Multiplicity of Solutions for $(p_1, ..., p_N)-$**polyharmonic system**

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Abstract This paper investigates the existence of weak solutions of the following Dirichlet $(p_1, ..., p_N)-$polyharmonic Kirchhoff problem.

$$\begin{cases} M_i\left(\int_\Omega \frac{1}{p_i}|D_L u_i|^{p_i} dx\right)\Delta_{p_i}^L u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_N) + \mu G_{u_i}(x, u_1, u_2, ..., u_N), \text{ in } \Omega, \\ D^\alpha u_i = 0, \text{ on } \partial\Omega, \end{cases}$$

for each mult-index α, with $|\alpha| \leq L - 1$, $L \in \mathbb{N}^*$, $1 \leq i \leq N$ and $N \in \mathbb{N}^*$. To the best of our knowledge, this paper is one of the initial contributions to the study of the p-polyharmonic operator. By using a recent three critical points theorem obtained by B. Ricceri, we give a result of the existence of at least three weak solutions.

1 Introduction

Recently, there has been substantial interest in studying partial differential equations associated with variational problems involving constant or variable exponents, as noted in [9], [27].

Interest in the study of these problems has been stimulated by their applications in physics, non-Newtonian fluid mechanics, such as electrorheological fluids, and elasticity theory, which supports image construction and restoration.

Our purpose is to study the following elliptic problem involving the $(p_1, ..., p_N)-$polyharmonic operator,

$$M_i(t_i)\Delta_{p_i}^L u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_N) + \mu G_{u_i}(x, u_1, u_2, ..., u_N), \text{ in } \Omega,$$
$$D^\alpha u_i = 0, \text{ on } \partial\Omega. \tag{1.1}$$

Where, $t_i = \int_\Omega \frac{1}{p_i}|D_L u_i|^{p_i} dx$. $\Omega \subseteq \mathbb{R}^N$, $(N \geq 1)$ is a bounded domain with a Lipschitz boundary, with $|\alpha| \leq L - 1$, $L \in \mathbb{N}^*$. The p_i-polyharmonic operator is defined by:

$$\Delta_{p_i}^L u = \begin{cases} -div\{ \Delta^{j-1}(|D_L u|^{p_i-2}D_L u)\} & L = 2j - 1, \\ \Delta^j(|D_L u|^{p_i-2}D_L u) & L = 2j, \end{cases}$$

$p_i > \frac{N}{L}$, $i = 1, ..., N$ which becomes the usual p_i-Laplacian for $L = 1$. With

$$D_L u = \begin{cases} \nabla\Delta^{j-1}u & \text{if} \quad L = 2j - 1, \\ \Delta^j u & \text{if} \quad L = 2j. \end{cases}$$

Note that D_L is an N-vectorial operator when L is odd and $N > 1$, while it is a scalar operator when L is even.

This type of system represents a natural extension of classical Kirchhoff problems, incorporating non-local terms that model physical phenomena in which diffusion or tension depends

on the global behavior of the solution (see, for example, Kirchhoff's original model for elastic strings in [16], as well as its modern generalizations in [7], [22]). Such nonlocal effects are relevant in various contexts, including elasticity, population dynamics, and electrorheological fluids [25].

Moreover, the presence of the polyharmonic operator $\Delta_{p_i}^L$ allows the modeling of higher-order anisotropic behaviors, which are crucial in the study of thin plates, composite materials, and other complex media (see [12], [23]). The coupling introduced by the nonlinear terms F and G also reflects realistic interactions between multiple fields or components, which makes the problem analytically rich and poses significant challenges from the point of view of variational analysis and critical point theory [26], [28].

Throughout this paper, we suppose the following assumptions: $Lp_i > N \ i = 1, ..., N$, the functionals $F, G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to $x \in \Omega$ for every $(t_1, t_2, ..., t_N) \in \mathbb{R}^N$ and are of class C^1 with respect to $(t_1, t_2, ..., t_N) \in \mathbb{R}^N$ for a.e. $x \in \Omega$, F_{u_i} and G_{u_i} denotes the partial derivatives of F and G with respect to u_i. Moreover, F and G satisfy the following:

- (F_1) for every $M > 0$ and every $1 \leq i \leq N$,

$$\sup_{|(t_1, t_2, ... t_N)| \leq M} \left| F_{t_i}(x, t_1, ..., t_N) \right| \in L^1(\Omega).$$

- (F_2) $F(x, 0, 0, .., 0) = 0$ for a.e. $x \in \Omega$.

