\delta_1,\delta_2), \tag{3.2}$$

for every $(x,y),(t,s) \in I^2$ and for any $\delta_1,\delta_2 > 0$. From the definition of $\Delta_{(x,y)}f[t,s;x,y]$, we get

$$f(x,s) + f(t,y) - f(t,s) = f(x,y) - \Delta_{(x,y)}f[t,s;x,y]. \tag{3.3}$$

This equality is then processed by the positive linear operators $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ and by the definition of $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ operators, we may write

$$S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2} = f(x,y)K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(e_{00};x,y) - K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(|\Delta_{(x,y)}f[t,s;x,y]|;x,y).$$

Since $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(e_{00};x,y) = 1$, Using the Cauchy-Schwarz inequality and taking into account the inequality (3.2) we obtain,

$$\begin{aligned}|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)| &\leq K^{\beta_1,\beta_2}_{n,m}(|\Delta_{(x,y)}f[t,s;x,y]|;x,y)\\ &\leq K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(e_{00};x,y) + \delta_1^{-1}\sqrt{K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)^2;x,y)}\\ &+\delta_2^{-1}\sqrt{K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y)^2;x,y)}\\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)^2;x,y)K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y)^2;x,y)}\ \omega_{mixed}(f;\delta_1,\delta_2).\end{aligned}$$

From Corollary (1.4), for all $(x,y) \in I^2$, we have the following inequalities

$$K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)^2;x,y) < \frac{1}{n+1}\left(x(1-x) + \frac{x}{n+1} + \frac{2}{n+1}\right) < \frac{3}{n+1}.$$
$$K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y)^2;x,y) < \frac{3}{m+1}.$$

By taking $\delta_1 = (n+1)^{\frac{-1}{2}}$ and $\delta_2 = (m+1)^{\frac{-1}{2}}$, we establish the intended relationship.

□

We will now outline the approximation order for B-differentiable functions employing the operators $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$. In [41], Pop established a notable theorem on the order of approximation for B-differentiable functions. A function $f : X \times Y \subset R^2 \to R$ is called a **B**-differentiable (Bögel differentiable) function at the point $(t,s) \in X \times Y$ if the limit,

$$\lim_{(x,y)\to(t,s)} \frac{\Delta_{(x,y)}\ f[t,s;x,y]}{(x-t)(y-s)}.$$

This limit exists and must be finite. The limit is defined as the B-differential of f at the point (t,s) and is represented as $D_{xy}f(t,s) := D_B(f;t,s)$. All B-differentiable functions are denoted by $D_b(X \times Y)$.

For $0 < d_1, d_2 \leq 1$, the Lipschitz class denoted by $\text{Lip}_M\,(d_1,d_2)$ for two variables is defined as :

$$\text{Lip}_M\,(d_1,d_2) := \{f : C(I^2) : |f(t,s) - f(x,y)| \leq M|t-x|^{d_1}|s-y|^{d_2}\},$$

where $M > 0$.

Theorem 3.2. *If $f \in \text{Lip}_M\,(d_1,d_2)$, then we have the following inequality*

$$\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)\right| \leq M\left[\gamma_{n,\beta_1}(x)\right]^{\frac{d_1}{2}}\left[\gamma_{m,\beta_2}(y)\right]^{\frac{d_2}{2}}$$

hold for all $(x,y) \in [0,1] \times [0,1]$,

where $\gamma_{n,\beta_1}(x) = ||S_n^{(\alpha_1)}((t-.)^2;)||$ and $\gamma_{m,\beta_2}(y) = ||S_n^{(\alpha_2)}((s-.)^2;)||$.

