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On Pell Sequence Spaces

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Abstract In this article, we introduce the recurrence relation of the sequence S_n obtained by the sum of the Pell numbers, and that the sequence of ratios of two consecutive S_n terms is equal to the *silver ratio*. We construct a new triangular analogue of the Pell matrix $\widehat{P} = \left(\widehat{P}_{nk}\right)$ defined by

$$\widehat{P} = \left(\widehat{P}_{nk}\right) = \begin{cases} \dfrac{P_k}{S_n}, & 1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases} \quad (k, n \in \mathbb{N})$$

Further, we introduce sequence spaces $c(\widehat{P})$, $c_0(\widehat{P})$, $\ell_\infty(\widehat{P})$, $\ell_p(\widehat{P})$ $(1 \leq p < \infty)$. Moreover, some inclusion relations for these spaces and examine a few topological characteristics. Furthermore, we construct a basis for the space $\ell_p(\widehat{P})$, calculate $\alpha-$, $\beta-$, $\gamma-$duals of the same space, characterize certain matrix classes, and look at some geometric properties.

1 Introduction and Preliminaries

In ancient times, the denominators of the closest rational approximations to the square root of two have been known as Pell numbers, an infinite sequence of integers have some interesting properties and applications in mathematics. They are named after the mathematician John Pell, although they were actually studied earlier by Indian mathematicians. The Pell numbers (P_n) [35] are defined by the recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2},$$

where $P_0 = 0$, $P_1 = 1$, and $n \geq 2$. A few initial Pell number expansions are as follows.

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots$$

Just as the sequence of ratios of two consecutive Fibonacci sequences converges to the *golden ratio*, the sequence of ratios of two consecutive Pell sequences converges to the *silver ratio*, which is an irrational number [19]

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 1 + \sqrt{2} \quad (\textit{silver ratio}).$$

Santana and Diaz-Barrero [37] defined the sum of Pell as

$$S_n = \frac{P_{n+1} + P_n - 1}{2} = \sum_{k=1}^{n} P_k. \tag{1.1}$$

Bradie [5] rewrite (1.1) equality as

$$P_{2n+1} - 1 = 2\sum_{k=1}^{n} P_{2k}.$$

Horadam [[20], equation(30), p. 249] defined Cassani identity for Pell numbers as follow

$$P_{n-1}P_{n+1} - P_n^2 = (-1)^n$$

and Bradie [5] obtained

$$2P_{n-1}P_n = P_n^2 - P_{n-1}^2 + (-1)^n$$

by substituting its equivalent $2P_n + P_{n-1}$ instead of P_{n+1}. Many researchers Santos and Sills [38], Schur [40], Pan [36], Mansour and Shattuck [22], Kac and Cheung [25], Güncan and Akduman [16], Akduman [1] have also studied on this topic. Let the set of all sequence spaces be represented by ω. The subspaces of ω that are ℓ_∞, c, c_0, and ℓ_p are characterized as $p-$absolutely summable, bounded, convergent and null sequences space, respectively. The spaces c_0, c, ℓ_∞ are Banach spaces under normed by

$$\|u\|_\infty = \sup_{k\in\mathbb{N}} |u_k|$$

and the ℓ_p $(1 \leq p < \infty)$ is a Banach space normed by

$$\|u\|_p = \left(\sum_k |u_k|^p\right)^{\frac{1}{p}}.$$

Moreover, we designate the spaces of all absolutely convergent series, convergent series, bounded series, and $p-$bounded variation sequences, respectively, by the notations ℓ_1, cs, bs, and bv_p. Let $U, V \subset \omega$ and $B = (b_{nk})$ is a real infinite matrix. The matrix B defines a matrix transformation from U to V if for every sequence $u \in U$,

$$Bu = (B_n(u)) = \left(\sum_{k=1}^{\infty} b_{nk}u_k\right) \in U$$

for each $n \in \mathbb{N}$. (U, V) represents the family of all matrices that map from U to V.

$$U_B = \{u \in \omega : Bu \in U\} \tag{1.2}$$

is a sequence space that defines the B's matrix domain U_B in a sequence space U. Moreover, the sequence space U_B is a $BK-$space normed by $\|u\|_{U_B} = \|Bu\|_U$ if B is a triangular matrix and U is a $BK-$space. Several authors have utilized sepcial numbers in summability theory, including Erdem et al. [9], Demiriz and Şahin [7], Yaying et al. [42, 43, 44, 45, 46], Mursaleen et al. [24], Bekar [4], Atabey et al. [2], Dağlı and Yaying [6], Karakaş [30], Karakaş and Et [31].