- (G) for every $M > 0$ and every $1 \leq i \leq N$,

$$\sup_{|(t_1, t_2, ... t_N)| \leq M} \left| G_{t_i}(x, t_1, ..., t_N) \right| \in L^1(\Omega).$$

- (M_0) $M(t) : \mathbb{R} \to (m_0, +\infty)$ is a continuous and increasing function, with $m_0 > 0$.

- (M_1) The exists $0 < \theta < 1$, such that $\tilde{M}(t) \geq (1 - \theta)M(t)t$, for $t \geq 0$.

Where, $\tilde{M}(t) = \int_0^t M(s)ds$.

Kratochvil and Necas introduced the p-biharmonic operator in [10, 15, 17] to study physical equations. It will be the $p-$biharmonic operator for $L = 2$ and the polyharmonic operator for $p = 2$.

In [5] N. Tsouli et al were studying the eigenvalue nonlinear problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega,$$

$$|\nabla u|^{p-2}\frac{\delta u}{\delta \nu} + \beta |u|^{p-2}u = \lambda |u|^{p-2}u \quad x \in \partial\Omega,$$

where ν denotes the unit exterior normal, $1 < p < \infty$ and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ indicates the p-Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary where $N \geq 2$ and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x \in \partial\Omega} \beta(x) > 0$. By applying the Ljusternik-Schnirelman theory, they prove the existence of a nondecreasing sequence of positive eigenvalues, and the first eigenvalue is simple and isolated. Moreover, they prove that the second eigenvalue coincides with the second variational eigenvalue obtained via the Ljusternik-Schnirelman principal.
In [8] F. Colasuonno and P. Pucci establish the existence of an unbounded sequence of solutions for a class of quasilinear elliptic p-polyharmonic Kirchhoff equations, including the new degenerate case. They use the symmetrical mountain passage theorem of Ambrosetti and Rabinowitz. They prove the existence of infinitely many solutions to Dirichlet problems involving the p-polyharmonic operators.

$$-M(\varphi(u))\Delta_p^L u = f(x, u) \quad \text{in } \Omega,$$

$$D^\alpha u(x) = 0 \quad x \in \partial\Omega, \quad \text{for each mult-index } \alpha, \text{ with } |\alpha| \leq L - 1.$$

where Ω is a bounded domain with a Lipschitz boundary.
Afrouzi et al in [4], established the existence of at least three weak solutions for the system:

$$\begin{cases} -\Delta_{p_i}u + a_i(x)|u_i|^{p_i-2}u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_N) + \mu G_{u_i}(x, u_1, u_2, ..., u_N), \text{ in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0, \text{ on } \partial\Omega. \end{cases}$$

For $1 \leq i \leq N$, where $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ is a nonempty bounded open domain with a smooth boundary $\partial\Omega$. $\Delta_{p_i} u_i := \text{div}\big(|\nabla u_i|^{p_i-2}\nabla u_i\big)$ is the p_i-Laplacian operator, $p_i > N$, $a_i \in L^\infty(\Omega)$ with ess $\inf_\Omega a_i > 0$, for $1 \leq i \leq N$, λ and μ are two positive parameters.

In [6], A. Bousgheiri and A. Ourraoui establish the existence of at least three weak solutions for a $p(x)-$Biharmonic and $p(x)-$Laplacian Kirchhoff problem:

$$M_1\big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\big)\Delta^2_{p(x)}u - M_2\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)\Delta_{p(x)}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega,$$

$$\Delta u = u = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$. with smooth boundary $\partial\Omega$.

Many authors consider the existence of nontrivial solutions for some fourth-order problems such as [11, 20, 21], which represent a generalization of the classical $p-$biharmonic operator obtained in the case when p is a positive constant.

Jaafri in [14] studies bi-nonlocal elliptic problem involving $p(x)-$Biharmonic operator. By applying variational methods and under adequate conditions, he proves the existence of nontrivial weak solutions to the following problem:

$$M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta^2_{p(x)}u = \Big[\int_\Omega F(x,u)dx\Big]^r \quad \text{on } \Omega,$$

$$\Delta u = u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Where, $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions, $F(x,u) = \int_0^u f(x,s)ds$, $\lambda \in \mathbb{R}$,$r > 0$.

