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ON ROUGH STATISTICAL CONVERGENCE IN FUZZY NORMED LINEAR SPACES

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Abstract. This paper introduces the concept of rough statistical convergence in fuzzy normed linear spaces(FNLS) and examines its fundamental properties. It is shown that the rough statistical limit set in such spaces is closed and convex. In addition, the notions of rough statistical cluster points and rough statistical limit points are defined and their relationships are established.

1 Introduction

The concept of fuzzy sets was first introduced by Zadeh [40] in 1965 which was an extension of the classical set-theoretical concept. Later, in 1994 Cheng and Mordeson [6], extended this idea to define fuzzy normed linear spaces, Fuzzy set deals with sets in which the membership value of an element lies between 0 and 1. The degree of membership reflects the uncertainty or fuzziness associated with whether an element belongs to the set. Over the years, fuzzy set theory has found wide applications in diverse fields such as engineering, business, medical and health sciences and the natural sciences.

In 1992, Felbin[14] introduced the concept of finite-dimensional fuzzy normed linear spaces and studied some of its basic properties. Subsequently, a noteworthy advancement was made by Golet [19], who expanded upon Felbin's work by using the concept of t-norms. This extension enriched the theoretical underpinning of fuzzy normed linear spaces.

On the other hand, in 1951 Fast [13] introduced the concept of statistical convergence and later in 1985 it was investigated from the sequence space point of view and linked with summability theory by Fridy [18]. Further extension of statistical convergence, many researchers work on it and do so many works, such as lacunary statistical convergence by Fridy [17], $\lambda-$statistical convergence by Mursaleen [26], deferred statistical convergence by Küçükaslan [21], and many more ([7],[8]). Further the concepts of statistical limit superior and limit inferior was introduced by Fridy and Orhan [15] in 1997. In 2004 the concepts of statistical limit points of the sequence of fuzzy numbers were introduced by Aytar [3]. For more on statistical convergence one may refer to [6, 16, 20, 22, 23, 27, 28, 29, 30, 25, 31, 32, 36, 37, 39]. Furthermore, in 2001 the idea of rough convergence was first introduced by Phu [33] in finite-dimensional normed spaces. This idea of rough convergence has motivated many authors and in 2008, the idea of rough statistical convergence was introduced by Aytar [4]. For more on rough convergence, one may refer to [9, 10, 11, 12, 38]. However, rough statistical convergence has not yet been studied in fuzzy normed linear spaces. Since fuzzy normed spaces are natural extensions of normed spaces, it is useful to investigate how rough statistical convergence behaves in this case. This forms the main motivation of the present work. In this paper, our aim is to introduce rough statistical convergence on fuzzy normed linear spaces and investigate some of its properties.

2 Definitions and Preliminaries

Definition 2.1. [18] Let $K \subseteq \mathbb{N}$. The *natural density* of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in K\}\right|,$$

provided the limit exists.

Definition 2.2. [18] A sequence of real number (x_k) is statistically convergent to l provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \ge \varepsilon\}$ has natural density zero, i.e.,

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\Big|\{k \le n : |x_k - l| \ge \varepsilon\}\Big| = 0, \;\; n \in \mathbb{N}.$$

Definition 2.3. [24] A binary operation $\star : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-norm if the following conditions are satisfied:

i. $\star$ is associative and commutative,

ii. $\star$ is continuous,

iii. $s \star 1 = s$, for all $s \in [0,1]$,

iv. $s \star t \le u \star v$ whenever $s \le u$ and $t \le v$, for all $s,t,u,v \in [0,1]$.

Example 2.4. (i) minimum t-norm is $s \star t = \min\{s,t\}$

(ii) product t-norm $s \star t = s \cdot t$

(iii) Łukasiewicz t-norm $s \star t = \max\{0, s+t-1\}$.

Definition 2.5. [34] Let X be a linear space over the real field F and $\star$ is a continuous t-norm. A fuzzy subset N on $X \times \mathbb{R}$ is called a fuzzy norm on X if and only if $x, y \in X$ and $c \in F$,

(i) $N(x,t) = 0, \forall\, t \le 0$;

(ii) $N(x,t) = 1, \forall\, t > 0$ iff $x = 0$;

(iii) $N(cx,t) = N\left(x, \dfrac{t}{|c|}\right) \forall\, t > 0$ if $c \ne 0$;

(iv) $N(x+y, t+s) \ge N(x,t) \star N(y,s)\; \forall\, x,y \in X$;

(v) $\lim_{t\to\infty} N(x,t) = 1$.

The triplet $\mathcal{N} = (X, N, \star)$ will be referred to as a fuzzy normed linear space.

Example 2.6. [19] Suppose $(X, ||\cdot||)$ be a norm space. For $u, v \in [0,1]$, define the t-norm $\star$ and the $u \star v = u.v$. For $\lambda > ||u||$, let

$$N(u,\lambda) = \frac{\lambda}{\lambda + ||u||}, \;\; \forall u(\ne 0) \in X$$

and for $\lambda \le 0$, let $N(u,\lambda) = 0$. Then $\mathcal{N} = (X, N, \star)$ is a FNS.

Let us recall the following definition based on fuzzy normed linear spaces which will be subsequently used in our main results.