Pell number sequence is a number sequence with interesting properties like Fibonacci number sequence. Matrix transformation sequence spaces formed by Cesàro, Abel and Borel matrices have an important place in summability theory. The aim of this study is to bring an innovation to summability theory by obtaining new transformation sequence spaces by means of analog matrix formed by using Pell sequences. We organise the article as follows: In Section 2, we introduce several Identities of Pell numbers. We present the sequence spaces $c(\widehat{P})$, $c_0(\widehat{P})$, $\ell_\infty(\widehat{P})$, $\ell_p(\widehat{P})$ $(1 \leq p < \infty)$ with Pell matrices. In Section 3, $\alpha-$, $\beta-$, $\gamma-$ duals of the $\ell_p(\widehat{P})$ are discussed. Several matrix classes $(\ell_p(\widehat{P}), U)$ are characterized in this section. Further, certain geometric properties of the space $\ell_p(\widehat{P})$ $(1 < p < \infty)$ are given in this section. In Section 4, we have discussed our conclusion.

2 Identities and Corollaries

Let's start our study by giving an identity that gives the sum of Pell numbers.

Theorem 2.1. *Let's define* $S_n = \dfrac{P_{n+1} + P_n - 1}{2}$, *then* $S_n = 2S_{n-1} + S_{n-2} + 1$ *recurrence relation is obtained as in Pell.*

Proof. Let's write S_{n-1} and S_{n-2} according to the definition of S_n and substitute them on the right side of the equation.

$$S_{n-1} = \frac{P_n + P_{n-1} - 1}{2} \quad \text{and} \quad S_{n-2} = \frac{P_{n-1} + P_{n-2} - 1}{2}$$

$$\begin{aligned} 2S_{n-1} + S_{n-2} + 1 &= 2\left(\frac{P_n + P_{n-1} - 1}{2}\right) + \frac{P_{n-1} + P_{n-2} - 1}{2} + 1 \\ &= P_n + P_{n-1} + \frac{P_{n-1} + P_{n-2} - 1}{2} = \frac{2P_n + 3P_{n-1} + P_{n-2} - 1}{2} \\ &= \frac{\overbrace{2P_n + P_{n-1}}^{P_{n+1}} + \overbrace{2P_{n-1} + P_{n-2}}^{P_n} - 1}{2} = \frac{P_{n+1} + P_n - 1}{2} \\ &= S_n = \{0, 1, 3, 8, 20, 49, 119 \ldots\} \end{aligned}$$

□

Corollary 2.2. *It can be easily shown that*

$$i) \lim_{n\to\infty} \frac{S_n}{S_{n-1}} = 1 + \sqrt{2} \quad (\textit{silver ratio})$$

and

$$ii) \lim_{n\to\infty} \frac{S_n}{P_n} = 2 + \sqrt{2}.$$

Proof.
i)

$$\lim_{n\to\infty} \frac{S_n}{S_{n-1}} = \lim_{n\to\infty} \frac{\frac{P_{n+1} + P_n - 1}{2}}{\frac{P_n + P_{n-1} - 1}{2}} = \frac{P_{n+1}\left(1 + \frac{P_n}{P_{n+1}} - \frac{1}{P_{n+1}}\right)}{P_n\left(1 + \frac{P_{n-1}}{P_n} - \frac{1}{P_n}\right)}$$