Proof. Let $f \in \text{Lip}_M\,(d_1,d_2)$, then we have

$$\begin{aligned}\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)\right| &\leq S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(|f\,(t,s) - f(x,y)|\,;x,y\right)\\ &\leq MS^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(|t-x|^{d_1}\,|s-y|^{d_2}\,;x,y\right)\\ &\leq MS^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(|t-x|^{d_1}\,;x,y\right)S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(|s-y|^{d_2}\,;x,y\right)\end{aligned}$$

Applying Hölder's inequality with $p_1 = \frac{2}{d_1}, q_1 = \frac{2}{2-d_1}$ and $p_2 = \frac{2}{d_2}, q_2 = \frac{2}{2-d_2}$, we get

$$\begin{aligned}\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)\right| &\le M S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((t-x)^2;x,y\right)^{\frac{d_1}{2}} S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((s-y)^2;x,y\right)^{\frac{d_2}{2}} \\ &= M\left[\gamma_{n,\beta_1}(x)\right]^{\frac{d_1}{2}}\left[\gamma_{m,\beta_2}(y)\right]^{\frac{d_2}{2}}.\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let the function $f \in D_b(I^2)$ with $D_B f \in B(I^2)$. Then, for each $(x,y) \in I^2$, we get*

$$|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)| \le N[\|D_B f\|_\infty + \omega_{mixed}(D_B f;(n+1)^{-1/2},(m+1)^{-1/2})]((n+1)(m+1))^{-1/2}$$

where $N > 0$ is any constant.

Proof. If $f \in D_b(I^2)$, then we have the identity

$$\Delta_{(x,y)} f[t,s;x,y] = (t-x)(s-y)D_B f(u,v) \ with\ x<u<t, y<v<s.$$

It is clear that

$$D_B f(u,v) = \Delta_{(x,y)} D_B f(u,v) + D_B f(u,y) + D_B f(x,v) - D_B f(x,y).$$

Since $D_B f \in B(I^2)$, by utilizing the relationships mentioned before, we can write

$$\begin{aligned}&|K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(\Delta_{(x,y)} f[t,s;x,y];x,y)| \\ &= |K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x)(s-y)D_B f(u,v));x,y)| \\ &\le K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(|t-x||s-y||\Delta_{(x,y)} D_B f(u,v)|;x,y) \\ &+ K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(|t-x||s-y|(|D_B f(u,y)| + |D_B f(x,v)| + |D_B f(x,y)|);x,y) \\ &\le K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(|t-x||s-y|\omega_{mixed}(D_B f;|u-x|,|v-y|);x,y)) \\ &+ 3||D_B||_\infty K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(|t-x||s-y|;x,y).\end{aligned}$$

Because the mixed modulus of smoothness ω_{mixed} is non decreasing, we have

$$\begin{aligned}&\omega_{mixed}(D_B f;|u-x|,|v-y|) \\ &\le \omega_{mixed}(D_B f;|t-x|,|s-y|) \\ &\le (1+\delta_1^{-1}|t-x|)(1+\delta_2^{-1}|s-y|)\omega_{mixed}(f;\delta_1,\delta_2).\end{aligned}$$

By substituting into the previously mentioned inequality, utilizing the linearity of the operators $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ and employing the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|S_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(f;x,y)-f(x,y)\right| = |K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(\Delta_{(x,y)}f[t,s;x,y];x,y)| \\
&\leq 3\|D_Bf\|_\infty\sqrt{K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^2(s-y)^2;x,y)} \\
&+\Big[K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(|t-x||s-y|;x,y)+\delta_1^{-1}K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^2|s-y|;x,y). \\
&+\delta_2^{-1}K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}(|t-x|(s-y)^2;x,y) \\
&+\delta_1^{-1}\delta_2^{-1}K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}((t-x)^2(s-y)^2;x,y)\Big]\omega_{\text{mixed}}\left(D_Bf;\delta_1,\delta_2\right) \\
&\leq 3\|D_Bf\|_\infty\sqrt{K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\Big((t-x)^2(s-y)^2;x,y\Big)} \\
&+\Bigg[\sqrt{K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\Big((t-x)^2(s-y)^2;x,y\Big)}. \\
&+\delta_1^{-1}\sqrt{K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\Big((t-x)^4(s-y)^2;x,y\Big)} \\
&+\delta_2^{-1}\sqrt{K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\Big((t-x)^2(s-y)^4;x,y\Big)} \\
&+\delta_1^{-1}\delta_2^{-1}K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\Big((t-x)^2(s-y)^2;x,y\Big)\Bigg]\omega_{\text{mixed}}\left(D_Bf;\delta_1,\delta_2\right).
\end{aligned}$$