Ourraoui in [21] establish the existence of a solution for a class of Robin problem involving fourth order operator:

$$\Delta^2_{p(x)}u = f(x,u) \quad \text{on } \Omega,$$

$$|\Delta u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Mazan in [19] investigates the existence of a weak solution to a class of nonlocal problems involving the variable exponent and concave-convex nonlinearities to the following problem:

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)(-\Delta_{p(x)}u) = f_\lambda(x,u) \quad \text{on } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Where, $f_\lambda = \lambda(m_q(x)|u|^{q(x)-2}u + m_h(x)|u|^{h(x)-2}u)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is bounded smooth domain, p is Lipschits continuous on $\overline{\Omega}$, $m_q, m_h \in C_+(\overline{\Omega})$, $m_q, m_h > 0$.

This work is mainly motivated by the study of the multiplicity of solutions for the p-polyharmonic elliptic in [8], the survey of eigenvalue problems for the p-Laplacian in [18] by An Lê, and the study of multiple solutions of the $(p_1, ..., p_N)-$system in [4]. However, it considers a nonlinear elliptic system involving $(p_1, ...p_N)-$polyharmonic operator, aiming at investigating the existence of at least three weak solutions. The approach is based on a recent three critical points theorem obtained by B. Ricceri [24].

Before presenting our results, we define weak solutions for Problem (1.1). For this purpose, we denote by X the space $W_0^{L,p_1}(\Omega) \times W_0^{L,p_2}(\Omega) \times ...... \times W_0^{L,p_N}(\Omega)$ and define the norm $\|.\|_{X_i}$ of $X_i = W_0^{L,p_i}(\Omega)$ by the formula

$$\|u_i\|_{X_i} = \big(\sum_{|\alpha| \leq L} \|D^\alpha u_i\|_{p_i}^{p_i}\big)^{\frac{1}{p_i}},$$

where $\|.\|_{p_i}$ denotes the standard $L^{p_i}(\Omega)$-norm, $1 \leq i \leq N$. It is well known that if $1 < p_i < \infty$ then $(X_i, \|.\|_{X_i})$ is a separable and reflexive Banach space.

The space X equipped with norm:

$$\|u\| = \sum_{i=1}^{N} \|u_i\|_{X_i}, \quad u = (u_1, .., u_N),$$

is a separable reflexive Banach space.

The paper is organized as follows. In Section 2, we present the basic preliminary results and mention some useful lemmas concerning the Lebesgue and Sobolev spaces. In Section 3, we define what is a weak solution of the problem (1.1), and give some propositions which are important for the proofs of our main results. Finally, we prove the main Theorem 3.2.

2   Preliminary notes

In order to deal with the problem, we need some theory of Sobolev Spaces. For convenience, we briefly recall some basic facts which will be used later, suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.
Define the Lebesgue space $L^{p_i}(\Omega)$,

$$L^{p_i}(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R} \text{ mesurable } : \int_{\Omega} |u|^{p_i} dx < \infty \} .$$

Then $L^p(\Omega)$ endowed with the norm :

$$\|u\|_{p_i} = (\int_{\Omega} |u|^{p_i} dx)^{\frac{1}{p_i}}, \tag{2.1}$$

becomes a Banach separable and reflexive space.

For any positive integer L. Define the Sobolev space $W^{L,p_i}(\Omega)$:

$$W^{L,p_i}(\Omega) = \{ u \in L^{p_i}(\Omega) : \ D^\alpha u \in L^{p_i}(\Omega), |\alpha| \leq L \} ,$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 .... \partial^{\alpha_N} x_N}$, with $\alpha = (\alpha_1, ...; \alpha_N)$ is a multi-index, and $|\alpha| = \sum_{i=1}^{N} \alpha_i$.