since

$$\lim_{n\to\infty} \frac{P_n}{P_{n+1}} = \frac{P_{n-1}}{P_n} = \sqrt{2} - 1,$$

we obtain

$$\lim_{n\to\infty} = \frac{P_{n+1}\left(1 + \sqrt{2} - 1\right)}{P_n\left(1 + \sqrt{2} - 1\right)} = \lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 1 + \sqrt{2}.$$

ii)

$$\lim_{n\to\infty} \frac{S_n}{P_n} = \frac{\frac{P_{n+1} + P_n - 1}{2}}{P_n} = \frac{P_{n+1} + P_n - 1}{2P_n} = \frac{P_n\left(\frac{P_{n+1}}{P_n} + 1 - \frac{1}{P_n}\right)}{2P_n} = \frac{\sqrt{2} + 2}{2}$$

□

2.1 Pell Matrix and Sequence Spaces

Using analogue of the Pell matrix, we present the sequence spaces $c(\widehat{P})$, $c_0(\widehat{P})$, $\ell_\infty(\widehat{P})$, $\ell_p(\widehat{P})$ $(1 \le p < \infty)$ in this section. After that, a Schauder basis for $\ell_p(\widehat{P})$ will be constructed and some inclusion relations will be given. The analogue of the Pell matrix is defined by

$$\widehat{P} = \left(\widehat{P}_{nk}\right) = \begin{cases} \frac{P_k}{S_n} & , 1 \le k \le n \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{bmatrix} \frac{P_1}{S_1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{P_1}{S_2} & \frac{P_2}{S_2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{P_1}{S_3} & \frac{P_2}{S_3} & \frac{P_3}{S_3} & 0 & 0 & 0 & \cdots \\ \frac{P_1}{S_4} & \frac{P_2}{S_4} & \frac{P_3}{S_4} & \frac{P_4}{S_4} & 0 & 0 & \cdots \\ \frac{P_1}{S_5} & \frac{P_2}{S_5} & \frac{P_3}{S_5} & \frac{P_4}{S_5} & \frac{P_5}{S_5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for $k, n = \{1, 2, \ldots\}$. The matrix transformation $y_n = (\widehat{P})_n(x)$ is denoted as

$$y_n = \frac{1}{S_n} \sum_{k=1}^{n} P_k x_k \tag{2.1}$$

and the sequence spaces $c_0(\widehat{P})$, $c(\widehat{P})$, $\ell_\infty(\widehat{P})$ and $\ell_p(\widehat{P})$ $(1 \leq p < \infty)$ are defined by

$$c_0(\widehat{P}) = \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} (\widehat{P})_n(x) = 0 \right\},$$

$$c(\widehat{P}) = \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} (\widehat{P})_n(x) \text{ exists} \right\},$$

$$\ell_\infty(\widehat{P}) = \left\{ x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \left| \frac{1}{S_n} \sum_{k=1}^{n} P_k x_k \right| < \infty \right\},$$

$$\ell_p(\widehat{P}) = \left\{ x = (x_k) \in \omega : \sum_n \left| \frac{1}{S_n} \sum_{k=1}^{n} P_k x_k \right|^p < \infty \right\}.$$

Considering the notation (1.2), the sequence spaces $\ell_p(\widehat{P})$, $\ell_\infty(\widehat{P})$, $c_0(\widehat{P})$ and $c(\widehat{P})$ can be redefined by

$$\ell_p(\widehat{P}) = (\ell_p)_{\widehat{P}} \, (1 \leq p < \infty), \; \ell_\infty(\widehat{P}) = (\ell_\infty)_{\widehat{P}}, \tag{2.2}$$

$$c_0(\widehat{P}) = (c_0)_{\widehat{P}} \quad \text{and} \quad c(\widehat{P}) = (c)_{\widehat{P}}, \tag{2.3}$$

respectively.

Theorem 2.3. *The $\ell_p(\widehat{P})$ is a $BK-$space normed by*

$$\|(\widehat{P})_n(x)\|_{\ell_p} = \|x\|_{\ell_p(\widehat{P})} = \left(\sum_n \left| (\widehat{P})_n(x) \right|^p \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

and the spaces $U(\widehat{P})$ are $BK-$spaces normed

$$\|(\widehat{P})_n(x)\|_U = \|x\|_{U(\widehat{P})} = \sup_{n\in\mathbb{N}} \left| (\widehat{P})_n(x) \right|,$$

where $U \in \{\ell_\infty, c, c_0\}$.