Definition 2.7. [2] Let (x_k) be a sequence in $\mathcal{N} = (X, N, \star)$. A sequence (x_k) in X is called convergent to $x_0 \in X$ with respect to the fuzzy norm $\mathcal{N}$ if there exist $k_0 \in \mathbb{N}$, for every $\varepsilon > 0$ and $a \in (0,1)$,

$$N(x_k - x_0, \varepsilon) \ge a, \text{for all } k \ge k_0.$$

Definition 2.8. [2] Let (x_k) be a sequence in $\mathcal{N} = (X, N, \star)$. Then an element $x \in X$ is said to be a statistical cluster point of the sequence (x_k) with respect to the fuzzy norm $\mathcal{N}$ provided that for every $t_0 > 0$ and $a \in (0,1)$,

$$\delta(\{k \in \mathbb{N} : N(x_k - x, t_0) > 1 - a\}) \neq 0.$$

In this case we say x is a $st_{\mathcal{N}}$-cluster point of the sequence (x_k). Let $\Gamma_{\mathcal{N}}(x_k)$ denotes the set of all $st_{\mathcal{N}}$-cluster points of the sequence (x_k).

Definition 2.9. [2] A sequence (x_k) in $\mathcal{N} = (X, N, \star)$ is said to be statistically bounded if there exists some $t_0 > 0$ and $b \in (0,1)$ such that $\delta(\{k \in \mathbb{N} : N(x_k, t_0) \geq b\}) = 0$.

Definition 2.10. [2] Let (x_k) be a sequence in $(X, \mathcal{N}, \star)$. Then for every $t > 0$, we define the sets $B^{\mathcal{N}}_{(x_k)}$ and $A^{\mathcal{N}}_{(x_k)}$ by

$$B^{\mathcal{N}}_{(x_k)} = \{b \in (0,1) : \delta(\{k : N(x_k, t) > b\}) \neq 0\},$$

$$A^{\mathcal{N}}_{(x_k)} = \{a \in (0,1) : \delta(\{k : N(x_k, t) < a\}) \neq 0\}.$$

The statistical limit superior of (x_k) with respect to the fuzzy norm $\mathcal{N}$ is defined by

$$st_{\mathcal{N}}\text{-lim sup}\,(x_k) = \begin{cases} \sup B^{\mathcal{N}}_{(x_k)}, & \text{if } B^{\mathcal{N}}_{(x_k)} \neq \phi \\ 0, & \text{if } B^{\mathcal{N}}_{(x_k)} = \phi \end{cases}$$

and the statistical limit inferior of (x_k) with respect to the fuzzy norm $\mathcal{N}$ is defined by

$$st_{\mathcal{N}}\text{-lim inf}\,(x_k) = \begin{cases} \inf A^{\mathcal{N}}_{(x_k)}, & \text{if } A^{\mathcal{N}}_{(x_k)} \neq \phi \\ 1, & \text{if } A^{\mathcal{N}}_{(x_k)} = \phi\,. \end{cases}$$

3 Main Results

In this section, using the concept of rough statistical convergence, we introduce the notion of rough statistical convergence in fuzzy normed linear spaces. Further, we have established the relationship between the rough statistical cluster point and the rough statistical limit set in fuzzy normed linear spaces.

Definition 3.1. Let $(X, N, \star)$ be a fuzzy normed linear space and r be a non-negative real number. A sequence (x_k) in $(X, N, \star)$ is said to be rough statistical convergent to $x^* \in X$ if for every $\lambda > 0$ and $\varepsilon \in (0,1)$ such that $\delta\{k \in \mathbb{N} : N(x_k - x^*, r + \lambda) \geq \varepsilon\} = 0$. It is denoted by $x_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x$.

Remark 3.2. For the case $r = 0$, the notion of rough statistical convergence coincides with statistical convergence in $(X, N, \star)$, which was studied by Sencimen and Pehlivan in 2008 [35].

In general, the rough statistical limit of a sequence (x_k) in $(X, N, \star)$ may not be unique for the roughness degree $r > 0$. Then the r-statistical limit set of a sequence (x_k) with respect to the fuzzy norm $\mathcal{N}$, defined as $st_{\mathcal{N}}\text{-LIM}^r(x_k) = \{x^* \in V : x_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x^*\}$. Moreover, the sequence (x_k) is said to be r-statistical convergent with respect to the fuzzy norm $\mathcal{N}$ provided that $st_{\mathcal{N}}\text{-LIM}^r(x_k) \neq \phi$. It is clear that if $st_{\mathcal{N}}\text{-LIM}^r(x_k) \neq \phi$ for a sequence (x_k), then we have $st_{\mathcal{N}}\text{-LIM}^r(x_k) = [st_{\mathcal{N}}\text{-lim sup}(x_k) - r, \quad st_{\mathcal{N}}\text{-lim inf}(x_k) + r]$. It can be shown that for an unbounded sequence in $(X, N, \star)$, the rough statistical limit set $st_{\mathcal{N}}\text{-LIM}^r(x_k)$ may not empty although the rough limit set $\text{LIM}^r_{\mathcal{N}}(x_k)$ is empty with respect to fuzzy norm $\mathcal{N}$.