Then $W^{L,p_i}(\Omega)$ endowed with the norm :

$$\|u\|_{W^{L,p_i}(\Omega)} = \big(\sum_{|\alpha| \leq L} \|D^\alpha u\|_{p_i} \big)^{\frac{1}{p_i}},$$

becomes a Banach separable and reflexive space.
The space $W_0^{L,p_i}(\Omega)$ denotes the completion of C_0^∞ with respect to the standard norm of $W^{L,p_i}(\Omega)$, that is $\|u\|_{W^{L,p_i}(\Omega)} = \sum_{|\alpha| \leq L} \|D^\alpha u\|_{p_i}$, where $\|.\|_{p_i}$ denotes the standard L^{p_i}-norm. Let $\mathbb{D}^{L,p_i}(\Omega)$ be the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\|_{\mathbb{D}^{L,p_i}(\Omega)} = \big(\sum_{|\alpha| = L} \|D^\alpha u\|_{p_i} \big)^{\frac{1}{p_i}}.$$

Then, by Poincaré inequality, see [2], there exists a positive constant $k = k(N, p_i, \Omega)$ such that

$$\|u\|_{W^{L,p_i}(\Omega)} \leq k\|u\|_{\mathbb{D}^{L,p_i}(\Omega)}, \text{ for all } u \in W_0^{L,p_i}(\Omega).$$

Hence, the two norms $\|.\|_{W^{L,p_i}(\Omega)}$ and $\|.\|_{\mathbb{D}^{L,p_i}(\Omega)}$ are equivalent, so that two completions of $C_0^\infty(\Omega)$, with respect to these norms, coincide.
For $N > 1$, we endow the vectorial space $[L^{p_i}(\Omega)]^N$ with the norm

$$\|v\|_{p_i} = \big(\sum_{i=1}^{N} \|v_i\|_{p_i}^{p_i} \big)^{\frac{1}{p_i}},$$

where $v = (v_1, v_2, ..., v_N)$. With abuse of notation, we shall use the same symbol $\|.\|_{p_i}$ to denote both the standard L^p-norm in the scalar space $L^{p_i}(\Omega)$ and the norm defined in (2.1) in the vectorial space $[L^{p_i}(\Omega)]^N$.

It is well known that for $L = 2$, $1 < p_i < \infty$ and Ω bounded, by a consequence of the Caldéron-Zygmund inequality, see [13], there exists a constant $k_2 = k_2(N, p_i) > 0$ such that

$$\|u\|_{\mathbb{D}^{2,p_i}(\Omega)} \leq k_2 \|\Delta u\|_{p_i} = k_2 \|D_2 u\|_{p_i}, \text{ for all } u \in W_0^{2,p_i}(\Omega).$$

It can be shown that for all $L = 1, 2, ...$ and $1 < p_i < \infty$ there exists $k_L = k_L(N, p_i) > 0$ such that

$$\|u\|_{\mathbb{D}^{L,p_i}(\Omega)} \leq k_L \|D_L u\|_{p_i}, \text{ for all } u \in W_0^{L,p_i}(\Omega).$$

Let introduce the Sobolev critical exponent $p_{i_L}^*$ defined by:

$$p_{i_L}^* = \begin{cases} \frac{Np_i}{N - Lp_i} & \text{if } N > Lp_i, \\ \infty & \text{if } N \leq Lp_i. \end{cases}$$

By the Rellich-Kondrachov theorem(see [3], Theorem 6.3) we have the following proposition.

Proposition 2.1. *The embedding $W_0^{L,p_i}(\Omega) \hookrightarrow L^h(\Omega)$, $i = 1, 2, ...N$ is compact for $1 \leq h < p_{i_L}^*$, in particular it is continuous and so there exist $k_h = k_h(N, p, L, \Omega) > 0$ such that*

$$\|u\|_h \leq k_h \|u\|_{W_0^{L,p_i}}, \text{ for all } u \in W_0^{L,p_i}(\Omega).$$

From now on we endow $W_0^{L,p_i}(\Omega)$ with the norm $\|.\|_{W_0^{L,p}(\Omega)} = \|D_L.\|_{p_i}$ which is equivalent to the standard Sobolev norm and the space X with the norm $\|u\| = \sum_{i=1}^{N} \|D_L u_i\|_{p_i}$, for all $u = (u_1, ..., u_N) \in X$.