Proof. The matrix $\widehat{P}$ is a triangle, and ℓ_∞ and ℓ_p are $BK-$spaces in terms of their natural norms, because (2.2) and (2.3) hold; Theorem 4.3.12 of [47, p. 63] states that the spaces $\ell_p(\widehat{P})$ and $\ell_\infty(\widehat{P})$ are $BK-$spaces with the given norms, where $(1 \leq p < \infty)$.

The spaces $c_0(\widehat{P})$ and $c(\widehat{P})$ are $BK-$spaces with the stated norms, as per [47, p. 61] Theorem 4.3.2 □

Theorem 2.4. *The space $\ell_p(\widehat{P})$ $(1 \leq p \leq \infty)$ is linearly isomorphic to the ℓ_p.*

Proof. To prove that $S : \ell_p(\widehat{P}) \to \ell_p$, $(x \to y = Vx = \widehat{P}x \in \ell_p)$, is a linear and bijection transformation for $(1 \leq p \leq \infty)$ is sufficient.

S is obviously linear. In addition, S is implied to be injective as it is evident that $x = 0$ whenever $Sx = 0$.

Let's get $y = (y_n) \in \ell_p$ to show that S is surjective. We have

$$y_n = \frac{1}{S_n} \sum_{k=1}^{n} P_k x_k.$$

and so

$$x_k = \frac{1}{S_k} \sum_{k-1 \leq i \leq k} (-1)^{k+i} P_i y_k \quad \text{or} \quad x_k = \frac{S_k}{P_k} y_k - \frac{S_{k-1}}{P_k} y_{k-1}$$

For $(1 \leq p < \infty)$ we consider

$$\begin{aligned}
&\|x\|_{\ell_p(\widehat{P})} = \\
&= \left(\sum_n \left| (\widehat{P})_n(x) \right|^p \right)^{\frac{1}{p}} = \left(\sum_n \left| \frac{1}{S_n} \sum_{k=1}^{n} P_k x_k \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_n \left| \frac{1}{S_n} \sum_{k=1}^{n} P_k \left(\frac{S_k}{P_k} y_k - \frac{S_{k-1}}{P_k} y_{k-1} \right) \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_n \left| \frac{1}{S_n} S_n y_n \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_n |y_n|^p \right)^{\frac{1}{p}} = \|y\|_p < \infty
\end{aligned}$$

and

$$\|x\|_{\ell_\infty(\widehat{P})} = \sup_{n \in \mathbb{N}} \left| (\widehat{P})_n(x) \right| = \|y\|_\infty < \infty.$$

The proof is now complete. □

Theorem 2.5. *The $c_0(\widehat{P})$ and the $c(\widehat{P})$ are linearly isomorphic to the c_0 and the c, respectively.*

Proof. A similar method may be used to prove the theorem using Theorem 2.4. □

Theorem 2.6. *The inclusions $c \subset c(\widehat{P})$ and $c_0 \subset c_0(\widehat{P})$ hold.*

Proof. For any real number l, let's get $x \in c$ and this means that $x \to l$. The method $\widehat{P}$ is regular and since the matrix $\widehat{P}$ satisfies the Silverman-Toeplitz criterias;

$$\begin{aligned}
&\sup_{n \in \mathbb{N}} \sum_k \left| \widehat{P}_{nk} \right| = \sup_{n \in \mathbb{N}} \left(\left| \frac{1}{S_n} \sum_{k=1}^{n} P_k \right| \right) = 1, \\
&\lim_{n \to \infty} \widehat{P}_n = 0, \\
&\lim_{n \to \infty} \sum_k \widehat{P}_{nk} = \lim_{n \to \infty} \left(\frac{1}{S_n} \sum_{k=1}^{n} P_k \right) = 1.
\end{aligned}$$

Then we can see that $\widehat{P}x \to l$. So $x \in c(\widehat{P})$. In order to prove the $c_0 \subset c_0(\widehat{P})$, $l = 0$ is necessary. For example, it is clear that the sequence $x = (x_k) = \left(\frac{(-1)^k}{P_k} \right)$ converges to "0". So $x \in c_0$.