Example 3.3. Let $V = \mathbb{R}$ and consider a real sequence (x_k) defined by

$$x_k = \begin{cases} 3k, & \text{if k is an odd square} \\ 1, & \text{if k is an even square} \\ 0, & \text{if k is an odd nonsquare} \\ \frac{1}{2}, & \text{if k is an even nonsquare.} \end{cases}$$

Let, $N(x_k,t)=\frac{t}{t+|x_k|}$.
The above sequence is unbounded. On the other hand, it is statistically bounded with respect to $\mathcal{N}$. For this we have

$$\delta(\{k:\mathcal{N}(x_k,t)\geq b\})=\delta\left(\left\{k:\frac{t}{t+|x_k|}\geq b\right\}\right)$$
$$=\delta\left(\left\{k:|x_k|\leq\frac{t(1-b)}{b}\right\}\right).$$

Since, $0<b<1$, $\frac{1}{b}-1>0$. Choose $t=\frac{b}{1-b}$. Then $t>0$ and

$$\delta(\{k:N(x_k,t)\leq b\})$$
$$=\delta(\{k:|x_k|\leq\frac{1-b}{b}\times\frac{b}{1-b}=1\})$$
$$=\delta(\{k:|x_k|\leq 1\})=\lim_{n\to\infty}\tfrac{1}{n}\times\sqrt{n}=0.$$

Hence it is statistically bounded with respect to $\mathcal{N}$. Since the given sequence is unbounded, therefore $\text{LIM}^r_{\mathcal{N}}(x_k)=\phi$. So the concepts of rough convergence does not work here and the sequence is not rough convergence with respect to the fuzzy norm $\mathcal{N}$. But the given sequence is statistically bounded. Now we try to find $st_{\mathcal{N}}\text{-LIM}^r(x_k)$, for every $t>0$ we have.
To find $B^{\mathcal{N}}_{(x_k)}$, we have to find those $b\in(0,1)$ such that, for all $t>0$

$$\delta(\{k:(x_k,t)>b\})\neq 0,$$

$$\text{Now,}\quad \delta(\{k:N(x_k,t)>b\})$$
$$=\delta\left(\left\{k:\frac{t}{t+|x_k|}>b\right\}\right)=\delta\left(\left\{k:|x_k|<\frac{t(1-b)}{b}\right\}\right)=0.$$

Hence, $B^{\mathcal{N}}_{(x_k)}=(0,1)$, and $st_{\mathcal{N}}$-lim sup $(x_k)=0$. Similarly, it can be shown that $st_{\mathcal{N}}$-lim inf $(x_k)=1$.

$$st_{\mathcal{N}}\text{-LIM}^r(x_k)=\begin{cases}\phi, & r<0.5\\ [-r,\ 1+r], & \text{otherwise.}\end{cases}$$

Theorem 3.4. *Let $(X,N,\star)$ be a fuzzy normed linear space. A sequence (x_k) in $(X,N,\star)$ is statistically bounded iff there exists a non-negative real number r such that* $\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)\neq\phi$.

Proof. Consider the sequence (x_k) be statistically bounded in $(X,N,\star)$. Then for every $b\in(0,1)$ there exists a real number $t_0>0$ such that

$$\delta\{k\in\mathbb{N}:N(x_k,t_0)\geq b\}=0.$$

Now define $r'=\sup\{N(x_k,t_0):k\in K^c\}$ where $K=\{k\in\mathbb{N}:N(x_k,t_0)\geq b\}$. Then the $\text{st}_{\mathcal{N}}\text{-LIM}^{r'}(x_k)$ contain the origin of $(X,N,\star)$. So we have $\text{st}_{\mathcal{N}}\text{-LIM}^{r'}(x_k)\neq\phi$.
Conversely, if $\text{st-LIM}^r(x_k)\neq\phi$ for some $r\geq 0$, then there exists x^* such that $x^*\in\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$, i.e., for every $\lambda>0$

$$\delta\{k\in\mathbb{N}:N(x_k-x^*,r+\lambda)\geq\varepsilon\}=0.$$

Then we say that almost all (x_k) are contained in some ball with any radius greater than r. So the sequence (x_k) is statistically bounded in $(X,N,\star)$. □

Theorem 3.5. *If (x_{k_j}) is a nonthin subsequence of (x_k), then* $\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)\subseteq\text{st}_{\mathcal{N}}\text{-LIM}^r(x_{k_j})$.

Proof. The proof is trivial. So omitted. □

But the above theorem is not true for a thin subsequence.

Example 3.6. Let $V=\mathbb{R}$ and consider a real sequence (x_k) defined by

$$x_k=\begin{cases}k, & k=n^2\\ 0, & \text{otherwise.}\end{cases}$$

Let, $N(x_k,t)=\frac{t}{t+|x_k|}$, $\forall\, x_k(\neq 0)\in\mathbb{R}$. The above sequence is unbounded.
On the other hand, it is statistically bounded with respect to the fuzzy norm $\mathcal{N}$.

$$\delta(\{k:\mathcal{N}(x_k,t)\geq b\})=\delta\left(\left\{k:\frac{t}{t+|x_k|}\geq b\right\}\right)$$

$$=\delta\left(\left\{k:|x_k|\leq\frac{t(1-b)}{b}\right\}\right)=0$$

Now we try to find $st_{\mathcal{N}}$-LIM$^r(x_k)$, for every $t>0$ we have.
To find $B^{\mathcal{N}}_{(x_k)}$, we have to find those $b\in(0,1)$ such that, for all $t>0$

$$\delta(\{k:(x_k,t)>b\})\neq 0,$$

$$Now,\quad \delta(\{k:N(x_k,t)>b\})$$

$$=\delta\left(\left\{k:\frac{t}{t+|x_k|}>b\right\}\right)=\delta\left(\left\{k:|x_k|<\frac{t(1-b)}{b}\right\}\right)=0.$$

