A Note On Milne's Formula-Type Inequalities For Various Function Classes

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Abstract: *The study of fractional calculus continues to be an active area of research with practical implications in real-world applications. Numerous mathematicians are interested in fractional operators with integral inequalities. In the present study, we have established some new identity within a Caputo-Fabrizio fractional integral operator. By using this novel identity some error estimations of Milne's formula-type inequalities are obtained for differentiable functions. Employing the new identity to Milne's Formula for s-convex and quasi-convex functions are presented. Moreover, we introduce several fractional Milne type inequalities for the case of Lipschitzian functions. Furthermore, we give fractional Milne type inequalities for functions of bounded variation. This is the first article for fractional version of Milne's Formula. Additionally, we have given a few application of our findings to quadrature formula. It is expected that the outcomes in this research will point to new developments in the study of fractional calculus..*

1 Introduction

In recent years, it is worth noting that the choice of numerical integration method and the estimation of error bounds depend on the properties of the integrand, the desired level of accuracy, and computational constraints. As research in numerical methods and error estimation techniques continues, new approaches and classes of functions may be explored to improve the accuracy and efficiency of numerical integration. Convexity is a useful property in established error bounds for certain numerical integration methods, as it provides a way to control the behavior of the integration over the integration interval. It plays a crucial role in various areas of pure and applied sciences, including optimization, functional analysis, numerical analysis, economics, physics, and engineering. The interested researchers on convexity and integral inequalities we refer to see these articles [11, 13, 27, 36, 42]. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality holds:

$$f(\delta\zeta_1 + (1-\delta)\zeta_2) \leq \delta f(\zeta_1) + (1-\delta) f(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in I$ and $\delta \in [0, 1]$.

In the past few years, several scholar have been interested in generalizing the different type inequalities, Simpson's type inequality is one of these. The Simpson-type inequality is important in many branches of mathematics. The classical Simpson-type inequality deals with the error term of Simpson's Rule and provides an upper bound for the error between the true value of the integral and the approximation obtained through Simpson's Rule. Some of these error bounds of Simpson's formula can be found in [1, 7, 21, 32, 34, 37, 38] utilizing convex functions.

(1) Simpson quadrature formula (Simpson's 1/3 rule) is stated as follows:

$$\int_{\zeta_1}^{\zeta_2} f(x)\,dx \approx \frac{\zeta_2 - \zeta_1}{6}\left[f(\zeta_1) + 4f\left(\frac{\zeta_1 + \zeta_2}{2}\right) + f(\zeta_2)\right].$$

(2) Simpson second formula (Simpson's 3/8 rule) is stated as follows:

$$\int_{\zeta_1}^{\zeta_2} f(x)\,dx \approx \frac{\zeta_2 - \zeta_1}{8}\left[f(\zeta_1) + 3f\left(\frac{2\zeta_1 + \zeta_2}{2}\right) + 3f\left(\frac{\zeta_1 + 2\zeta_2}{2}\right) + f(\zeta_2)\right].$$

The most important Newton-cotes quadrature attained three-point Simpson-type inequality is described as follows:

Theorem 1.1. *Suppose $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ be a four times differentiable function and continuous on (ζ_1, ζ_2) and let $\left\|f^{(4)}\right\|_\infty = \sup_{x \in (\zeta_1, \zeta_2)} \left|f^{(4)}(x)\right| < \infty$, then the classical inequality as follows:*

$$\left|\left[\frac{f(\zeta_1)}{6} + \frac{4}{6}f\left(\frac{\zeta_1 + \zeta_2}{2}\right) + \frac{f(\zeta_2)}{6}\right] - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (\zeta_2 - \zeta_1)^4.$$

Overall, Simpson-type inequalities have been a subject of interest for researchers across different mathematical fields. The study of Simpson-type inequalities for convex functions is an important area of research that contributes to the broader field of convex analysis. Milne's formula, which is of open type, is parallel to Simpson's formula, which is of closed type, in terms of Newton-Cotes formulas, because they hold under the same conditions.

Theorem 1.2. *A function $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a four times differentiable function mapping on (ζ_1, ζ_2) and let $\left\|f^{(4)}\right\|_\infty = \sup_{x \in (\zeta_1, \zeta_2)} \left|f^{(4)}(x)\right| < \infty$, then the inequality as follows:*

$$\left| \frac{2}{3} f(\zeta_1) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2}\right) + \frac{2}{3} f(\zeta_2) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(x)\, dx \right| \leq \frac{7}{23040} \left\| f^{(4)} \right\|_\infty.$$

Fractional calculus, also known as fractional order calculus or calculus of arbitrary order, is a mathematical framework that extends the traditional integer-order calculus to allow for differentiation and integration of non-integer orders. This field has indeed gained prominence and relevance across various scientific and industrial disciplines over the past few decades. Here are some of the areas where fractional calculus has found applications for example (electromagnetic, photoelasticity, fluid mechanics, elec-trochemistry, and biological population models). These techniques and concepts demonstrate the wide range of mathematical tools that fractional calculus brings to various fields, enabling researchers to tackle complex problems involving memory, non-locality, and intricate dynamics. The applications and utility of these techniques continue to grow as researchers explore new mathematical properties and their practical implications in different disciplines. Given the significance of fractional calculus (see [4, 12, 17–19, 29, 30]), academics have developed a variety of fractional integral inequalities using Riemann-Liouville fractional integrals that have been shown to be particularly useful in approximation theory. These inequalities are crucial for making informed decisions when applying numerical integration techniques in practice, as they allow for efficient and effective utilization of these methods across a wide range of functions and integration domains. Set in [39] was demonstrated a fractional version of Ostrowski's type inequality for differentiable function. Using harmonic convexity on Hermite-Hadamard type inequalities is presented in [22]. For further inequalities that can be solved using fractional integrals, and the references therein [2, 5, 6, 9, 10, 14, 15, 24–26, 28, 31, 33, 35, 40, 41, 43].

2 Preliminaries

We give definitions of generalized convex functions, Riemann-Liouville and Caputo-Fabrizio fractional integral operators to use these definitions in our main results. In the year 1994, Hudzik [20] introduced and investigated the term s-convexity.

Definition 2.1. A function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex function if the inequality holds:

$$f(\delta \zeta_1 + (1 - \delta) \zeta_2) \leq \delta^s f(\zeta_1) + (1 - \delta)^s f(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in I$, $\delta \in [0, 1]$ and some fixed $s \in (0, 1]$.

Definition 2.2. [23] Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex function if the inequality holds:

$$f(\delta \zeta_1 + (1 - \delta) \zeta_2) \leq \max[f(\zeta_1), f(\zeta_2)],$$

for all $\zeta_1, \zeta_2 \in I$ and $\delta \in [0, 1]$.

Definition 2.3. [16] Suppose $f \in L[\zeta_1, \zeta_2]$. The left and right-sided Riemann-Liouville fractional integrals of order $\Omega > 0$ defined by:

$$I_{\zeta_1^+}^\Omega f(\delta) = \frac{1}{\Gamma(\Omega)} \int_{\zeta_1}^x (x - \delta)^{\Omega - 1} f(\delta)\, d\delta, \; x > \zeta_1$$

$$I_{\zeta_2^-}^\Omega f(\delta) = \frac{1}{\Gamma(\Omega)} \int_x^{\zeta_2} (x - \delta)^{\Omega - 1} f(\delta)\, d\delta, \; x < \zeta_2,$$

where $\Gamma(.)$ is the gamma function and $I_{\zeta_1^+}^0 f(\delta) = I_{\zeta_2^-}^0 f(\delta) = f(\delta)$.