When the matrix transformation sequence is considered, it is

$$\lim_{n\to\infty}\left(\frac{1}{S_n}\sum_{k=1}^{n}P_k\left(\frac{(-1)^k}{P_k}\right)\right)=0.$$

Therefore, $\widehat{P}x\in c_0(\widehat{P})$. □

Theorem 2.7. *The inclusion $\ell_p\subset\ell_p(\widehat{P})$ holds, where $1\le p\le\infty$.*

Proof. It is sufficient to prove that there is a constant $K>0$ that satisfies the inequality $\|x\|_{\ell_p(\widehat{P})}\le K\|x\|_p$ for every $x\in\ell_p$.

For $(1<p<\infty)$, let's get $x\in\ell_p$. Applying From Hölder's inequality for $\forall n\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left|(\widehat{P})_n(x)\right|^p &=\sum_{n=0}^{\infty}\left|\sum_{k=1}^{n}\frac{P_k}{S_n}x_k\right|^p\le\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{P_k}{S_n}|x_k|\Big)^p\\ &\le\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{P_k}{S_n}|x_k|^p\Big)\Big(\sum_{k=0}^{n}\frac{P_k}{S_n}\Big)^{p-1}\\ &=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{P_k}{S_n}|x_k|^p\Big)\\ &=\sum_{k=0}^{\infty}|x_k|^p\left(P_k\sum_{n=k}^{\infty}\frac{1}{S_n}\right).\end{aligned}$$

So this means

$$\|x\|_{\ell_p(\widehat{P})}\le K\|x\|_p, \tag{2.4}$$

where $K=\sup_{k\in\mathbb{N}}\left(\sum_{n=k}^{\infty}\frac{P_k}{S_n}\right)$. Also for $p=\infty$, we take $(x_k)\in\ell_\infty$. Then, for all $k\in\mathbb{N}$, there exists a constant $K>0$ such that $|x_k|\le K$. Using triangle inequality, we get

$$|(\widehat{P})_n(x)|\le\sum_{k=0}^{n}\frac{P_k}{S_n}|x_k|\le\sum_{k=0}^{n}\frac{P_k}{S_n}K=K.$$

So $x\in\ell_p(\widehat{P})$.

A similar proof can be done for $p=1$. □

We give the following two theorems without proof.

Theorem 2.8. *The $\ell_p(\widehat{P})\subset\ell_s(\widehat{P})$, if $1\le p<s$.*

Theorem 2.9. *The inclusion $c_0(\widehat{P})\subset c(\widehat{P})$ is strict.*

Theorem 2.10. *The inclusion $\ell_p(\widehat{P})\subset\ell_\infty(\widehat{P})$ is strict.*

Proof. Let take $x=(x_n)\in\ell_p(\widehat{P})$. Then we have $\widehat{P}x\in\ell_p$. Since $\ell_p\subset\ell_\infty$, we can conclude $\widehat{P}x\in\ell_\infty$. So $x=(x_n)\in\ell_\infty(\widehat{P})$ which means $\ell_p(\widehat{P})\subset\ell_\infty(\widehat{P})$. The sequence $x=(x_n)=(1^n)$ be examined for the inclusion's strict. Since

$$\sup_{n\in\mathbb{N}}\left|\sum_{k=1}^{n}\frac{P_k}{S_n}(1^n)\right|=1<\infty,$$

we have $x\in\ell_\infty(\widehat{P})$. But since

$$\sum_{n}\left|\sum_{k=1}^{n}\frac{P_k}{S_n}(1^n)\right|^p=\sum_{n}|1|^p\to\infty$$

we have $x\notin\ell_p(\widehat{P})$. □

Theorem 2.11. *The space $\ell_p(\widehat{P})$ is not a Hilbert space where $p \in [1,\infty] - \{2\}$.*