For this we have, st$_{\mathcal{N}}$-LIM$^r(x_k)=[st_{\mathcal{N}}$-lim sup $(x_k)-r,\ st_{\mathcal{N}}$-lim inf $(x_k)+r]=[-r,\ r+1]$ i.e.,

$$st_{\mathcal{N}}\text{-LIM}^r(x_k)=\begin{cases}\phi, & r<0.5\\ [-r,\ 1+r], & \text{otherwise.}\end{cases}$$

Now for thin subsequence $(x_{k_j})=(1,4,9,16,...)$ of (x_k), which is clearly unbounded, so we have st$_{\mathcal{N}}$-LIM$^r(x_{k_j})=\phi$, so st$_{\mathcal{N}}$-LIM$^r(x_k)\nsubseteq$ st$_{\mathcal{N}}$-LIM$^r(x_{k_j})$, which contradicts the above theorem.

Theorem 3.7. *The set* st$_{\mathcal{N}}$-LIM$^r(x_k)$ *of a sequence* (x_k) *in* $(X,N,\star)$ *is a closed set.*

Proof. If st$_{\mathcal{N}}$-LIM$^r(x_k)=\phi$, then there is nothing to prove. Let us assume that st$_{\mathcal{N}}$-LIM$^r(x_k)\neq\phi$, then we can choose $y^*\in X$, for a sequence $(y_k)\subseteq$ st$_{\mathcal{N}}$-LIM$^r(x_k)$ with respect to the fuzzy norm $\mathcal{N}$, such that $y_k\to y^*$. If we show that $y^*\in$ st$_{\mathcal{N}}$-LIM$^r(x_k)$, then the proof will be complete. Now for every $\lambda>0$ and $\varepsilon\in(0,1)$ there exists $k_1\in\mathbb{N}$ such that

$$N\left(y_k-y^*,\frac{\lambda}{2}\right)\geq\varepsilon,\ \text{for all } k\geq k_1.$$

Let us choose $y_m\in$ st$_{\mathcal{N}}$-LIM$^r(x_k)$ with $m>k_1$ such that

$$\delta\left\{k\in\mathbb{N}:N\left(x_k-y_m,r+\frac{\lambda}{2}\right)\geq\varepsilon\right\}=0. \tag{3.1}$$

$$\text{For } j\in\left\{k\in\mathbb{N}:N\left(x_k-y_m,r+\frac{\lambda}{2}\right)\geq\varepsilon\right\},$$

$$\text{We have } N\left(x_j-y_m,r+\frac{\lambda}{2}\right)\geq\varepsilon,$$

$$\text{Then we have, } N(x_j-y^*,r+\lambda)\geq min\left\{N\left(x_j-y_m,r+\frac{\lambda}{2}\right),N\left(y_m-y^*,\frac{\lambda}{2}\right)\right\}\geq\varepsilon$$

$$\text{Hence, } j\in\{k\in\mathbb{N}:N(x_k-y^*,r+\lambda)\geq\varepsilon\}.$$

Now we have the following inclusion

$$\{k\in\mathbb{N}:N(x_k-y_m,r+\frac{\lambda}{2})\geq\varepsilon\}\subseteq\{k\in\mathbb{N}:N(x_k-y^*,r+\lambda)\geq\varepsilon\}.$$

Therefore, $\delta\{k \in \mathbb{N} : N(x_k - y^*, r + \lambda) \leq \varepsilon\} \leq \delta\{k \in \mathbb{N} : N(x_k - y_m, r + \frac{\lambda}{2}) \geq \varepsilon\}$.

Using (3.1) we get

$$\delta\{k \in \mathbb{N} : N(x_k - y^*, r + \lambda) \leq \varepsilon\} = 0.$$

Therefore, $y^* \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$. Which completes the proof. □

Theorem 3.8. *Let (x_k) be a sequence in $(X, N, \star)$. Then the set $\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ with respect to fuzzy norm $\mathcal{N}$ is convex.*

Proof. Assume that $y_0, y_1 \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$. For convex we have to show that $(1-\alpha)y_0 + \alpha y_1 \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ for some $\alpha \in (0, 1)$. Now for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we define

$$M_1 = \left\{k \in \mathbb{N} : N\left(x_k - y_0, \frac{r+\lambda}{2(1-\alpha)}\right) \geq \varepsilon\right\}, M_2 = \left\{k \in \mathbb{N} : N\left(x_k - y_1, \frac{r+\lambda}{2\alpha}\right) \geq \varepsilon\right\}.$$

As $y_0, y_1 \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$, we have $\delta(M_1) = \delta(M_2) = 0$. For $k \in M_1^c \cap M_2^c$ with $\delta(M_1^c \cap M_1^c) = 1$, we have

$$\begin{aligned} N(x_k - ((1-\alpha)y_0 + \alpha y_1), r + \lambda) &= N((1-\alpha)(x_k - y_0) + \alpha(x_k - y_1), r + \lambda) \\ &\geq min\left\{N\left((1-\alpha)(x_k - y_0), \frac{r+\lambda}{2}\right), N\left(\alpha(x_k - y_1), \frac{r+\lambda}{2}\right)\right\} \\ &= min\left\{N\left((x_k - y_0), \frac{r+\lambda}{2(1-\alpha)}\right), N\left((x_k - y_1), \frac{r+\lambda}{2\alpha}\right)\right\} \geq \varepsilon, \end{aligned}$$