Taking the inequalities into account

$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\left((t-x)^2;x,y\right) < 3(n+1)^{-1},$$
$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\left((s-y)^2;x,y\right) < 3(m+1)^{-1},$$

and using the following equality, for $(x,y),(t,s)\in I^2$ and $p,q\in\{1,2\}$,

$$K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\left((t-x)^{2p}(s-y)^{2q};x,y\right) = K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\left((t-x)^{2p};x,y\right) \times K_{n,m,\alpha_1,\alpha_2}^{\beta_1,\beta_2}\left((s-y)^{2q};x,y\right).$$

$\delta_1=(n+1)^{-1/2}$ and $\delta_2=(m+1)^{-1/2}$,we get our intended outcome as in Theorem 3.3.

□

In [6, 19], the mixed K-functional was developed to improve the evaluation of smoothness. For $f\in C_b\left(I^2\right)$, we define the mixed K-functional as follows:

$$K_{\text{mixed}}\left(f;t,s\right)=\inf_{\mathsf{g}_1,\mathsf{g}_2,\mathsf{h}}\left\{\|f-\mathsf{g}_1-\mathsf{g}_2-\mathsf{h}\|_\infty+t\left\|D_B^{2,0}\mathsf{g}_1\right\|_\infty +s\left\|D_B^{0,2}\mathsf{g}_2\right\|_\infty+ts\left\|D_B^{2,2}\mathsf{h}\right\|_\infty\right\}$$

where $\mathsf{g}_1\in C_B^{2,0}$, $\mathsf{g}_2\in C_B^{0,2}$, $\mathsf{h}\in C_B^{2,2}$ and for $0\leq p,q\leq 2$, $C_B^{p,q}$ denotes the space of the functions $f\in C_b(I^2)$ with continuous mixed partial derivatives $D_B^{l,r}f$, $0\leq l\leq p$, $0\leq r\leq q$. The partial derivatives are

$$D_xf(x_0,y_0):=D_B^{1,0}(f;x_0,y_0)=\lim_{x\to x_0}\frac{\Delta_xf([x_0,x];y_0)}{x-x_0}$$

and

$$D_yf(x_0,y_0):=D_B^{0,1}(f;x_0,y_0)=\lim_{y\to y_0}\frac{\Delta_yf(x_0;[y_0,y])}{y-y_0}.$$

where $\Delta_x f([x_0,x];y_0) = f(x,y_0) - f(x_0,y_0)$ and $\Delta_y f(x_0;[y_0,y]) = f(x_0,y) - f(x_0,y_0)$. Regular derivatives are equivalent to second-order partial derivatives. Given a point (x_0,y_0), the derivative of $D_x f(x_0,y_0)$ with respect to the variable y is defined as:

$$D_y D_x f(x_0,y_0) := D_B^{0,1} D_B^{1,0}(f;x_0,y_0) = \lim_{y\to y_0} \frac{\Delta_y (D_x f)(x_0;[y_0,y])}{y-y_0}.$$

The sequence $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f)$ is approximated to the function $f \in C_b(I^2)$ and we provide an estimate for its order in terms of the mixed K-functional.

Theorem 3.4. *Let $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ be GBS operators of $K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ given by (2.1). Then,*

$$\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)\right| \le 3 \cdot K_{mixed}\left(f;(n+1)^{-1},(m+1)^{-1}\right)$$

for each $f \in C_b\left(I^2\right)$.