Proof. We use the sequences

$$v = (v_n) = \left(\frac{S_1}{P_1}, \frac{-S_1+S_2}{P_2}, \frac{-S_2}{P_3}, 0, \ldots\right)$$

and

$$u = (u_n) = \left(\frac{S_1}{P_1}, \frac{-S_1-S_2}{P_2}, \frac{S_2}{P_3}, 0, \ldots\right)$$

for proof. The $\tilde{P}_q$ transformations of these sequences are as follows, respectively:

$$\widehat{P}v = (1,1,0,0,\ldots) \text{ and } \widehat{P}u = (1,-1,0,0,\ldots).$$

Thus, $\tilde{P}_q(v+u) = (2,0,0,0,\ldots)$ and $\widehat{P}(v-u) = (0,2,0,0,\ldots)$ are obtained. Hence, the expression for $p \neq 2$ that results is as follows

$$\|v+u\|^2_{\ell_p(\widehat{P})} + \|v-u\|^2_{\ell_p(\widehat{P})} = 8 \neq 2^{2+\frac{2}{p}} = 2\left(\|v\|^2_{\ell_p(\widehat{P})} + \|u\|^2_{\ell_p(\widehat{P})}\right).$$

This implies that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(\widehat{P})$. □

We now provide a basis for $\ell_p(\widehat{P})$ $(1 \le p < \infty)$. For detailed discussions on basis for $\ell_p(\widehat{P})$ $(1 \le p < \infty)$, one can refer the studies [17, 18].

Theorem 2.12. *For $1 \le p < \infty$ and each fixed $k \in \mathbb{N}$, define a sequence $\xi^{(k)} \in \ell_p(\widehat{P})$ as*

$$(\xi^{(k)})_n = \begin{cases} (-1)^{n+k}\dfrac{S_k}{P_n} & , n-1 \le k \le n \\ 0 & , \textit{otherwise} \end{cases} \quad (k,n \in \mathbb{N}). \tag{2.5}$$

Later, $\{\xi^{(k)}\}_{k\in\mathbb{N}}$ is a Schauder basis for the space $\ell_p(\widehat{P})$ and each $u \in \ell_p(\widehat{P})$ has a unique representation of the form

$$u = \sum_k (\widehat{P})_k(u)\xi^{(k)} \tag{2.6}$$

for each $k \in \mathbb{N}$.

Proof. Let consider $1 \le p < \infty$. Afterward, it is clear by (2.5) that $(\widehat{P})(\xi^{(k)}) = e^{(k)} \in \ell_p$ and hence $\xi^{(k)} \in \ell_p(\widehat{P})$.

Let us take $u \in \ell_p(\widehat{P})$ and for each non-negative integer m and all $k \in \mathbb{N}$ we put

$$u^{(m)} = \sum_k (\widehat{P})_k(u)\xi^{(k)}.$$

Then we can obtain

$$\widehat{P}(u^{(m)}) = \sum_{k=0}^{m} (\tilde{P})_k(u)(\widehat{P})(\xi^{(k)}) = \sum_{k=0}^{m} (\tilde{P})_k(u)e^{(k)}$$

and then

$$(\widehat{P})_n(u-u^{(m)}) = \begin{cases} 0 & , (0 \le n \le m) \\ (\widehat{P})_n(x) & , (n > m) \end{cases} \quad (n,m \in \mathbb{N}). \tag{2.7}$$

For any given $\varepsilon > 0$, there is a $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m_0+1}^{\infty} \left|(\tilde{P}_q)_n(u)\right|^p = \left(\frac{\varepsilon}{2}\right)^p.$$

As a result, for every $m > m_0$, we acquire

$$\begin{aligned}\|u - u^{(m)}\|_{\ell_p(\widehat{P})} &= \left(\sum_{k=m+1}^{\infty} \left|(\widehat{P})_n(u)\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{k=m_0+1}^{\infty} \left|(\widehat{P})_n(u)\right|^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

demonstrating that $\lim_{m\to\infty} \|u - u^{(m)}\|_{\ell_p(\widehat{P})} = 0$ and as a result, u can be stated as in (2.6).