So we have,

$$\delta\{k \in \mathbb{N} : N(x_k - ((1-\alpha)y_0 + \alpha y_1), r + \lambda) \geq \varepsilon\} = 0.$$

Hence, $[(1-\alpha)y_0 + \alpha y_1] \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ i.e., $\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ is a convex set, which completes the proof. □

Theorem 3.9. *Let $r > 0$ and $\lambda > 0$. Then a sequence (x_k) in $(X, N, \star)$ is rough statistically convergent to $x^* \in X$ with respect to the fuzzy norm $\mathcal{N}$ if and only if there exists a sequence (y_k) such that $\text{st}_{\mathcal{N}}\text{-}LIM^r(y_k) = x^*$ and for every $\varepsilon \in (0, 1)$, $\delta\{k \in \mathbb{N} : N(x_k - y_k, \lambda) > \varepsilon\} = 0$.*

Proof. Assume that $y_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x^*$. Then, by Definition 3.1, for every $\varepsilon \in (0, 1)$ and $\lambda > 0$ we have

$$\delta\{k \in \mathbb{N} : N(y_k - x^*, r + \lambda) \geq \varepsilon\} = 0.$$

Now we define, $A = \{k \in \mathbb{N} : N(y_k - x^*, r + \lambda) \geq \varepsilon\}$, $B = \{k \in \mathbb{N} : N(x_k - y_k, \lambda) \geq \varepsilon\}$.

Clearly, $\delta(A) = 0$ and $\delta(B) = 0$. From the properties of the fuzzy norm we get

$$N(x_k - x^*, r + 2\lambda) \geq \min\{N(x_k - y_k, \lambda),\ N(y_k - x^*, r + \lambda)\}.$$

$$\text{Thus,} \quad \{k \in \mathbb{N} : N(x_k - x^*, r + 2\lambda) \geq \varepsilon\} \subseteq A \cup B.$$

Since $\delta(A) = 0$ and $\delta(B) = 0$, we obtain $\delta\{k \in \mathbb{N} : N(x_k - x^*, r + 2\lambda) \geq \varepsilon\} = 0$. Therefore $x_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x^*$. □

Definition 3.10. Let (x_k) be a sequence in $(X, N, \star)$. An element $x^* \in X$ is said to be a rough statistical limit point of the sequence (x_k) with respect to the fuzzy norm $\mathcal{N}$ provided that there is a nonthin subsequence of (x_k) that rough converges to x^*. In this case, we say x^* is a $r\text{-}st_{\mathcal{N}}$-limit point of sequence (x_k). Let $\Lambda^r_{\mathcal{N}}(x_k)$ denotes the set of all $r\text{-}st_{\mathcal{N}}$-limit points of the sequence (x_k).

Definition 3.11. Let (x_k) be a sequence in $(X, N, \star)$. An element $x^* \in V$ is said to be a rough statistical cluster point of the sequence (x_k) with respect to the fuzzy norm $\mathcal{N}$ provided that for every $\lambda > 0$ and $\varepsilon \in (0, 1)$,

$$\delta(\{k \in \mathbb{N} : N(x_k - x^*, r + \lambda) < \varepsilon\}) \neq 0.$$

In this case, we say x^* is a $r\text{-}st_{\mathcal{N}}$-cluster point of the sequence (x_k). Let $\Gamma^r_{\mathcal{N}}(x_k)$ denotes the set of all $r\text{-}st_{\mathcal{N}}$-cluster points of the sequence (x_k).

Theorem 3.12. *Let $\Gamma^r_{\mathcal{N}}(x_k)$ be the set of all rough statistical cluster points of a sequence (x_k) with respect to the fuzzy norm $\mathcal{N}$ in $(X, N, \star)$ and r be some non-negative real number. Then, for an arbitrary $x^* \in \Gamma^r_{\mathcal{N}}(x_k)$ we have $\delta\{k \in \mathbb{N} : N(y_k - x^*, r + \lambda) < \varepsilon\} \neq 0$ for all $y_k \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$.*

Proof. Assume on the contrary that there exists a point $x^* \in \Gamma^r_{\mathcal{N}}(x_k)$ and $y_k \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ for every $\varepsilon \in (0, 1)$ we can write from theorem 3.9
$\delta\{k \in \mathbb{N} : N(x_k - y_k, r + \lambda) > \varepsilon\} = 0$ for all $k \in \mathbb{N}$.

$$\text{Now, } \{k \in \mathbb{N} : N(x_k - y_k, r + \lambda) > \varepsilon\} \supseteq \{k \in \mathbb{N} : N(x_k - x^*, \lambda) < \varepsilon\}.$$

As $x^* \in \Gamma^r_{\mathcal{N}}(x_k)$, we have $\delta\{k \in \mathbb{N} : N(x_k - x^*, \lambda) < \varepsilon\} \neq 0$. Hence we get $\delta\{k \in \mathbb{N} : N(x_k - y_k, r + \lambda) < \varepsilon\} \neq 0$, which contradicts the fact $y_k \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$. □

Definition 3.13. Let $(X, N, \star)$ be a fuzzy normed linear space with norm $\mathcal{N}$ and $x^* \in X$ for some non-negative real number r, then $B_{(r,\varepsilon)}(x) = \{x \in X : N(x - x^*, r) > \varepsilon\}$ is called an open ball.