Proposition 2.2. *[29] Let $f : M \subseteq X \to \mathbb{R}$ be a function on the convex closed set M of the $B-$space X. Then f is weakly sequentially lower semicontinuous if one of the following three conditions is satisfied:*

(i) *f is continuous and convex.*

(ii) *f is is lower semicontinuous and convex.*

(iii) *f is $G-$differentiable on M and f' is monotone on M.*

Theorem 2.3. *[24] Let X be a real reflexive Banach space, $\phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, bounded on each bounded subsets of X, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\tilde{u} \in X$ with , $\phi(\tilde{u}) > r$, such that

(I_1) $\dfrac{\sup_{\phi(u) \leq r} \Psi(u)}{r} < \dfrac{\Psi(\tilde{u})}{\phi(\tilde{u})}.$

(I_2) *The functional $I = \phi - \lambda\Psi$ coercive, for each $\lambda \in \Lambda_r :=]\dfrac{\phi(\tilde{u})}{\Psi(\tilde{u})}, \dfrac{r}{\sup_{\phi(u) \leq r} \Psi(u)}[.$*

Then, for each compact interval $[a, b] \subset \Lambda_r$, there exist $\rho > 0$, for every $\lambda \in [a, b]$ and for each C^1 functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\phi'(u) - \lambda\Psi'(u) - \mu J'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ.

In the upcoming section, we shall present the definition of a weak solution to the problem (1.1), along with the pertinent lemmas essential to deduce our principal results.

3 Main results.

In this section, we begin by defining the weak solution to the problem (1.1).

Definition 3.1. Let $u = (u_1, u_2, ...u_N) \in X$, u is called a weak solution of the Problem (1.1) if

$$\sum_{i=1}^{N} M_i(t_i) \int_{\Omega} |D_L u_i|^{p_i-2} D_L u_i D_L v_i dx - \lambda \int_{\Omega} \sum_{i=1}^{N} F_{u_i}(x, u)v_i dx - \mu \int_{\Omega} \sum_{i=1}^{N} G_{u_i}(x, u)v_i dx = 0, \quad (3.1)$$

for all $v = (v_1, v_2, ..., v_N) \in X$, where $(x, u) = (x, u_1, u_2, ..., u_N)$.

We now present our main result as follows:

Theorem 3.2. *Assume that the conditions* (F_1), (F_2), (G) *are satisfied, and there exist a positive constant* r *and a function* $w = (w_1, w_2, ...w_N) \in X$ *such that*

(i) $\sum_{i=1}^{N} \tilde{M}_i\left(\frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i}\right) > r$.

(ii) $r \frac{\int_{\Omega} F(x, w_1, w_2, ..., w_N) dx}{\sum_{i=1}^{N} \tilde{M}_i\left(\frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i}\right)} - \int_{\Omega} \sup_{(t_1, t_2, ..., t_N) \in \Gamma(cr)} F(x, t_1, t_2, ..., t_N) dx > 0$,

(iii) $\lim \sup_{(|t_1|, ..., |t_N|) \longrightarrow (+\infty, ..., +\infty)} \frac{F(x, t_1, ..., t_N)}{\sum_{i=1}^{N} \frac{|t_i|^{p_i}}{p_i}} \leq 0$.

Let $r_1 = \frac{\sum_{i=1}^{N} (\Pi_{j=1, j\neq i}^{N} p_j) \|D_L w_i\|_{p_i}^{p_i}}{\left(\Pi_{i=1}^{N} p_i\right) \int_{\Omega} F(x, w_1, w_2, ..., w_N) dx}$ *and* $r_2 = \frac{r}{\int_{\Omega} \sup_{(t_1, t_2, ..t_N) \in \Gamma(cr)} F(x, t_1, ...t_N) dx}$ *be such that*

$\Gamma(\gamma) = \{ (t_1, ..., t_N) \in \mathbb{R}^N : \sum_{i=1}^{N} \frac{|t_i|^{p_i}}{p_i} \leq \gamma \}$. *Then, for each compact interval* $[a, b] \subset \Lambda =]r_1, r_2[$, *there exists* $\rho > 0$, *for every* $\lambda \in [a, b]$, *there exists* $\delta > 0$ *such that for each* $\mu \in [0, \sigma]$, *system* (1.1) *admits at least three weak solutions in* X *whose norms are less than* ρ.