Proof. Based on the Taylor formula for the function $\mathsf{g}_1 \in C_B^{2,0}\left(I^2\right)$, we get

$$\mathsf{g}_1(t,s) = \mathsf{g}_1(x,y) + (t-x)D_B^{1,0}\mathsf{g}_1(x,y) + \int_x^t (t-u)D_B^{2,0}\mathsf{g}_1(u,y)du$$

[14], p. 67-69. Since the operators $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ reproduces linear functions

$$S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(\mathsf{g}_1;x,y) = \mathsf{g}_1(x,y) + S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\int_x^t (t-u)D_B^{2,0}\mathsf{g}_1(u,y)du;x,y\right)$$

and by definition of $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$, we may write

$$\begin{aligned}
&\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(\mathsf{g}_1;x,y) - \mathsf{g}_1(x,y)\right| \\
&= \left|K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\int_x^t (t-u)\left[D_B^{2,0}\mathsf{g}_1(u,y) - D_B^{2,0}\mathsf{g}_1(u,s)\right]du;x,y\right)\right| \\
&\le K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\left|\int_x^t |t-u||D_B^{2,0}\mathsf{g}_1(u,y) - D_B^{2,0}\mathsf{g}_1(u,s)|du;x,y\right|\right) \\
&\le \left\|D_B^{2,0}\mathsf{g}_1\right\|_\infty K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((t-x)^2;x,y\right) < \left\|D_B^{2,0}\mathsf{g}_1\right\|_\infty \frac{3}{n+1}.
\end{aligned}$$

Similarly, we may write

$$\begin{aligned}
&\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(\mathsf{g}_2;x,y) - \mathsf{g}_2(x,y)\right| \\
&\le \left\|D_B^{0,2}\mathsf{g}_2\right\|_\infty K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((s-y)^2;x,y\right) \\
&< \left\|D_B^{0,2}\mathsf{g}_2\right\|_\infty \frac{3}{m+1},
\end{aligned}$$

for $\mathsf{g}_2 \in C_B^{0,2}\left(I^2\right)$. For $\mathsf{h} \in C_B^{2,2}\left(I^2\right)$,

$$\begin{aligned}
\mathsf{h}(t,s) =& \mathsf{h}(x,y) + (t-x)D_B^{1,0}\mathsf{h}(x,y) + (s-y)D_B^{0,1}\mathsf{h}(x,y) \\
&+ (t-x)(s-y)D_B^{1,1}\mathsf{h}(x,y) \\
&+ \int_x^t (t-u)D_B^{2,0}\mathsf{h}(u,y)du + \int_y^s (s-v)D_B^{0,2}\mathsf{h}(x,v)dv \\
&+ \int_x^t (s-y)(t-u)D_B^{2,1}\mathsf{h}(u,y)du \\
&+ \int_y^s (t-x)(s-v)D_B^{1,2}\mathsf{h}(x,v)dv \\
&+ \int_x^t \int_y^s (t-u)(s-v)D_B^{2,2}\mathsf{h}(u,v)dvdu.
\end{aligned}$$

Referring to the operators's definition $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}$ into account and using $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((t-x);x,y)=0$, $S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}((s-y);x,y)=0$, we have

$$\begin{aligned}
&\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(\mathsf{h};x,y)-\mathsf{h}(x,y)\right|\\
&\leq \left|K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\int_x^t\int_y^s (t-u)(s-v)D_B^{2,2}\mathsf{h}(u,v)dvdu;x,y\right)\right|\\
&\leq K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\left|\int_x^t\int_y^s (t-u)(s-v)D_B^{2,2}\mathsf{h}(u,v)dvdu\right|;x,y\right)\\
&\leq K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left(\int_x^t\int_y^s |t-u||s-v|\left|D_B^{2,2}\mathsf{h}(u,v)\right|dvdu;x,y\right)\\
&\leq \frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty K^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((t-x)^2(s-y)^2;x,y\right)\\
&< 3\left\|D_B^{2,2}h\right\|_\infty \frac{1}{n+1}\frac{1}{m+1}.
\end{aligned}$$