Definition 3.14. Let $(X, N, \star)$ be a fuzzy normed linear space with norm $\mathcal{N}$ and $x^* \in X$ for some non-negative real number r, then $\bar{B}_{(r,\varepsilon)}(x) = \{x \in X : N(x - x^*, r) \geq \varepsilon\}$ is the closure of an open ball.

Theorem 3.15. *Let $(X, N, \star)$ be a fuzzy normed linear space with norm $\mathcal{N}$. If a sequence (x_k) in $(X, N, \star)$ is rough statistically convergent to $x^* \in X$ for some non-negative real number r, then $\Gamma^r_{\mathcal{N}}(x_k) = \bigcup_{x^* \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$.*

Proof. Let $z^* \in \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$, for some $r > 0$ and given $\varepsilon \in (0, 1)$ such that $N(x - z^*, r) \geq \varepsilon$. For $\lambda > 0$ and $x \in \Gamma_{\mathcal{N}}(x_k)$, we have $\delta\{k : N(x_k - x, \lambda) < \varepsilon\} \neq 0$.

$$\text{Now,} \qquad N(x_k - z^*, r + \lambda) \geq min\{N(x_k - x, \lambda), N(x - z^*, r)\} \geq \varepsilon.$$

This implies that $\delta\{k : N(x_k - z^*, r + \lambda) < \varepsilon\} \neq 0$. Hence $z^* \in \Gamma^r_{\mathcal{N}}(x_k)$. Therefore

$$\bigcup_{x^* \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x) \subseteq \Gamma^r_{\mathcal{N}}(x_k). \tag{3.2}$$

Again, let $z^* \in \Gamma^r_{\mathcal{N}}(x_k)$. Then we have to show that $z^* \in \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x^*)$. Let if possible, $z^* \notin \bigcup_{x^* \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$ that is $z^* \notin \bar{B}_{(r,\varepsilon)}(x)$ for all $x \in \Gamma_{\mathcal{N}}(x_k)$. Then $\delta\{k \in \mathbb{N} : N(x - z^*, r) \geq \varepsilon\} = 0$, then from theorem 3.12 for $\lambda > 0$, we have $\delta\{k \in \mathbb{N} : N(z^* - x, r + \lambda) < \varepsilon\} \neq 0$, which is a contradiction to the assumption. Therefore $z^* \in \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$.

$$\text{Hence } \Gamma^r_{\mathcal{N}}(x_k) \subseteq \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x). \tag{3.3}$$

Combaining (3.2) and (3.3) we get $\Gamma^r_{\mathcal{N}}(x_k) = \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$. □

Theorem 3.16. *Let (x_k) be a sequence in $(X, N, \star)$ then for every $\varepsilon \in (0, 1)$*
(i)*If $x \in \Gamma_{\mathcal{N}}(x_k)$ then $\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x)$;*
(ii)$\text{st}_{\mathcal{N}}\text{-LIM}^r(x_k) = \bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$.

Proof. (i) Consider $y^* \in \text{st}_{\mathcal{N}}\text{-LIM}^r(x_k)$ and $x \in \Gamma_{\mathcal{N}}(x_k)$. For every $\lambda > 0$ and $\varepsilon \in (0, 1)$ define the sets
$A = \{k \in \mathbb{N} : N(x_k - y^*, r + \frac{\lambda}{2}) \geq \varepsilon\}$

$B = \{k \in \mathbb{N} : N(x_k - x, \frac{\lambda}{2}) \geq \varepsilon\}$ with $\delta(A^c) = 0$ and $\delta(B^c) = 0$. Now $k \in A \cap B$ we have

$$N(y^* - x, r + \lambda) \geq min\left\{N\left(x_k - y^*, r + \frac{\lambda}{2}\right), N\left(x_k - x, \frac{\lambda}{2}\right)\right\} \geq \varepsilon.$$

Therefore, we have $\bar{B}_{(r,\varepsilon)}(x) = \{N(y^* - x, r + \lambda) \geq \varepsilon\}$ then $y^* \in \bar{B}_{(r,\varepsilon)}(x)$ for every $\lambda > 0$, that implies $\bar{B}_{(r,\varepsilon)}(x) = \{N(y^* - x, r) \geq \varepsilon\}$ then $y^* \in \bar{B}_{(r,\varepsilon)}(x)$. Hence st$_{\mathcal{N}}$-LIM$^r(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x)$.