Definition 2.4. [8] Let $H^1(\zeta_1, \zeta_2)$ be the Sobolev space of order one defined as:

$$H^1(\zeta_1, \zeta_2) := \left\{ g \in L^2(\zeta_1, \zeta_2) : g' \in L^2(\zeta_1, \zeta_2) \right\},$$

where

$$L^2(\zeta_1, \zeta_2) := \left\{ g(z) : \left(\int_{\zeta_1}^{\zeta_2} g^2(z)\, dz \right)^{\frac{1}{2}} < \infty \right\}.$$

Let $f \in H^1(\zeta_1, \zeta_2)$, $\zeta_1 < \zeta_2$, $\Omega \in (0, 1)$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}_{\zeta_1}^{CFD} D^\Omega f \right)(x) = \frac{\beta(\Omega)}{1 - \Omega} \int_{\zeta_1}^x f'(\delta)\, e^{\frac{-\Omega(x - \delta)^\Omega}{1 - \Omega}}\, d\delta,$$

$x > \zeta_1$ and the associated integral operator is

$$\left({}_{\zeta_1}^{CF} I^\Omega f \right)(x) = \frac{1 - \Omega}{\beta(\Omega)} f(x) + \frac{\Omega}{\beta(\Omega)} \int_{\zeta_1}^x f(\delta)\, d\delta,$$

where $\beta(\Omega) > 0$ is the normalization function satisfying the condition $\beta(0) = \beta(1) = 1$. For $\Omega = 0$, $\Omega = 1$, the left derivative is defined as follows, respectively

$$\left({}_{\zeta_1}^{CFD} D^0 f \right)(x) = f'(x)$$

$$\left({}_{\zeta_1}^{CFD} D^1 f \right)(x) = f(x) - f(\zeta_1).$$

For the right derivative operator

$$\left({}_{\zeta_2}^{CFD} D^\Omega f\right)(x) = \frac{\beta(\Omega)}{1-\Omega} \int_x^{\zeta_2} f'(\delta) e^{\frac{-\Omega(\delta-x)\Omega}{1-\Omega}} d\delta,$$

$x < \zeta_2$ and the associated integral operator is

$$\left({}^{CF} I_{\zeta_2}^\Omega f\right)(x) = \frac{1-\Omega}{\beta(\Omega)} f(x) + \frac{\Omega}{\beta(\Omega)} \int_x^{\zeta_2} f(\delta) d\delta.$$

We established an identity for Caputo-Fabrizio fractional integral operator. By utilizing this identity, we show some Milne's formula-type inequalities for differentiable mapping whose derivatives are convex. Based on a identity several new error bounds are obtained using for different classes of functions such as s-convex, quasi-convex and bounded. Furthermore, we give some application to quadrature formula.

3   An Identity for Differentiable Function

In this section, we have derived a fractional version of Milne's Formula type identity to the case of a differentiable function involving the Caputao-Fabrizio integral operator.

Lemma 3.1. *A function* $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ *is a differentiable function on* $[\zeta_1, \zeta_2]$. *If* $f' \in L_1 [\zeta_1, \zeta_2]$, *then the following equality holds:*

$$\left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4}\right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2}\right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4}\right)\right]$$

$$- \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}_{\zeta_1}^{CF} I^\Omega f\right)(k) + \left({}^{CF} I_{\zeta_2}^\Omega f\right)(k)\right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k)$$

$$= \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} \delta \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta\right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3}\right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3}\right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$\left. + \int_{\frac{3}{4}}^{1} (\delta - 1) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta\right].$$

Proof. Let

$$\int_0^{\frac{1}{4}} \delta \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3}\right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3}\right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$+ \int_{\frac{3}{4}}^{1} (\delta - 1) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) - f'(\delta\zeta_1 + (1-\delta)\zeta_2)\right] d\delta$$

$$= \sum_{i-1}^{8} I_i.$$

Integration by parts, we get

$$I_1 = \int_0^{\frac{1}{4}} \delta f'(\delta\zeta_2 + (1-\delta)\zeta_1) d\delta$$

$$= \frac{\delta}{\zeta_2 - \zeta_1} f(\delta\zeta_2 + (1-\delta)\zeta_1) \Big|_0^{\frac{1}{4}} - \frac{1}{\zeta_2 - \zeta_1} \int_0^{\frac{1}{4}} f(\delta\zeta_2 + (1-\delta)\zeta_1) d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4}\right) - \frac{1}{\zeta_2 - \zeta_1} \int_0^{\frac{1}{4}} f(\delta\zeta_2 + (1-\delta)\zeta_1) d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4}\right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} f(u) du. \tag{3.1}$$

$$I_2 = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3} \right) f'(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \left. \frac{\delta - \frac{2}{3}}{\zeta_2 - \zeta_1} f(\delta\zeta_2 + (1-\delta)\zeta_1) \right|_{\frac{1}{4}}^{\frac{1}{2}} - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{1}{4}}^{\frac{1}{2}} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{1}{4}}^{\frac{1}{2}} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{3\zeta_1 + \zeta_2}{4}}^{\frac{\zeta_1 + \zeta_2}{2}} f(u)\, du. \tag{3.2}$$

$$I_3 = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3} \right) f'(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \left. \frac{\delta - \frac{1}{3}}{\zeta_2 - \zeta_1} f(\delta\zeta_2 + (1-\delta)\zeta_1) \right|_{\frac{1}{2}}^{\frac{3}{4}} - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{1}{2}}^{\frac{3}{4}} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{1}{2}}^{\frac{3}{4}} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\frac{\zeta_1 + 3\zeta_2}{4}} f(u)\, du, \tag{3.3}$$

and

$$I_4 = \int_{\frac{3}{4}}^{1} (\delta - 1) f'(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \left. \frac{\delta - 1}{\zeta_2 - \zeta_1} f(\delta\zeta_2 + (1-\delta)\zeta_1) \right|_{\frac{3}{4}}^{1} - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{3}{4}}^{1} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\frac{3}{4}}^{1} f(\delta\zeta_2 + (1-\delta)\zeta_1)\, d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{\zeta_1 + 3\zeta_2}{4}}^{\zeta_2} f(u)\, du. \tag{3.4}$$

Similarly, we have

$$I_5 = \int_{0}^{\frac{1}{4}} -\delta f'(\delta\zeta_1 + (1-\delta)\zeta_2)\, d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{\zeta_1 + 3\zeta_2}{4}}^{\zeta_2} f(u)\, du. \tag{3.5}$$

$$I_6 = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{2}{3} - \delta \right) f'(\delta\zeta_1 + (1-\delta)\zeta_2)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\frac{\zeta_1 + 3\zeta_2}{4}} f(u)\, du. \tag{3.6}$$

$$I_7 = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\frac{1}{3} - \delta \right) f'(\delta\zeta_1 + (1-\delta)\zeta_2)\, d\delta$$

$$= \frac{-1}{6(\zeta_2 - \zeta_1)} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{5}{12(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\frac{3\zeta_1 + \zeta_2}{4}}^{\frac{\zeta_1 + \zeta_2}{2}} f(u)\, du, \tag{3.7}$$

and

$$I_8 = \int_{\frac{3}{4}}^{1} (1 - \delta) f'(\delta\zeta_1 + (1-\delta)\zeta_2)\, d\delta$$

$$= \frac{1}{4(\zeta_2 - \zeta_1)} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{(\zeta_2 - \zeta_1)^2} \int_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} f(u)\, du. \tag{3.8}$$

Adding the equalities (3.1)-(3.8), we get

$$\sum_{i-1}^{8} I_i$$

$$
\begin{aligned}
= \quad & \frac{1}{4\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} f\left(u\right) du + \frac{-1}{6\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right) \\
+ & \frac{5}{12\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} f\left(u\right) du + \frac{-1}{6\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right) \\
+ & \frac{5}{12\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} f\left(u\right) du + \frac{1}{4\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) \\
- & \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} f\left(u\right) du + \frac{1}{4\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} f\left(u\right) du \\
+ & \frac{-1}{6\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right) + \frac{5}{12\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{\zeta_1+3\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} f\left(u\right) du \\
+ & \frac{-1}{6\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right) + \frac{5}{12\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{3\zeta_1+\zeta_2}{4}} f\left(u\right) du \\
+ & \frac{1}{4\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{\left(\zeta_2-\zeta_1\right)^2}\int_{\zeta_1}^{\frac{\zeta_1+3\zeta_2}{4}} f\left(u\right) du \\
= \quad & \frac{4}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) + \frac{4}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{2}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right) \\
- & \frac{2}{\left(\zeta_2-\zeta_1\right)^2}\int_{\zeta_1}^{\zeta_2} f\left(u\right) du. \tag{3.9}
\end{aligned}
$$