To demonstrate the uniqueness of the expression, we assume the existence of another form (2.6), similar to

$$u = \sum_k (\widehat{M})_k(u)\xi^{(k)}.$$

By using the continuous transform S, we have proved its isomorphism in Theorem 2.4, the equation that follows may be written as

$$(\widehat{P})_n(u) = \sum_k (\widehat{M_q})_k(u)(\widehat{P})_n(\xi^{(k)}) = \sum_k (\widehat{M})_k(u)\delta_{nk} = (\widehat{M})_n(u).$$

This proves that the form (2.6) is unique. This concludes the proof. □

3 $\alpha-, \beta-, \gamma-$ duals of the $\ell_p(\widehat{P})$

The $\alpha-$, $\beta-$, $\gamma-$ duals of the $\ell_p(\widehat{P})$ are given in this section. Since $p = 1$ can be demonstrated by analogy, we will focus on the case $1 < p \leq \infty$. We serve the lemmas in Stieglitz and Tietz [41] to prove Theorem 3.5 and Theorem 3.6. Many researchers have examined sequence spaces, dual spaces, and matrix transforms utilizing the domain of certain matrices, such as [3, 10, 11, 27, 28, 29, 32, 14, 15].

Take note that $(p^{-1} + r^{-1}) = 1$ for $(1 < p \leq \infty)$ and that F represents the family of all finite subsets of $\mathbb{N}$.

Lemma 3.1. $B = (b_{nk}) \in (\ell_p, \ell_1) \Leftrightarrow$

$$\sup_{K\in F} \sum_k \left|\sum_{n\in K} b_{nk}\right|^r < \infty.$$

Lemma 3.2. $B = (b_{nk}) \in (\ell_p, c) \Leftrightarrow$

$$\textit{For } (\forall k \in \mathbb{N}) \lim_{n\to\infty} b_{nk} \textit{ exists} \tag{3.1}$$

$$\sup_{n\in\mathbb{N}} \sum_k |b_{nk}|^r < \infty. \tag{3.2}$$

Lemma 3.3. $B = (b_{nk}) \in (\ell_\infty, c) \Leftrightarrow$ *(3.1) holds and*

$$\lim_{n\to\infty} \sum_k |b_{nk}| = \sum_k \left|\lim_{n\to\infty} b_{nk}\right|. \tag{3.3}$$

Lemma 3.4. $B = (b_{nk}) \in (\ell_p, \ell_\infty) \Leftrightarrow$ *(3.2) holds with* $(1 < p \leq \infty)$.

Theorem 3.5. *The set*

$$D_1 = \left\{ b = (b_k) \in \omega : \sup_{K \in F} \sum_k \left| \sum_{n \in K} (-1)^{n+k} \frac{S_k}{P_n} b_n \right|^r < \infty \right\}$$

is the $\alpha-$dual of the space $\ell_p(\widehat{P})$, where $1 < p \le \infty$.

Proof. For $1 < p \le \infty$ and any sequence $b = (b_n) \in \omega$, let's define a matrix G by

$$G = (g_{nk}) = \begin{cases} (-1)^{n+k} \dfrac{S_k}{P_n} b_n & , n-1 \le k \le n \\ 0 & , otherwise \end{cases} .$$

Furthermore, for each $x = (x_n) \in \omega$, we get $y = \widehat{P}x$. After it tracks by (2.1)

$$b_n x_n = \sum_{k=n-1}^{n} (-1)^{n+k} \frac{S_k}{P_n} b_n y_k = G_n(y) \quad (n \in \mathbb{N}). \tag{3.4}$$

Because of (3.4), we obtain that $bx = (b_n x_n) \in \ell_1$ whenever $x \in \ell_p(\widehat{P})$ if and only if $Gy \in \ell_1$ whenever $y \in \ell_p$.