(ii) By previous part st$_{\mathcal{N}}$-LIM$^r(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x)$ implies st$_{\mathcal{N}}$-LIM$^r(x_k) \subseteq \bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$. Assume $y^* \in \bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$ then we have $N(y^* - x, r) \geq \varepsilon$ for all $x \in \Gamma_{\mathcal{N}}(x_k)$. This implies that $\Gamma_{\mathcal{N}}(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x)$. Further, let $y^* \notin$ st$_{\mathcal{N}}$-LIM$^r(x_k)$ then for $\lambda > 0$ we have $\delta\{k \in \mathbb{N} : N(x_k - y^*, r + \lambda) \geq \varepsilon\} \neq 0$, which implies that a statistical cluster point x exists for the sequence (x_k) with $N(y^* - x, r + \lambda) \geq \varepsilon$. Thus, $\Gamma_{\mathcal{N}}(x_k) \nsubseteq \bar{B}_{(r,\varepsilon)}(x)$ and $y^* \notin \Gamma_{\mathcal{N}}(x_k)$. This implies that $y^* \in \Gamma_{\mathcal{N}}(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x) \subseteq$ st$_{\mathcal{N}}$-LIM$^r(x_k)$ and we get $\bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x) \subseteq$ st$_{\mathcal{N}}$-LIM$^r(x_k)$. Therefore, st$_{\mathcal{N}}$-LIM$^r(x_k) = \bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$. □

Theorem 3.17. *Let (x_k) be a sequence in $(X, N, \star)$ which is rough statistically convergent to $x \in X$ with respect to fuzzy norm $\mathcal{N}$ such that* st$_{\mathcal{N}}$-LIM$^r(x_k) = \bar{B}_{(r,\varepsilon)}(x)$ *for some $r > 0$.*

Proof. Let $\lambda > 0$. Since $x_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x$ then we have $\delta\left\{k \in \mathbb{N} : N\left(x_k - x, r + \frac{\lambda}{2}\right) \geq \varepsilon\right\} = 0$ and
$\delta\left\{k \in \mathbb{N} : N\left(y^* - x, \frac{\lambda}{2}\right) \geq \varepsilon\right\} = 0$, Consider $y^* \in \bar{B}_{(r,\varepsilon)}(x)$ and for all $\lambda > 0$

$$N(x_k - y^*, r + \lambda) \geq min\left\{N\left(x_k - x, r + \frac{\lambda}{2}\right), N\left(y^* - x, \frac{\lambda}{2}\right)\right\} \geq \varepsilon.$$

Therefore, we have $N(x_k - y^*, r + \lambda) \geq \varepsilon$. This implies that $N(x_k - y^*, r) \geq \varepsilon$, then $y^* \in$ st$_{\mathcal{N}}$-LIM$^r(x_k)$ i.e., $\bar{B}_{(r,\varepsilon)}(x) \subseteq$ st$_{\mathcal{N}}$-LIM$^r(x_k)$. Also from theorem 3.16 we have st$_{\mathcal{N}}$-LIM$^r(x_k) \subseteq \bar{B}_{(r,\varepsilon)}(x)$. Hence st$_{\mathcal{N}}$-LIM$^r(x_k) = \bar{B}_{(r,\varepsilon)}(x)$. □

Theorem 3.18. *Let (x_k) be a sequence in $(X, N, \star)$ which is rough statistically convergent to x in X with respect to fuzzy norm $\mathcal{N}$, such that $\Gamma^r_{\mathcal{N}}(x_k) =$* st$_{\mathcal{N}}$-LIM$^r(x_k)$ *for some $r > 0$.*

Proof. Necessary part: Suppose $x_k \xrightarrow{r\text{-}st_{\mathcal{N}}} x$. Then $\Gamma^r_{\mathcal{N}}(x_k) = \{x\}$. By theorem 3.15, for some $r > 0$ and $\varepsilon \in (0, 1)$ we have $\Gamma^r_{\mathcal{N}}(x_k) = \bar{B}_{(r,\varepsilon)}(x)$. Also by theorem 3.17, we get st$_{\mathcal{N}}$-LIM$^r(x_k) = \bar{B}_{(r,\varepsilon)}(x)$. Hence $\Gamma^r_{\mathcal{N}}(x_k) =$ st$_{\mathcal{N}}$-LIM$^r(x_k)$.

Sufficient part: Let $\Gamma^r_{\mathcal{N}}(x_k) =$ st$_{\mathcal{N}}$-LIM$^r(x_k)$. From theorem 3.15 and 3.16, we have $\bigcap_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x) = \bigcup_{x \in \Gamma_{\mathcal{N}}(x_k)} \bar{B}_{(r,\varepsilon)}(x)$. This implies that either $\Gamma^r_{\mathcal{N}}(x_k) = \phi$ or $\Gamma^r_{\mathcal{N}}(x_k)$ is a singleton set. Then st$_{\mathcal{N}}$-LIM$^r(x_k) = \Gamma^r_{\mathcal{N}}(x_k) = \bar{B}_{(r,\varepsilon)}(x)$ for some $x \in \Gamma_{\mathcal{N}}(x_k)$, Further by theorem 3.17, we get st$_{\mathcal{N}}$-LIM$^r(x_k) = \{x\}$. □

4 Conclusion

In this paper, we studied rough statistical convergence in fuzzy normed linear spaces. We showed that the rough statistical limit set is closed and convex. We also introduced rough statistical cluster points and limit points and explained their relation. In future, this study can be extended to double sequences for further results.

References

[1] P. Agarwal, J. Choi and J. Shilpi, *Extended hypergeometric functions of two and three variables*, Commun. Korean Math. Soc., **30** , 403–414, (2015).

[2] M. A. Alghamdi, A. Alotaibi, Q. M. D. Lohani and M. Mursaleen, *Statistical limit superior and limit inferior in intuitionistic fuzzy normed spaces*, J. Inequal. Appl., **2012** 2012:96.

[3] S. Aytar, *Statistical limit points of sequence of fuzzy numbers*, Inf. Sci., **165(1-2)**, 129–138, (2004).