Multiplying the equality (3.9) with $\frac{\zeta_2-\zeta_1}{2}$ and subtracting $\frac{2(1-\Omega)}{\beta(\Omega)} f(k)$, we obtain

$$
\begin{aligned}
& \sum_{i-1}^{8} I_i \frac{\zeta_2-\zeta_1}{2} - \frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)} f\left(k\right) \\
= \quad & \frac{4}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{3\zeta_1+\zeta_2}{4}\right)\frac{\zeta_2-\zeta_1}{2} + \frac{4}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\frac{\zeta_2-\zeta_1}{2} \\
- \quad & \frac{2}{3\left(\zeta_2-\zeta_1\right)} f\left(\frac{\zeta_1+\zeta_2}{2}\right)\frac{\zeta_2-\zeta_1}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f\left(u\right) du - \frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)} f\left(k\right) \\
= \quad & \frac{2}{3} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) + \frac{2}{3} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{1}{3} f\left(\frac{\zeta_1+\zeta_2}{2}\right) \\
- \quad & \frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left(\frac{\Omega}{\beta\left(\Omega\right)}\int_{\zeta_1}^{k} f\left(u\right) du - \frac{\left(1-\Omega\right)}{\beta\left(\Omega\right)} f\left(k\right) + \frac{\Omega}{\beta\left(\Omega\right)}\int_{k}^{\zeta_2} f\left(u\right) du - \frac{\left(1-\Omega\right)}{\beta\left(\Omega\right)} f\left(k\right)\right) \\
= \quad & \frac{2}{3} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) + \frac{2}{3} f\left(\frac{\zeta_1+3\zeta_2}{4}\right) - \frac{1}{3} f\left(\frac{\zeta_1+\zeta_2}{2}\right) \\
- \quad & \frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left[\left(_{\zeta_1}^{CF}I^{\Omega}f\right)\left(k\right) + \left(^{CF}I_{\zeta_2}^{\Omega}f\right)\left(k\right)\right].
\end{aligned}
$$

The proof of Lemma 3.1 is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4   Milne's-formula-type inequalities for s-convex functions

In this section, the Milne's formula-type inequalities are presented for s-convex functions.

Theorem 4.1. *Let assumptions of Lemma 3.1 hold. If $|f'|$ is s-convex on $[\zeta_1, \zeta_2]$ and some fixed $s \in (0, 1]$, then the following fractional inequality holds:*

$$
\begin{aligned}
& \left|\left[\frac{2}{3} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{3} f\left(\frac{\zeta_1+\zeta_2}{2}\right) + \frac{2}{3} f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\right]\right. \\
& \left. - \frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left[\left(_{\zeta_1}^{CF}I^{\Omega}f\right)\left(k\right) + \left(^{CF}I_{\zeta_2}^{\Omega}f\right)\left(k\right)\right] + \frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)} f\left(k\right)\right| \\
& \leq \quad \frac{\zeta_2-\zeta_1}{2}\left[\frac{s\left(3^{s+1}-1\right)+3\times 2^{s+2}-7\times 3^{s+1}+3\times 2^{2s+3}-5}{3\times 4^{s+1}\left(s+1\right)\left(s+2\right)}\right]\left(\left|f'\left(\zeta_1\right)\right|+\left|f'\left(\zeta_2\right)\right|\right).
\end{aligned}
$$

Proof. By using the modulus of Lemma 3.1, since s-convexity of $|f'|$, we get

$$
\left|\left[\frac{2}{3} f\left(\frac{3\zeta_1+\zeta_2}{4}\right) - \frac{1}{3} f\left(\frac{\zeta_1+\zeta_2}{2}\right) + \frac{2}{3} f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\right]\right.
$$

$$-\frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left[\left(_{\zeta_1}^{CF}I^\Omega f\right)(k)+\left(^{CF}I_{\zeta_2}^\Omega f\right)(k)\right]+\frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)}f\left(k\right)\Bigg|$$

$$\leq\quad\frac{\zeta_2-\zeta_1}{2}\left[\int_0^{\frac{1}{4}}|\delta|\left[\left|f'\left(\delta\zeta_2+(1-\delta)\zeta_1\right)\right|+\left|f'\left(\delta\zeta_1+(1-\delta)\zeta_2\right)\right|\right]d\delta\right.$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\delta-\frac{2}{3}\right|\left[\left|f'\left(\delta\zeta_2+(1-\delta)\zeta_1\right)\right|+\left|f'\left(\delta\zeta_1+(1-\delta)\zeta_2\right)\right|\right]d\delta$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\delta-\frac{1}{3}\right|\left[\left|f'\left(\delta\zeta_2+(1-\delta)\zeta_1\right)\right|+\left|f'\left(\delta\zeta_1+(1-\delta)\zeta_2\right)\right|\right]d\delta$$

$$+\int_{\frac{3}{4}}^{1}|\delta-1|\left[\left|f'\left(\delta\zeta_2+(1-\delta)\zeta_1\right)\right|+\left|f'\left(\delta\zeta_1+(1-\delta)\zeta_2\right)\right|\right]d\delta\Bigg]$$

$$\leq\quad\frac{\zeta_2-\zeta_1}{2}\left[\int_0^{\frac{1}{4}}|\delta|\,\delta^s\left|f'\left(\zeta_2\right)\right|+\int_0^{\frac{1}{4}}|\delta|\left(1-\delta\right)^s\left|f'\left(\zeta_1\right)\right|\right.$$

$$+\int_0^{\frac{1}{4}}|\delta|\,\delta^s\left|f'\left(\zeta_1\right)\right|+\int_0^{\frac{1}{4}}|\delta|\left(1-\delta\right)^s\left|f'\left(\zeta_2\right)\right|$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\delta-\frac{2}{3}\right|\delta^s\left|f'\left(\zeta_2\right)\right|+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\delta-\frac{2}{3}\right|\left(1-\delta\right)^s\left|f'\left(\zeta_1\right)\right|$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\delta-\frac{2}{3}\right|\delta^s\left|f'\left(\zeta_1\right)\right|+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\delta-\frac{2}{3}\right|\left(1-\delta\right)^s\left|f'\left(\zeta_2\right)\right|$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\delta-\frac{1}{3}\right|\delta^s\left|f'\left(\zeta_2\right)\right|+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\delta-\frac{1}{3}\right|\left(1-\delta\right)^s\left|f'\left(\zeta_1\right)\right|$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\delta-\frac{1}{3}\right|\delta^s\left|f'\left(\zeta_1\right)\right|+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\delta-\frac{1}{3}\right|\left(1-\delta\right)^s\left|f'\left(\zeta_2\right)\right|$$

$$+\int_{\frac{3}{4}}^{1}|\delta-1|\,\delta^s\left|f'\left(\zeta_2\right)\right|+\int_{\frac{3}{4}}^{1}|\delta-1|\left(1-\delta\right)^s\left|f'\left(\zeta_1\right)\right|$$

$$+\int_{\frac{3}{4}}^{1}|\delta-1|\,\delta^s\left|f'\left(\zeta_1\right)\right|+\int_{\frac{3}{4}}^{1}|\delta-1|\left(1-\delta\right)^s\left|f'\left(\zeta_2\right)\right|\Bigg]$$

$$\leq\quad\frac{\zeta_2-\zeta_1}{2}\left[\frac{s\left(3^{s+1}-1\right)+3\times2^{s+2}-7\times3^{s+1}+3\times2^{2s+3}-5}{3\times4^{s+1}\left(s+1\right)\left(s+2\right)}\right]\left(\left|f'\left(\zeta_1\right)\right|+\left|f'\left(\zeta_2\right)\right|\right).$$