Define on X the functionals

$$\phi(u) = \sum_{i=1}^{N} \tilde{M}_i\left(\int_{\Omega} \frac{1}{p_i} |D_L u_i|^{p_i} dx\right), \quad (3.2)$$

$$\Psi(u) = \int_{\Omega} F(x, u_1, ..., u_N) dx, \quad (3.3)$$

and

$$J(u) = \int_{\Omega} G(x, u_1, ..., u_N) dx. \quad (3.4)$$

To derive the Theorem 3.2 we need the following results.

Proposition 3.3. *Consider* ϕ *and* Ψ *the two functionals defined as in* (3.2) *and* (3.3), *then*

(i) ϕ *is bounded on each bounded subset of* X *and* ϕ, Ψ *are continuously Gâteaux differentiable functionals.*

(ii) ϕ *is sequentially weakly lower semi-continuous.*

(iii) ϕ' *admits a continuous inverse* ϕ'^{-1} *on* X^*.

Proof.   (i) It is well known that ϕ and $\Psi \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\langle \phi'(u), v \rangle = \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} |D_L u_i|^{p_i-2} D_L u_i D_L v_i dx, \quad (3.5)$$

$$\langle \Psi'(u), v \rangle = \sum_{i=1}^{N} \int_{\Omega} F_{u_i}(x, u_1, u_2, ..., u_N) v_i dx, \quad (3.6)$$

for all $v = (v_1, ...v_N) \in X$.

(ii) First, let us verify that ϕ' is monotone. Suppose, $(u, v) \in X^2$, then

$$
\begin{aligned}
\langle \phi'(u) - \phi'(v), w \rangle &= \langle \phi'(u), w \rangle - \langle \phi'(v), w \rangle \\
&= \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} \Big[|D_L u_i|^{p_i-2} D_L u_i D_L(w_i) - |D_L v_i|^{p_i-2} D_L v_i D_L(w_i) \Big] dx \\
&= \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} \Big(|D_L u_i|^{p_i} + |D_L v_i|^{p_i} - |D_L u_i|^{p_i-2} D_L u_i D_L v_i \Big) dx \\
&\quad - \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} |D_L v_i|^{p_i-2} D_L v_i D_L u_i dx \\
&= \sum_{i=1}^{N} M_i(t_i) \Big(\|D_L u_i\|_{p_i}^{p_i} + \|D_L v_i\|_{p_i}^{p_i} \Big) \\
&\quad - \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} \Big(|D_L u_i|^{p_i-2} D_L u_i D_L v_i + |D_L v_i|^{p_i-2} D_L v_i D_L u_i \Big) dx.
\end{aligned}
$$

Where, $w = u - v$ and $w_i = u_i - v_i$. Using Hölder's inequality, we obtain

$$
\begin{aligned}
\int_{\Omega} |D_L u_i|^{p_i-2} D_L u_i D_L v_i dx &\leq \Big(\int_{\Omega} |D_L u_i|^{p_i} dx \Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\Omega} |D_L v_i|^{p_i} dx \Big)^{\frac{1}{p_i}} \\
&\leq \|D_L u_i\|_{p_i}^{p_i-1} \|D_L v_i\|_{p_i}.
\end{aligned}
\tag{3.7}
$$

Similarly, we have

$$
\begin{aligned}
\int_{\Omega} |D_L v_i|^{p_i-2} D_L v_i D_L u_i dx &\leq \Big(\int_{\Omega} |D_L v_i|^{p_i} dx \Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\Omega} |D_L u_i|^{p_i} dx \Big)^{\frac{1}{p_i}} \\
&\leq \|D_L v_i\|_{p_i}^{p_i-1} \|D_L u_i\|_{p_i}.
\end{aligned}
\tag{3.8}
$$

According to assumption (M_0) and since $1 < p_i < \infty$ for all $i = 1, \dots, N$, we apply Young's inequality (see [29]) to obtain:

$$
\begin{aligned}
\langle \phi'(u) - \phi'(v), u - v \rangle &\geq m_0 \sum_{i=1}^{N} \Big(\|D_L u_i\|_{p_i}^{p_i} + \|D_L v_i\|_{p_i}^{p_i} \Big) \\
&\quad - m_0 \sum_{i=1}^{N} \Big(\|D_L u_i\|_{p_i}^{p_i-1} \|D_L v_i\|_{p_i} + \|D_L v_i\|_{p_i}^{p_i-1} \|D_L u_i\|_{p_i} \Big) \\
&\geq m_0 \sum_{i=1}^{N} (\|D_L u_i\|_{p_i}^{p_i-1} - \|D_L v_i\|_{p_i}^{p_i-1})(\|D_L u_i\|_{p_i} - \|D_L v_i\|_{p_i}) \\
&\geq 0.
\end{aligned}
$$

Drawing on item (i) of Proposition 3.3, in conjunction with Proposition 2.2, it follows that the functional ϕ possesses the property of being sequentially weakly lower semi-continuous.