Therefore, for $f\in C_b\left(I^2\right)$, we get

$$\begin{aligned}
&\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y)-f(x,y)\right|\\
&\quad\leq |(f-\mathsf{g}_1-\mathsf{g}_2-\mathsf{h})(x,y)|+\left|\left(\mathsf{g}_1-S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\mathsf{g}_1\right)(x,y)\right|\\
&\quad+\left|\left(\mathsf{g}_2-S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\mathsf{g}_2\right)(x,y)\right|+\left|\left(\mathsf{h}-S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\mathsf{h}\right)(x,y)\right|\\
&\quad+\left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}\left((f-\mathsf{g}_1-\mathsf{g}_2-\mathsf{h});x,y\right)\right|\\
&\leq 2||f-\mathsf{g}_1-\mathsf{g}_2-\mathsf{h}||_\infty+3\left\|D_B^{2,0}\mathsf{g}_1\right\|_\infty\frac{1}{n+1}\\
&\quad+3\left\|D_B^{0,2}\mathsf{g}_2\right\|_\infty\frac{1}{m+1}+3\left\|D_B^{2,2}\mathsf{h}\right\|_\infty\frac{1}{n+1}\frac{1}{m+1}.
\end{aligned}$$

Calculating the infimum across all $\mathsf{g}_1\in C_B^{2,0}$, $\mathsf{g}_2\in C_B^{0,2}$, $\mathsf{h}\in C_B^{2,2}$, we achieve the desired relationship. □

4 NUMERICAL EXAMPLES

In this section, we will prove the convergence of the operators by choosing different values of the introduced shape parameters α_1,α_2 and β_1,β_2, and m,n taking into different examples.

Example 4.1. Consider $f(x,y)=ysin(x)$(blue). In the Figure 1, we show the convergence of the operators (2.1) by choosing various values of α_1,α_2. It represents the efffect of the introduced shape parameters. The value of the parameters are as n=m=20, $\beta_1=\beta_2=0.5$ and $\alpha_1=0.4$, $\alpha_2=0.5$(red), $\alpha_1=0.6$, $\alpha_2=0.7$ (green).

Example 4.2. Consider $f(x,y)=x^7y^3+x^6y^2-5x^2y^4$(blue). In the Figure 2, we show the convergence of the operators (2.1) for the different values of n=m=40 (green) and n=m=30 (red), where other parameters are fixed as $\alpha_1=0.3$, $\alpha_2=0.4$, $\beta_1=0.5$, $\beta_2=0.5$. From the Figure 2, it is clear that as we increase the values of n and m the operators converge faster to the given function f(x,y).

Example 4.3. Consider $f(x,y)=y^3sin(x)$(blue). In this example, we have studied the convergence of operators (2.1) by varying the introduced parameters β_1 and β_2. We consider n=m=20, $\alpha_1=0.8$, $\alpha_2=0.8$, and $\beta_1=0.5$, $\beta_2=0.4$(red), $\beta_1=0.7$, $\beta_2=0.5$(green) in Figure 3.

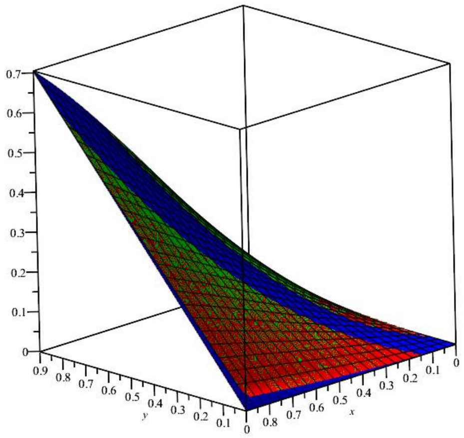

Figure 1. Convergence of the operators $S^{0.5,0.5}_{20,20,0.4,0.5}$(red) and $S^{0.5,0.5}_{20,20,0.6,0.7}$(green).