The proof of Theorem 4.1 is finished. □

Corollary 4.2. *If we put $s=1$ in Theorem 4.1, then we have*

$$\left|\left[\frac{2}{3}f\left(\frac{3\zeta_1+\zeta_2}{4}\right)-\frac{1}{3}f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2}{3}f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\right]\right.$$

$$\left.-\frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left[\left(_{\zeta_1}^{CF}I^\Omega f\right)(k)+\left(^{CF}I_{\zeta_2}^\Omega f\right)(k)\right]+\frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)}f\left(k\right)\right|$$

$$\leq\quad\frac{5\left(\zeta_2-\zeta_1\right)}{48}\left(\left|f'\left(\zeta_1\right)\right|+\left|f'\left(\zeta_2\right)\right|\right). \tag{4.1}$$

Remark 4.3. If we put $\Omega=1$ and $\beta\left(0\right)=\beta\left(1\right)=1$ in inequality (4.1) which is proved by Ali et. al in [3].

Theorem 4.4. *Suppose the assumptions of Lemma 3.1. If $|f'|^q$ is s-convex on $[\zeta_1,\zeta_2]$, some fixed $s\in(0,1]$ and $q>1$, then the following fractional inequality holds:*

$$\left|\left[\frac{2}{3}f\left(\frac{3\zeta_1+\zeta_2}{4}\right)-\frac{1}{3}f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{2}{3}f\left(\frac{\zeta_1+3\zeta_2}{4}\right)\right]\right.$$

$$\left.-\frac{\beta\left(\Omega\right)}{\Omega\left(\zeta_2-\zeta_1\right)}\left[\left(_{\zeta_1}^{CF}I^\Omega f\right)(k)+\left(^{CF}I_{\zeta_2}^\Omega f\right)(k)\right]+\frac{2\left(1-\Omega\right)}{\beta\left(\Omega\right)}f\left(k\right)\right|$$

$$\leq\quad\frac{\zeta_2-\zeta_1}{2}\left[\left(\frac{1}{4^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{4^{s+1}\left(1+s\right)}\left|f'\left(\zeta_2\right)\right|^q+\frac{3^s\times4^{-1-s}\left(-3+3^{-s}\times4^{1+s}\right)}{1+s}\left|f'\left(\zeta_1\right)\right|^q\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{1}{4^{s+1}\left(1+s\right)}\left|f'\left(\zeta_1\right)\right|^q+\frac{3^s\times4^{-1-s}\left(-3+3^{-s}\times4^{1+s}\right)}{1+s}\left|f'\left(\zeta_2\right)\right|^q\right)^{\frac{1}{q}}\right\}$$

$$+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\frac{2^{-2-2s}\left(-1+2^{1+s}\right)}{1+s} \left| f'(\zeta_2) \right|^q - \frac{2^{-2-2s}\left(2^{1+s}-3^{1+s}\right)}{1+s} \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\frac{2^{-2-2s}\left(-1+2^{1+s}\right)}{1+s} \left| f'(\zeta_1) \right|^q - \frac{2^{-2-2s}\left(2^{1+s}-3^{1+s}\right)}{1+s} \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(-\frac{2^{-2-2s}\left(2^{1+s}-3^{1+s}\right)}{1+s} \left| f'(\zeta_2) \right|^q + \frac{2^{-2-2s}\left(-1+2^{1+s}\right)}{1+s} \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(-\frac{2^{-2-2s}\left(2^{1+s}-3^{1+s}\right)}{1+s} \left| f'(\zeta_1) \right|^q + \frac{2^{-2-2s}\left(-1+2^{1+s}\right)}{1+s} \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{3^s \times 4^{-1-s}\left(-3+3^{-s}\times 4^{1+s}\right)}{1+s} \left| f'(\zeta_2) \right|^q + \frac{1}{4^{s+1}(1+s)} \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\frac{3^s \times 4^{-1-s}\left(-3+3^{-s}\times 4^{1+s}\right)}{1+s} \left| f'(\zeta_1) \right|^q + \frac{1}{4^{s+1}(1+s)} \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\} \right].$$

Proof. By employing the Hölder inequality since s-convexity of $|f'|^q$, we get

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| \left[\left| f'(\delta\zeta_2 + (1-\delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right| \right] d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left[\left| f'(\delta\zeta_2 + (1-\delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right| \right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left[\left| f'(\delta\zeta_2 + (1-\delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right| \right] d\delta$$

$$+ \left. \int_{\frac{3}{4}}^{1} |\delta - 1| \left[\left| f'(\delta\zeta_2 + (1-\delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right| \right] d\delta \right]$$

$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\left(\int_0^{\frac{1}{4}} |\delta|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\int_0^{\frac{1}{4}} \delta^s \left| f'(\zeta_2) \right|^q + \int_0^{\frac{1}{4}} (1-\delta)^s \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right. \right.$$

$$+ \left. \left(\int_0^{\frac{1}{4}} \delta^s \left| f'(\zeta_1) \right|^q + \int_0^{\frac{1}{4}} (1-\delta)^s \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \delta^s \left| f'(\zeta_2) \right|^q + \int_{\frac{1}{4}}^{\frac{1}{2}} (1-\delta)^s \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \delta^s \left| f'(\zeta_1) \right|^q + \int_{\frac{1}{4}}^{\frac{1}{2}} (1-\delta)^s \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \delta^s \left| f'(\zeta_2) \right|^q + \int_{\frac{1}{2}}^{\frac{3}{4}} (1-\delta)^s \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \delta^s \left| f'(\zeta_1) \right|^q + \int_{\frac{1}{2}}^{\frac{3}{4}} (1-\delta)^s \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{3}{4}}^{1} |\delta - 1|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\int_{\frac{3}{4}}^{1} \delta^s \left| f'(\zeta_2) \right|^q + \int_{\frac{3}{4}}^{1} (1-\delta)^s \left| f'(\zeta_1) \right|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left. \left(\int_{\frac{3}{4}}^{1} \delta^s \left| f'(\zeta_1) \right|^q + \int_{\frac{3}{4}}^{1} (1-\delta)^s \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}} \right\} \right]$$

$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{1}{4^{s+1}(1+s)} |f'(\zeta_2)|^q + \frac{3^s \times 4^{-1-s}(-3+3^{-s} \times 4^{1+s})}{1+s} |f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right. \right.$$

$$+ \left. \left(\frac{1}{4^{s+1}(1+s)} |f'(\zeta_1)|^q + \frac{3^s \times 4^{-1-s}(-3+3^{-s} \times 4^{1+s})}{1+s} |f'(\zeta_2)|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(\frac{2^{-2-2s}(-1+2^{1+s})}{1+s} |f'(\zeta_2)|^q - \frac{2^{-2-2s}(2^{1+s}-3^{1+s})}{1+s} |f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\frac{2^{-2-2s}(-1+2^{1+s})}{1+s} |f'(\zeta_1)|^q - \frac{2^{-2-2s}(2^{1+s}-3^{1+s})}{1+s} |f'(\zeta_2)|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right|^p d\delta \right)^{\frac{1}{p}} \left\{ \left(-\frac{2^{-2-2s}(2^{1+s}-3^{1+s})}{1+s} |f'(\zeta_2)|^q + \frac{2^{-2-2s}(-1+2^{1+s})}{1+s} |f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(-\frac{2^{-2-2s}(2^{1+s}-3^{1+s})}{1+s} |f'(\zeta_1)|^q + \frac{2^{-2-2s}(-1+2^{1+s})}{1+s} |f'(\zeta_2)|^q \right)^{\frac{1}{q}} \right\}$$

$$+ \left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{3^s \times 4^{-1-s}(-3+3^{-s} \times 4^{1+s})}{1+s} |f'(\zeta_2)|^q + \frac{1}{4^{s+1}(1+s)} |f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left. \left(\frac{3^s \times 4^{-1-s}(-3+3^{-s} \times 4^{1+s})}{1+s} |f'(\zeta_1)|^q + \frac{1}{4^{s+1}(1+s)} |f'(\zeta_2)|^q \right)^{\frac{1}{q}} \right\} \right].$$