(iii) At this point, it is necessary to establish the coercivity of ϕ'. Let $u = (u_1, .. u_N) \in X$ such

that $\|u\| \geq 1$. Then

$$
\begin{aligned}
\langle \phi'(u), u \rangle &= \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} |D_L u_i|^{p_i-2} D_L u_i D_L u_i dx \\
&= \sum_{i=1}^{N} M_i(t_i) \int_{\Omega} |D_L u_i|^{p_i} dx \\
&\geq \frac{m_0(1-\theta)}{p_M} \sum_{i=1}^{N} \|D_L u_i\|_{p_i}^{p_i},
\end{aligned}
$$
(3.9)

where, $p_M = \max\{ p_i, i = 1, ..., N \}$. Hence, $\lim_{\|u\| \to +\infty} \langle \phi'(u), u \rangle = +\infty$. Therefore, since ϕ' is coercive by using Theorem 26.A[29] ϕ' admits a continuous inverse ϕ'^{-1} in X^*.

$\square$

Proof of Theorem 3.2

According to Proposition 3.3, ϕ' admits a continuous inverse on X^*. Moreover,

$$\phi(0) = \Psi(0) = 0.$$

Alternatively, by choosing $\tilde{u} = w$, we obtain $\phi(\tilde{u}) > r$.

Let $u_i \in W^{L,p_i}(\Omega)$. Since $Lp_i > N$ and according to the compact embedding $W^{L,p_i}(\Omega)$ in to $C(\overline{\Omega})$(see [1]), there exists a constant $c > 0$ such that

$$\sup_{x \in \Omega} |u_i|^{p_i} \leq c \|D_L u_i\|_{p_i}^{p_i},$$
(3.10)

for $1 \leq i \leq N$. Hence,

$$\sup_{x \in \Omega} \sum_{i=1}^{N} \frac{|u_i|^{p_i}}{p_i} \leq c \sum_{i=1}^{N} \frac{\|D_L u_i\|_{p_i}^{p_i}}{p_i},$$
(3.11)

for each $u = (u_1, .., u_N) \in X$. Let $r > 0$ we have

$$
\begin{aligned}
\phi^{-1}\Big((-\infty, r]\Big) &= \{ u = (u_1, u_2, ..., u_n) \in X : \phi(u) \leq r\} \\
&= \{ u = (u_1, u_2, ..., u_n) \in X : \sum_{i=1}^{N} \tilde{M}_i(\frac{\|D_L u_i\|_{p_i}^{p_i}}{p_i}) \leq r\} \\
&\subseteq \{ u = (u_1, u_2, ..., u_n) \in X : \sum_{i=1}^{N} \tilde{M}_i(\frac{|D_L u_i|^{p_i}}{p_i}) \leq cr\},
\end{aligned}
$$

where, $\delta = \frac{cr}{m_0(1-\theta)}$. Hence,

$$
\begin{aligned}
\sup_{u \in \phi^{-1}((-\infty, r])} \Psi(u) &= \sup_{u \in \phi^{-1}((-\infty, r])} \int_{\Omega} F(x, u_1, ..., u_N) dx \\
&\leq \int_{\Omega} \sup_{\{ u \in X : \sup_{x \in \Omega} \sum_{i=1}^{N} \tilde{M}_i(\frac{\|D_L u_i\|_{p_i}^{p_i}}{p_i}) \leq cr\}} F(x, u_1, ..., u_N) dx \\
&\leq r \frac{\int_{\Omega} F(x, w_1, w_2, ..., w_N) dx}{\sum_{i=1}^{N} \tilde{M}_i(\frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i})}, \\
&\leq r \frac{\int_{\Omega} F(x, w_1, w_2, ..., w_N) dx}{\sum_{i=1}^{N} \tilde{M}_i(\frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i})} \\
&\leq r \frac{\Psi(w)}{\phi(w)}.
\end{aligned}
$$