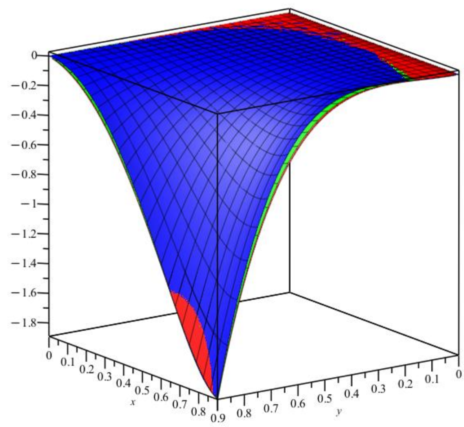

Figure 2. Convergence of the operators $S^{0.5,0.5}_{30,30,0.3,0.4}$(red) and $S^{0.5,0.5}_{40,40,0.3,0.4}$(green).

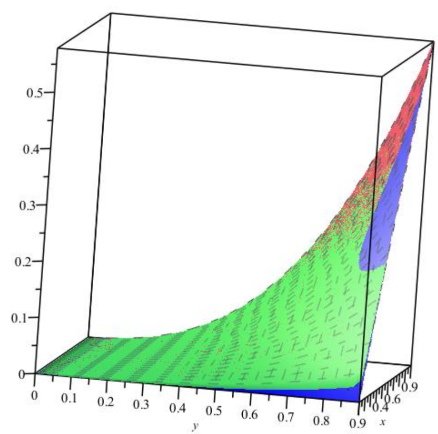

Figure 3. Convergence of the operators $S^{0.5,0.4}_{20,20,0.8,0.8}$(red) and $S^{0.7,0.5}_{20,20,0.8,0.8}$(green).

Example 4.4. Consider $f(x,y) = yx^2$. In this example, we have studied the error of approximation (2.1) by varying the introduced parameters α_1 and α_2. We consider n=m=20, $\beta_1 = 0.5$, $\beta_2 = 0.4$, and $\alpha_1 = 0.8$, $\alpha_2 = 0.8$(red), $\alpha_1 = 0.4$, $\alpha_2 = 0.5$(green) in Figure 4. Similarly, Figure 5 represents the error of approximation $H^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) = \left|S^{\beta_1,\beta_2}_{n,m,\alpha_1,\alpha_2}(f;x,y) - f(x,y)\right|$ of the operators for the certain values of the parameters. The results from this figure are related to Figure 3 that is by increasing the values of n and m the induced error is converging to zero fastly.

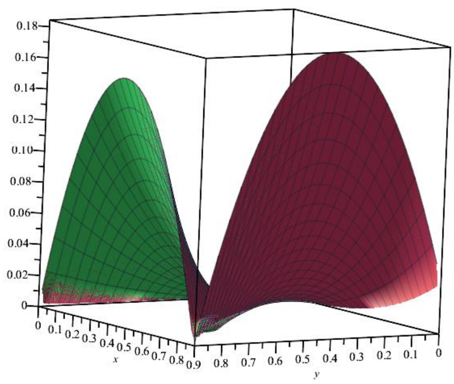

Figure 4. Error of approximation for $S^{0.5,0.4}_{20,20,0.8,0.8}$(red) and $S^{0.5,0.4}_{20,20,0.4,0.5}$(green).

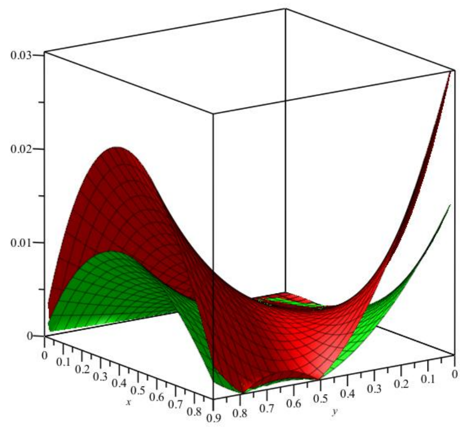

Figure 5. Error of approximation for $S^{0.5,0.4}_{20,20,0.8,0.8}$(red) and $S^{0.5,0.4}_{40,40,0.8,0.8}$(green).

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