The proof of Theorem 4.4 is finished. □

Corollary 4.5. *If we chosen $s = 1$ in Theorem 4.4, then we have*

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right| \qquad (4.2)$$

$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{7|f'(\zeta_2)|^q + |f'(\zeta_1)|^q}{32} \right)^{\frac{1}{q}} + \left(\frac{7|f'(\zeta_1)|^q + |f'(\zeta_2)|^q}{32} \right)^{\frac{1}{q}} \right\} \right.$$

$$+ \left. \left(\frac{5^{p+1}}{12^{p+1}(p+1)} - \frac{1}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{3|f'(\zeta_2)|^q + 5|f'(\zeta_1)|^q}{32} \right)^{\frac{1}{q}} + \left(\frac{3|f'(\zeta_1)|^q + 5|f'(\zeta_2)|^q}{32} \right)^{\frac{1}{q}} \right\} \right].$$

Remark 4.6. If we put $\Omega = 1$ and $\beta(0) = \beta(1) = 1$ in inequality (4.2) which is proved by Ali et al. in [3].

5 Milne's-formula-type inequalities for quasi-convex functions

In this section, the Milne's formula-type inequalities are obtained for quasi-convex functions.

Theorem 5.1. *Under the assumptions of Lemma 3.1. If $|f'|$ is quasi-convex on $[\zeta_1, \zeta_2]$, then the following fractional inequality holds:*

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \quad \frac{5(\zeta_2 - \zeta_1)}{48} \max \left(|f'(\zeta_1)|, |f'(\zeta_2)| \right).$$

Proof. From Lemma 3.1, by utilizing properties of modulus and $|f'|$ is quasi-convex, we get

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta$$

$$+ \left. \int_{\frac{3}{4}}^{1} |\delta - 1| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta \right]$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\left\{ \int_0^{\frac{1}{4}} |\delta| \max\left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta + \int_0^{\frac{1}{4}} |\delta| \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\} \right.$$

$$+ \left\{ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \max\left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\}$$

$$+ \left\{ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \max\left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\}$$

$$+ \left. \left\{ \int_{\frac{3}{4}}^{1} |\delta - 1| \max\left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta + \int_{\frac{3}{4}}^{1} |\delta - 1| \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\} \right]$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| + \int_{\frac{3}{4}}^{1} |\delta - 1| \right] \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right)$$

$$\leq \frac{5(\zeta_2 - \zeta_1)}{48} \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right).$$

This completes the proof. □

Theorem 5.2. *Under the assumptions of Lemma 3.1. If $|f'|$ is quasi-convex on $[\zeta_1, \zeta_2]$ and $q > 1$, then the following fractional inequality holds:*

$$\left| \left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}_{\zeta_1}^{CF} I^\Omega f \right)(k) + \left({}^{CF} I_{\zeta_2}^\Omega f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} + \left(\frac{5^{p+1}}{12^{p+1}(p+1)} - \frac{1}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \right] \max\left(\left| f'(\zeta_1) \right|^q, \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}}.$$

Proof. From Lemma 3.1, by utilizing Hölder inequality and $|f'|^q$ is quasi-convex, we get

$$\left| \left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}_{\zeta_1}^{CF} I^\Omega f \right)(k) + \left({}^{CF} I_{\zeta_2}^\Omega f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta$$

$$+ \left. \int_{\frac{3}{4}}^{1} |\delta - 1| \left[\left| f'\left(\delta\zeta_2 + (1-\delta)\zeta_1 \right) \right| + \left| f'\left(\delta\zeta_1 + (1-\delta)\zeta_2 \right) \right| \right] d\delta \right]$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\left\{ \left(\int_0^{\frac{1}{4}} |\delta|^p \, d\delta \right)^{\frac{1}{p}} \max\left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta \right. \right.$$

$$+ \left. \left(\int_0^{\frac{1}{4}} |\delta|^p \, d\delta \right)^{\frac{1}{p}} \max\left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\}$$

$$+ \left\{ \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta \right.$$

$$+ \left. \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\}$$

$$+ \left\{ \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta \right.$$

$$+ \left. \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\}$$

$$\left\{ \left(\int_{\frac{3}{4}}^{1} \left| \delta - 1 \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_2) \right|, \left| f'(\zeta_1) \right| \right) d\delta \right.$$

$$+ \left. \left. \left(\int_{\frac{3}{4}}^{1} \left| \delta - 1 \right|^p d\delta \right)^{\frac{1}{p}} \max \left(\left| f'(\zeta_1) \right|, \left| f'(\zeta_2) \right| \right) d\delta \right\} \right]$$

$$\leq \frac{\zeta_2 - \zeta_1}{2} \left[\left(\frac{1}{4^{p+1}(p+1)} \right)^{\frac{1}{p}} + \left(\frac{5^{p+1}}{12^{p+1}(p+1)} - \frac{1}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \right] \max \left(\left| f'(\zeta_1) \right|^q, \left| f'(\zeta_2) \right|^q \right)^{\frac{1}{q}}.$$

This completes the proof. □

6 Milne's-formula-type ineqalities for Bounded functions

In this section, the Milne's formula-type inequalities we are established for bounded functions.

Theorem 6.1. *Under the assumptions of Lemma 3.1. If there exist constants $-\infty < m < M < +\infty$ such that $m < f'(x) < M$ for all $x \in [\zeta_1, \zeta_2]$, then the following fractional inequality holds:*

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \frac{5(\zeta_2 - \zeta_1)(M - m)}{48}.$$

Proof. From Lemma 3.1, we get

$$\left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right]$$

$$- \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1-\Omega)}{\beta(\Omega)} f(k)$$

$$= \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} \delta \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) + f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right] d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3} \right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) + f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right] d\delta$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3} \right) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) + f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right] d\delta$$

$$+ \left. \int_{\frac{3}{4}}^{1} (\delta - 1) \left[f'(\delta\zeta_2 + (1-\delta)\zeta_1) + f'(\delta\zeta_1 + (1-\delta)\zeta_2) \right] d\delta \right]$$

$$= \frac{\zeta_2 - \zeta_1}{2} \left[\left\{ \int_0^{\frac{1}{4}} \delta \left(f'(\delta\zeta_2 + (1-\delta)\zeta_1) - \frac{m+M}{2} \right) d\delta \right. \right.$$

$$+ \left. \int_0^{\frac{1}{4}} \delta \left(f'(\delta\zeta_1 + (1-\delta)\zeta_2) - \frac{m+M}{2} \right) d\delta \right\}$$

$$+ \left\{ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3} \right) \left(f'(\delta\zeta_2 + (1-\delta)\zeta_1) - \frac{m+M}{2} \right) d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\delta - \frac{2}{3} \right) \left(f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right) d\delta \Bigg\}$$

$$+ \left\{ \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3} \right) \left(f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right) d\delta \right.$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\delta - \frac{1}{3} \right) \left(f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right) d\delta \Bigg\}$$

$$+ \left\{ \int_{\frac{3}{4}}^{1} \left(\delta - 1 \right) \left(f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right) d\delta \right.$$

$$+ \int_{\frac{3}{4}}^{1} \left(\delta - 1 \right) \left(f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right) d\delta \Bigg\} \Bigg]. \tag{6.1}$$