Thanks to assumption $iii)$ in Theorem 3.2, the function $I = \phi - \lambda\Psi$ is coercive, for every λ. Therefore, I is coercive for each $\lambda \in \Lambda_r :=]\frac{\phi(w)}{\Psi(w)}, \frac{r}{\sup_{\phi(u) \leq r} \Psi(u)}[$. Moreover, since $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for every $(t_1, t_2, ..., t_N) \in \mathbb{R}^N$ and is (C^1) with respect to $(t_1, t_2, ...t_N) \in \mathbb{R}^N$ for a.e. $x \in \Omega$ satisfying condition (G), the functional

$$J(u) = \int_\Omega G(x, t_1, t_2, ...t_N)dx,$$

is well defined and continuously Gâteaux differentiable on X with a compact derivative, and

$$\langle J'(u), v \rangle = \sum_{i=1}^{N} \int_\Omega G_{u_i}(x, u_1, u_2, ..., u_N)v_i dx,$$

for all $u = (u_1, u_2, ..., u_N) \in X$ and $v = (v_1, v_2, ..., v_N) \in X$. Thus, all the hypotheses of Theorem 2.3 are satisfied. Therefore, the problem (1.1) admits at least three weak solutions.

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that there exist two positive constants θ and τ, with $\theta < \tau$ such that*

(i) $\max_{t \in [-\theta, \theta]} F(t) \leq 0$.

(ii) $\frac{\theta^p}{c} \frac{F(\tau)}{\tau^p} > 0$.

(iii) $\lim_{|t| \longrightarrow +\infty} \frac{F(t)}{|t|^p} \leq 0$.

Then setting, $\Lambda :=]\frac{\tau^p}{pm(\Omega)F(\tau)}, \frac{\theta^p}{\max_{t \in [-\theta, \theta]} F(t)}[$, for each compact interval $K = [a, b] \subseteq \Lambda$ there exist $\rho > 0$, for every $\lambda \in K$, there exist $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the problem

$$\begin{cases} \Delta_p^L u = \lambda F_u(u) + \mu G_u(u), \text{ in } \Omega, \\ D^\alpha u = 0, \text{ on } \partial\Omega. \end{cases}$$

admits at least three weak solutions in $W^{L,p}(\Omega)$ whose norms are less than ρ.

Proof. Let $w(x) = \tau x^L$, for every $x \in \Omega \subset \mathbb{R}$. Then we have:

If $L = 2j - 1$,

$$D_L w = \nabla\Delta^{j-1}w = \begin{cases} \tau L & \text{if } j = 1 \\ \tau L x^{L-1} & \text{if } j > 1. \end{cases}$$

Hence,

$$\sum_{i=1}^{N} \frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i} = \frac{1}{p}\|w\|_p^p = \frac{1}{p}\begin{cases} \tau L m(\Omega) & \text{if } j = 1, \\ \tau L |x|^p & \text{if } j > 1. \end{cases}$$

If $L = 2j$, then we have

$$D_L w = \Delta^j w = L(L-1).....(L-j)x^{L-j}.$$

Therefore, $\sum_{i=1}^{N} \frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i} = \frac{1}{p}\|w\|_p^p = \frac{L(L-1).....(L-j)x^{L-j}\tau}{p}|x|^{(L-j)p}$.

Choosing, $r = \min\{\frac{1}{p}\tau L\theta^p, \frac{L(L-1).....(L-j)\tau}{p}\theta^{(L-j)p}\}$. Then, we get $i)$ in Theorem 3.2.

Also, we have

$$r\frac{\int_\Omega F(x, w_1, w_2, ..., w_N)dx}{\sum_{i=1}^{N} \frac{\|D_L w_i\|_{p_i}^{p_i}}{p_i}} = rp\frac{\int_\Omega F(w(x))dx}{\|D_L w(x)\|_p^p}$$

$$= \frac{p\theta^p}{\|D_L w(x)\|_p^p}\int_\Omega F(L\tau x^L)dx. \quad (3.12)$$

According to (ii) in Corollary 3.4 and (3.12) implies $ii)$ in Theorem 3.2. Thus, all the assumptions of Theorem 3.2 are satisfied and the proof is complete. $\qquad\square$

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