[4] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim, **29(3-4)**, 291–303, (2008).

[5] C. Belen, S. A. Mohiuddine, *Generalized weighted statistical convergence and application*, Appl. Math. Comput. **219(18)**, 9821–9826, (2013).

[6] S. Cheng and J. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*. Bull. Calcutta Math. Soc., **86**, 429–436, (1994).

[7] C. Choudhury and S. Debnath, *On some properties of $\mathcal{I}^{\mathcal{K}}$-convergence*. Palestine J. Math, **11(2)**, 129–135, (2022).

[8] C. Choudhury and S. Debnath, *On $\mathcal{I}^{K-st}$ convergence of sequence of real numbers*. Palestine J. Math, **11(2)**, 505–510, (2022).

[9] S. Debnath and D. Rakshit, *Rough statistical convergence of sequences of fuzzy numbers*, Mathematica, **61**, 33–39, (2019).

[10] S. Debnath and D. Rakshit, *Rough convergence in metric*, New Trends in Analysis and Interdisciplinary Applications, Birkhäuser, 449–454, (2017).

[11] S. Debnath and N. Subramanian, Rough statistical convergence on triple sequences, Proyecciones, **36(4)**, 685–699, (2017).

[12] A. Esi, N. Subramanian, A. Esi, *Triple rough statistical convergence of sequence of Bernstein operators*, Int. J. Adv. Appl. Sci, **4(2)**, 28–34, (2017).

[13] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2(3-4)**, 241–244, (1951).

[14] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets Syst., **48**, 239–248, (1992).

[15] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Am. Math. Soc., **125(12)**, 3625–3631, (1997).

[16] J. A. Fridy, *Statistical limit point*, Proc. Am. Math. Soc, **118(4)**, 1187–1192, (1993).

[17] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pac. J. Math., **160(1)**, 43—51, (1993).

[18] J. A. Fridy, *On statistical convergence*, Analysis, **5(4)**, 301–313, (1985).

[19] I. Golet, *On fuzzy normed spaces*, Southeast Asian Bull. Math., **31**, 245–254, (2007).

[20] B. Hazarika, A. Alotaibi, S. A. Mohiuddine, *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput. **24**, 6613–6622, (2020).

[21] M. Küçükaslan and M. Yılmaztürk, *On deferred statistical convergence of sequences*, Kyungpook Math. J. **56(2)**, 357—366, (2016).

[22] Ö. Kişi, C. Choudhury, *Rough statistical convergence of sequences in gradual normed linear spaces*, J. Anal., **31(2)**, 1511–1525, (2023).

[23] Ö. Kişi, C. Choudhury, *Rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence of double sequences in gradual normed linear Spaces*, Suleyman Demirel Universitesi Fen Edebiyat Fakultesi Fen Dergisi, **17(2)**, 405–428, (2022).

[24] K. Menger, Statistical metrics, *Proc. Natl. Acad. Sci. U.S.A.*, **28(12)**, 535–537, (1942).

[25] U. Kadak, S. A. Mohiuddine, *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-gamma function and related approximation theorems*, Results Math. **73**, Article 9, (2018).

[26] M. Mursaleen, *$\lambda-$Statistical convergence*, Math. Slovaca, **50(1)**, 111–115, (2000).

[27] S. A. Mohiuddine, A. Alotaibi, M. Mursaleen, *A new variant of statistical convergence*, J. Inequalities Appl., **2013(1)**, 1–8, (2013).

[28] S. A. Mohiuddine, B. A. S. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM **113(3)**, 1955–1973, (2019).

[29] S. A. Mohiuddine, *Some new results on approximation in fuzzy 2-normed spaces*, Math. Comput. Model., **53(5-6)**, 574–580, (2011).

[30] S. A. Mohiuddine, *Statistical weighted A-summability with application to Korovkin's type approximation theorem*, J. Inequal. Appl. **2016**, Article 101, (2016).

[31] S. A. Mohiuddine, A. Asiri, B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst. **48(5)**, 492–506, (2019).

[32] S. A. Mohiuddine, B. Hazarika, M. A. Alghamdi, *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems*, Filomat **33(14)**, 4549–4560, (2019).

[33] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22(1-2)**, 199–222, (2001).

[34] G. Rano and T. Bag, *Fuzzy normed linear spaces*, Int. J. Math. Comput Sci., **2(2)**, 16–19, (2012).

[35] C. Sencimen and S. Pehlivan, *Statistical convergence in fuzzy normed linear spaces*, Fuzzy Sets Syst., **159(3)**, 361–370, (2008).

[36] E. Savas, P. Das, *A generalized statistical convergence via ideals*, Appl. Math. Lett., **24(6)**, 826–830, (2011).

[37] E. Savas, *On statistically convergent sequences of fuzzy numbers*, Inf. Sci., **137(1-4)**, 277–282, (2001).

[38] N. Subramanian, A. Esi, *Wijsman rough convergence of triple sequences*, Mat. Stud., **48(2)**, 171–179, (2017).

[39] B. C. Tripathy and M. Sen, *On generalized statistically convergent sequences*, Indian J. Pure Appl. Math., **32(11)**, 1689–1694, (2001).

[40] L. A. Zadeh, *Fuzzy sets*, Inf. Control., **8(3)**, 338–353, (1965).

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