Taking the absolute value in both sides of equality (6.1), we get

$$\left| \left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}_{\zeta_1}^{CF} I^{\Omega} f \right)(k) + \left({}^{CF} I_{\zeta_2}^{\Omega} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\left\{ \int_0^{\frac{1}{4}} |\delta| \left| f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right| d\delta \right. \right.$$

$$+ \left| \int_0^{\frac{1}{4}} |\delta| \right| \left| f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right| d\delta \Bigg\}$$

$$+ \left\{ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left| f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right| d\delta \right.$$

$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left| f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right| d\delta \Bigg\}$$

$$+ \left\{ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left| f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right| d\delta \right.$$

$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left| f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right| d\delta \Bigg\}$$

$$+ \left\{ \int_{\frac{3}{4}}^{1} |\delta - 1| \left| f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right| d\delta \right.$$

$$+ \int_{\frac{3}{4}}^{1} |\delta - 1| \left| f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right| d\delta \Bigg\} \Bigg]. \tag{6.2}$$

Since $m < f'(x) < M$ for all $x \in [\zeta_1, \zeta_2]$, we get

$$\left| f'\left(\delta \zeta_2 + (1 - \delta) \zeta_1 \right) - \frac{m + M}{2} \right| \leq \frac{M - m}{2} \tag{6.3}$$

and

$$\left| f'\left(\delta \zeta_1 + (1 - \delta) \zeta_2 \right) - \frac{m + M}{2} \right| \leq \frac{M - m}{2}. \tag{6.4}$$

Using the inequalities (6.3)-(6.4) in (6.2), we have

$$\left| \left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}_{\zeta_1}^{CF} I^{\Omega} f \right)(k) + \left({}^{CF} I_{\zeta_2}^{\Omega} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$

$$\leq \quad \frac{(\zeta_2 - \zeta_1)(M - m)}{2} \left[\int_0^{\frac{1}{4}} |\delta| + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| + \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| + \int_{\frac{3}{4}}^{1} |\delta - 1| \right]$$

$$\leq \quad \frac{5(M - m)(\zeta_2 - \zeta_1)}{48}.$$

This completes the proof.   □

7   Milne's-formula-type inequalities for Lipschitzain functions

In this section, the Milne's formula-type inequalities are presented for Lipschitzain functions.

Theorem 7.1. *Under the assumptions of Lemma 3.1. If f' is a Lipschitzain function on $[\zeta_1, \zeta_2]$, then the following fractional inequality holds:*

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$
$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$
$$\leq \quad \frac{(\zeta_2 - \zeta_1)^2}{24} L.$$

Proof. From Lemma 3.1, f' is Lipschitzain function, we get

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$
$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$
$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| \left[\left| f'(\delta\zeta_2 + (1 - \delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1 - \delta)\zeta_2) \right| \right] d\delta \right.$$
$$+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| \left[\left| f'(\delta\zeta_2 + (1 - \delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1 - \delta)\zeta_2) \right| \right] d\delta$$
$$+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| \left[\left| f'(\delta\zeta_2 + (1 - \delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1 - \delta)\zeta_2) \right| \right] d\delta$$
$$+ \left. \int_{\frac{3}{4}}^{1} |\delta - 1| \left[\left| f'(\delta\zeta_2 + (1 - \delta)\zeta_1) \right| + \left| f'(\delta\zeta_1 + (1 - \delta)\zeta_2) \right| \right] d\delta \right]$$
$$\leq \quad \frac{\zeta_2 - \zeta_1}{2} \left[\int_0^{\frac{1}{4}} |\delta| L\delta(\zeta_2 - \zeta_1) d\delta + \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \delta - \frac{2}{3} \right| L\delta(\zeta_2 - \zeta_1) d\delta \right.$$
$$+ \left. \int_{\frac{1}{2}}^{\frac{3}{4}} \left| \delta - \frac{1}{3} \right| L\delta(\zeta_2 - \zeta_1) d\delta + \int_{\frac{3}{4}}^{1} |\delta - 1| L\delta(\zeta_2 - \zeta_1) d\delta \right]$$
$$\leq \quad \frac{(\zeta_2 - \zeta_1)^2}{24} L.$$

This completes the proof. □

8 Milne's-formula-type inequalities for Functions of Bounded Variation

In this section, we represent Milne's formaula type inequalities for functions of bounded variation.

Theorem 8.1. *Consider that $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ is a function of bounded variation on $[\zeta_1, \zeta_2]$. Then, we have*

$$\left| \left[\frac{2}{3} f \left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f \left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f \left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right.$$
$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$
$$\leq \frac{5}{12} \bigvee_{\zeta_1}^{\zeta_2} (f).$$

Here, $\bigvee_{\zeta_1}^{\zeta_2} (f)$ denotes the total variation of f on $[\zeta_1, \zeta_2]$.

Proof. Define the function $K_\Omega(x)$ by

$$K_\Omega(x) = \begin{cases} x - \zeta_1, & \zeta_1 \leq x < \frac{3\zeta_1 + \zeta_2}{4}, \\ x - \frac{\zeta_1 + 2\zeta_2}{3}, & \frac{3\zeta_1 + \zeta_2}{4} \leq x < \frac{\zeta_1 + \zeta_2}{2}, \\ x - \frac{2\zeta_1 + \zeta_2}{3}, & \frac{\zeta_1 + \zeta_2}{2} \leq x < \frac{\zeta_1 + 3\zeta_2}{4}, \\ x - \zeta_2, & \frac{\zeta_1 + 3\zeta_2}{4} \leq x \leq \zeta_2. \end{cases}$$

With the aid of the integrating by parts, we have

$$\frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} K_\Omega(x) df(x) \tag{8.1}$$

$$= \frac{1}{\zeta_2 - \zeta_1} \left(\int_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} (x - \zeta_1) \, df(x) + \int_{\frac{3\zeta_1 + \zeta_2}{6}}^{\frac{\zeta_1 + \zeta_2}{2}} \left(x - \frac{\zeta_1 + 2\zeta_2}{3} \right) df(x) \right.$$

$$\left. + \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\frac{\zeta_1 + 3\zeta_2}{4}} \left(x - \frac{2\zeta_1 + \zeta_2}{3} \right) df(x) + \int_{\frac{\zeta_1 + 3\zeta_2}{4}}^{\zeta_2} (x - \zeta_2) \, df(x) \right)$$

$$= \frac{1}{\zeta_2 - \zeta_1} \left((x - \zeta_1) f(x)\Big|_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} - \int_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} f(x) dx + \left(x - \frac{\zeta_1 + 2\zeta_2}{3} \right)\Big|_{\frac{3\zeta_1 + \zeta_2}{4}}^{\frac{\zeta_1 + \zeta_2}{2}} - \int_{\frac{3\zeta_1 + \zeta_2}{4}}^{\frac{\zeta_1 + \zeta_2}{2}} f(x) dx \right.$$

$$\left. + \left(x - \frac{2\zeta_1 + \zeta_2}{3} \right) f(x)\Big|_{\frac{\zeta_1 + \zeta_2}{2}}^{\frac{\zeta_1 + 3\zeta_2}{4}} - \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\frac{\zeta_1 + 3\zeta_2}{4}} f(x) dx + (x - \zeta_2) f(x)\Big|_{\frac{\zeta_1 + 3\zeta_2}{4}}^{\zeta_2} - \int_{\frac{\zeta_1 + 3\zeta_2}{4}}^{\zeta_2} f(x) dx \right)$$

$$= \frac{1}{\zeta_2 - \zeta_1} \left(\left(\frac{\zeta_2 - \zeta_1}{4} \right) f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \left(\frac{\zeta_2 - \zeta_1}{6} \right) f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + 5 \left(\frac{\zeta_2 - \zeta_1}{12} \right) f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) \right.$$

$$\left. + 5 \left(\frac{\zeta_2 - \zeta_1}{12} \right) f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \left(\frac{\zeta_2 - \zeta_1}{6} \right) f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \left(\frac{\zeta_2 - \zeta_1}{4} \right) f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \int_{\zeta_1}^{\zeta_2} f(x) dx \right)$$

$$= \frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(x) dx.$$

In other words, we have

$$\left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right]$$

$$- \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k)$$

$$\leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} K_\Omega(x) df(x).$$

It is known that if $g, f : [\zeta_1, \zeta_2] \to \mathbb{R}$ are such that g is continuous on $[\zeta_1, \zeta_2]$ and f is of bounded variation on $[\zeta_1, \zeta_2]$, then $\int_{\zeta_1}^{\zeta_2} g(t) df(t)$ exist and

$$\left| \int_{\zeta_1}^{\zeta_2} g(t) df(t) \right| \leq \sup_{t \in [\zeta_1, \zeta_2]} |g(t)| \bigvee_{\zeta_1}^{\zeta_2}(f). \tag{8.2}$$

By using (8.2), it yields

$$\left| \left[\frac{2}{3} f\left(\frac{3\zeta_1 + \zeta_2}{4} \right) - \frac{1}{3} f\left(\frac{\zeta_1 + \zeta_2}{2} \right) + \frac{2}{3} f\left(\frac{\zeta_1 + 3\zeta_2}{4} \right) \right] \right. \tag{8.3}$$

$$\left. - \frac{\beta(\Omega)}{\Omega(\zeta_2 - \zeta_1)} \left[\left({}^{CF}_{\zeta_1} I^\Omega f \right)(k) + \left({}^{CF} I^\Omega_{\zeta_2} f \right)(k) \right] + \frac{2(1 - \Omega)}{\beta(\Omega)} f(k) \right|$$

$$= \frac{1}{\zeta_2 - \zeta_1} \left| \int_{\zeta_1}^{\zeta_2} K_\Omega(x) df(x) \right|$$

$$\leq \frac{1}{\zeta_2 - \zeta_1} \left\{ \left| \int_{\zeta_1}^{\frac{3\zeta_1 + \zeta_2}{4}} (x - \zeta_1) \, df(x) \right| + \left| \int_{\frac{3\zeta_1 + \zeta_2}{6}}^{\frac{\zeta_1 + \zeta_2}{2}} \left(x - \frac{\zeta_1 + 2\zeta_2}{3} \right) df(x) \right| \right.$$

$$+ \left| \int_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} \left(x - \frac{2\zeta_1+\zeta_2}{3} \right) df(x) \right| + \left| \int_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} (x-\zeta_2) df(x) \right| \Bigg\}$$

$$\leq \frac{1}{\zeta_2-\zeta_1} \left\{ \sup_{x \in \left[\zeta_1, \frac{3\zeta_1+\zeta_2}{6}\right]} |x-\zeta_1| \bigvee_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} (f) + \sup_{x \in \left[\frac{3\zeta_1+\zeta_2}{4}, \frac{\zeta_1+\zeta_2}{2}\right]} \left| x - \frac{\zeta_1+2\zeta_2}{3} \right| \bigvee_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+3\zeta_2}{4}} (f) \right.$$

$$\left. + \sup_{x \in \left[\frac{\zeta_1+\zeta_2}{2}, \frac{\zeta_1+3\zeta_2}{4}\right]} \left| x - \frac{2\zeta_1+\zeta_2}{3} \right| \bigvee_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} (f) + \sup_{x \in \left[\frac{\zeta_1+3\zeta_2}{4}, \zeta_2\right]} |x-\zeta_2| \bigvee_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} (f) \right\}$$

$$= \frac{1}{\zeta_2-\zeta_1} \left\{ \frac{\zeta_2-\zeta_1}{4} \bigvee_{\zeta_1}^{\frac{3\zeta_1+\zeta_2}{4}} (f) + \frac{5\zeta_2-5\zeta_a}{12} \bigvee_{\frac{3\zeta_1+\zeta_2}{4}}^{\frac{\zeta_1+\zeta_2}{2}} (f) \right.$$

$$\left. + \frac{5\zeta_2-5\zeta_a}{12} \bigvee_{\frac{\zeta_1+\zeta_2}{2}}^{\frac{\zeta_1+3\zeta_2}{4}} (f) + \frac{\zeta_2-\zeta_1}{4} \bigvee_{\frac{\zeta_1+3\zeta_2}{4}}^{\zeta_2} (f) \right\}$$

$$\leq \frac{5}{12} \bigvee_{\zeta_1}^{\zeta_2} (f).$$

Thus the proof is completed. $\square$

9 Quadrature formula

Let d is the partition of the interval $[\zeta_1, \zeta_2]$, $d : \zeta_1 = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = \zeta_2$ and Let the quadrature formula

$$\int_{\zeta_1}^{\zeta_2} f(x)\, dx = Z(f,d) + R(f,d),$$

where

$$Z(f,d) = \sum_{i=0}^{n-1} \frac{x_{i+1}-x_i}{3} \left[2f\left(\frac{3x_i+x_{i+1}}{4}\right) - f\left(\frac{x_i+x_{i+1}}{2}\right) + 2f\left(\frac{x_i+3x_{i+1}}{4}\right) \right].$$

That the approximation error $R(f,d)$ of the interval I by Milne Formula:

Proposition 9.1. *Under the assumptions of Lemma 3.1, then in Theorem 4.1 for every division d of $[\zeta_1, \zeta_2]$, then the following inequality holds:*

$$|R(f,d)| \leq \sum_{i=0}^{n-1} \frac{5(x_{i+1}-x_i)^2}{48} \left[\left(|f'(x_i)| + |f'(x_{i+1})| \right) \right].$$

Proof. Applying the Theorem 4.1 on the subinterval $[x_i, x_{i+1}]$, $(i = 0, 1, 2, 3....n-1)$, we have $\square$

$$\left| \frac{1}{3} \left[2f\left(\frac{3x_i+x_{i+1}}{4}\right) - f\left(\frac{x_i+x_{i+1}}{2}\right) + 2f\left(\frac{x_i+3x_{i+1}}{4}\right) \right] - \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \qquad (9.1)$$

$$\leq \quad \frac{5(x_{i+1}-x_i)}{48} \left[|f'(x_i)| + |f'(x_{i+1})| \right].$$

Summing over i from 0 to $n-1$ and multiplying by $(x_{i+1}-x_i)$ on both sides (9.1), we deduce by the triangle inequality, we attained the result.

This completes the proof.

10 Conclusion

In the present study, we established a new Milne's formula-type identity for differentiable convex function through well know Caputo-Fabrizio fractional integral operator. Employing the new identity to Milne's formula for s-convex, quasi-convex functions, bounded function, and Lipschitzain are proved. Furthermore, we have discussed some applications to the quadrature formula. We know that much research in this intriguing area of inequalities and analysis will focus on our recently identified approach. In future work, it would be fascinating to apply these findings to other convexities, and fractional operators for example modified Caputo-Fabrizio and Atangana Baleanu.

References

[1] *Alomari, M., Darus, M., & Dragomir, S. S. (2009). New inequalities of Simpson's type for s-convex functions with applications. Research report collection, 12(4).*

[2] *Ali, M. A., Kara, H., Tariboon, J., Asawasamrit, S., Budak, H., & Hezenci, F. (2021). Some new Simpson's-formula-type inequalities for twice-differentiable convex functions via generalized fractional operators. Symmetry, 13(12), 2249.*

[3] *Ali, M. A., Zhang, Z., & Feckan, M. (2022). On Some Error Bounds for Milne's Formula in Fractional Calculus. Mathematics, 11(1), 146.*

[4] *Budak, H., Kara, H., & Hezenci, F. (2023). Fractional Simpson-Type Inequalities for Twice Differentiable Functions. Sahand communications in mathematical analysis, 20(3), 97-108.*

[5] *Budak, H., Kara, H., & Hezenci, F. (2023). Fractional Simpson-Type Inequalities for Twice Differentiable Functions. Sahand communications in mathematical analysis, 20(3), 97-108.*

[6] *Budak, H., Sarıkaya, M. Z., & Qayyum, A. (2021). New refinements and applications of Ostrowski type inequalities for mappings whose nth derivatives are of bounded variation. TWMS Journal of Applied and Engineering Mathematics, 11(2), 424-435.*

[7] *Chen, J., & Huang, X. (2017). Some new inequalities of Simpson's type for s-convex functions via fractional integrals. Filomat, 31(15), 4989-4997.*

[8] *Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in fractional differentiation applications, 1(2), 73-85.*

[9] *İ. Demir, A new approach of Milne-type inequalities based on proportional Caputo-Hybrid operator: A new approach for Milne-type inequalities, Journal of Advances in Applied and Computational Mathematics, 10, 2023, 102-119.*

[10] *H. D. Desta, H. Budak, and H. Kara, New perspectives on fractional Milne-type inequalities: Insights from twice-differentiable functions, Universal Journal of Mathematics and Applications, 7(1), 2024, 30-37.*

[11] *Dragomir, S. S., Pecaric, J., & Persson, L. E. (1995). Some inequalities of Hadamard type. Soochow journal of mathematics, 21(3), 335-341.*

[12] *Ertugral, F., & Sarikaya, M. Z. (2019). Simpson type integral inequalities for generalized fractional integral. Revista de la real academia de ciencias exactas, Físicas y naturales. serie a. matemáticas, 113, 3115-3124.*

[13] *Farid, G., Mehmood, S., Rathour, L., Mishra, L. N., & Mishra, V. N.: Fractional Hadamard and Fejer-Hadamard inequalities associated with $exp.(\alpha, h - m)$-convexity. Dyn. Contin. Discr. Impuls. Syst. Ser. A: Math. Anal, **30**, 353-367 (2023).*

[14] *Farid, G., Akbar, S. B., Rathour, L., Mishra, L. N., & Mishra, V. N.: Riemann-Liouville fractional versions of Hadamard inequality for strongly m-convex functions. International Journal of Analysis and Applications, **20**, 5-5(2022).*

[15] *Mishra, V. N., Khatri, K., Mishra, L. N., & Deepmala.: Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators. Journal of inequalities and applications, 2013, 1-11 (2013).*

[16] *Gorenflo, R., & Mainardi, F. (1997). Fractional calculus: integral and differential equations of fractional order (pp. 223-276). Springer Vienna.*

[17] *Gairola, A. R., Maindola, S., Rathour, L., Mishra, L. N., & Mishra, V. N.: Better uniform approximation by new Bivariate Bernstein Operators. International Journal of Analysis and Applications, **20**, 60-60(2022).*

[18] *Gairola, A. R., Bisht, N., Rathour, L., Mishra, L. N., & Mishra, V. N.: Order of Approximation by a New Univariate Kantorovich Type Operator. International journal of analysis and applications, **21**, 106-106 (2023).*

[19] *Hussain, S., Khalid, J., & Chu, Y. M. (2020). Some generalized fractional integral Simpson's type inequalities with applications. Aims math, 5(6), 5859-5883.*

[20] *Hudzik, H., & Maligranda, L. (1994). Some remarks on s-convex functions. Aequationes mathematicae, 48, 100-111.*

[21] *Iqbal, M., Qaisar, S., & Hussain, S. (2017). On Simpson's type inequalities utilizing fractional integrals. J. comput. anal. appl, 23(6), 1137-1145.*

[22] *Iscan, I., & Wu, S. (2014). Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. Applied mathematics and computation, 238, 237-244.*

[23] *Ion, D. A. (2007). Some estimates on the Hermite-Hadamard inequality through quasi-convex functions. Annals of the university of craiova-mathematics and computer science series, 34, 82-87.*

[24] *Junjua, M. U. D., Qayyum, A., Munir, A., Budak, H., Saleem, M. M., & Supadi, S. S. (2024). A study of some new Hermite Hadamard inequalities via specific convex functions withaApplications. Mathematics, 12(3), 478.*

[25] H. Kadakal, *Generalization of Some Integral Inequalities for Arithmetic Harmonically Convex Functions.* Cumhuriyet Science Journal, 43(3), 2022, 497-503.

[26] M. Kadakal, H. Kadakal, I. Iscan, *Semi P-geometric-arithmetically functions and some new related inequalities.* Filomat, 37(21), 2023, 7017-7028.

[27] Kaijser, S., Nikolova, L., Persson, L. E., & Wedestig, A. (2005). *Hardy-type inequalities via convexity.* Mathematical inequalities applications, 8(3), 403-417.

[28] Kashif, A. R., Khan, T. S., Qayyum, A., and Faye, I. (2018). *A comparison and error analysis of error bounds.* International journal of analysis and applications, 16(5), 751-762.

[29] Kermausuor, S. E. T. H. (2021). *Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions.* Kragujevac journal of mathematics, 45(5), 709-720.

[30] Lei, H., Hu, G., Nie, J., & Du, T. (2020). *Generalized Simpson-type inequalities considering first derivatives through the k-Fractional Integrals.* Iaeng Int. J. appl. math, 50(3), 1-8.

[31] Miller, K. S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations.*

[32] Munir, A.,Budak, H., Faiz, I., Qaisar, S. (2024). *Generalizations of Simpson type inequality for (α, m)-convex functions.* Fiomat,(2024), 3295–3312.

[33] Munir, A, A., Qayyum, A., Supadi, S. S., Budak, H., Faiz, I. (2024). *Astudy of improved error bounds for Simpson type inequality via fractional integral operator.* Fiomat, (2024), 3415–3427.

[34] Ozdemir, M. E., Akdemir, A. O., & Kavurmaci, H. (2014). *On the Simpson's inequality for convex functions on the coordinates.* Turkey. J. anal. number theory, 2(5), 165-169.

[35] Patel, P., & Mishra, V. N. (2015). *Approximation properties of certain summation integral type operators.* Demonstratio Mathematica, 48(1), 77-90.

[36] Saker, S. H., Abdou, D. M., & Kubiaczyk, I. (2018). *Opial and Polya type inequalities via convexity.* Fasciculi mathematici.

[37] Sarikaya, M. Z., Set, E., & Ozdemir, M. E. (2010). *On new inequalities of Simpson's type for s-convex functions.* Computers mathematics with applications, 60(8), 2191-2199.

[38] Sarikaya, M. Z., Set, E., & Ozdemir, M. E. (2010) *On new inequalities of Simpson's type for convex functions,* RGMIA Res. Rep. Coll, 13.

[39] Set, E., Akdemir, A. O., & Özdemir, E. M. (2017). *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals.* Filomat, 31(14), 4415-4420.

[40] Qayyum, A., Faye, I., and Shoaib, M. (2015). *On new generalized inequalities via Riemann-Liouville fractional integration.* J. Fract. Calc. Appl, 6(1), 91-100.

[41] Rathour, L., Rehman, A. U., Bibi, S., Farid, G., Mishra, L. N., & Mishra, V. N.: *k-Fractional integral inequalities of Hadamard type for strongly exponentially $(\alpha, h-m)$-convex functions.* Applied mathematics E-Notes, 23, 393-411 (2023).

[42] Vivas-Cortez, M., Abdeljawad, T., Mohammed, P. O., & Rangel-Oliveros, Y. (2020). *Simpson's integral inequalities for twice differentiable convex functions.* Mathematical problems in engineering, 2020, 1-15.

[43] Yang, H., Qaisar, S., Munir, A., & Naeem, M. (2023). *New inequalities via Caputo-Fabrizio integral operator with applications.* Aims mathematics, 8(8), 19391-19412.

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