We can see from Lemma 3.1 that

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} (-1)^{n+k} \frac{S_k}{P_n} b_n \right|^r < \infty$$

and so $\left(\ell_p(\tilde{P}_q)\right)^\alpha = D_1$. □

Theorem 3.6. *Define the following sets D_2, D_3, D_4 as:*

$$D_2 = \left\{ b = (b_k) \in \omega : \sum_{j=k}^{\infty} (-1)^{j+k} \frac{S_k}{P_j} b_j \quad \textit{exists}, \forall k \in \mathbb{N} \right\},$$

$$D_3 = \left\{ b = (b_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=1}^{n} \left| \sum_{j=k}^{n} (-1)^{j+k} \frac{S_k}{P_j} b_j \right|^r < \infty \right\},$$

$$D_4 = \left\{ b = (b_k) \in \omega : \lim_{n \to \infty} \sum_{k=1}^{n} \left| \sum_{j=k}^{n} (-1)^{j+k} \frac{S_k}{P_j} b_j \right| \right.$$
$$\left. = \sum_k \left| \sum_{j=k}^{\infty} (-1)^{j+k} (-1)^{j+k} \frac{S_k}{P_j} b_j \right| < \infty \right\}.$$

Then we have

a) $\left(\ell_p(\widehat{P})\right)^\beta = D_2 \cap D_3$ *and*

b) $\left(\ell_\infty(\widehat{P})\right)^\beta = D_2 \cap D_4$

for $1 < p < \infty$.

Proof. Let's get $b = (b_k) \in \omega$ and look at the equality

$$\sum_{k=1}^{n} b_k x_k = \sum_{k=1}^{n} b_k \left(\sum_{j=n-1}^{n} (-1)^{j+k} \frac{S_k}{P_j} y_j \right)$$
$$= \sum_{k=1}^{n} \left(\sum_{j=k}^{n} (-1)^{j+k} \frac{S_k}{P_j} b_j \right) y_k = D_n(y), \tag{3.5}$$

where $D = (d_{nk})$ is determined by

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{j+k} \frac{S_k}{P_j} b_j & , n-1 \leq k \leq n \\ 0 & , k > n \end{cases} .$$

After, we deduce from Lemma 3.2 using (2.1) that $Dy \in c$ whenever $y = (y_k) \in \ell_p$ if and only if $bx = (b_k x_k) \in cs$ whenever $x \in \ell_p(\widehat{P})$. Therefore, $(b_k) \in \left(\ell_p(\widehat{P})\right)^{\beta}$ if and only if $(b_k) \in D_2$ and $(b_k) \in D_3$ are defined by (3.1) and (3.2), respectively. Consequently $\left(\ell_p(\widehat{P})\right)^{\beta} = D_2 \cap D_3$.

An equivalent proof can be formulated when $p = \infty$ by utilizing Lemma 3.3 in place of Lemma 3.2 through analogous approaches. □

Theorem 3.7. $\left(\ell_p(\widehat{P})\right)^{\gamma} = D_3$*, for* $1 < p \leq \infty$.

Proof. One may utilize (3.5) to produce the proof by using Lemma 3.4. □

3.1 Matrix transformations associated with the space $\ell_p(\widehat{P})$

The matrix classes $(\ell_p(\widehat{P}), U)$ are characterized in this section, where $1 < p \leq \infty$, and $U \in \{\ell_\infty, \ell_1, c, c_0\}$. We utilize

$$\tilde{b}_{nk} = \sum_{j=k}^{\infty} (-1)^{j+k} \frac{S_k}{P_j} b_{nj}$$

in order to achieve brevity. The following lemma forms the basis of our findings.

Lemma 3.8. *(see [33], Theorem 4.1]) Let μ be an arbitrary subset of ω, U a triangular matrix, V its inverse, and λ a $FK-$space. Define $H^{(n)} = (h_{mk}^{(n)})$ and $H = (h_{nk})$ by*

$$H^{(n)} = h_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} b_{nj} v_{jk} & , 1 \leq k \leq m \\ 0 & , k > m \end{cases} , \qquad H = (h_{nk}) = \sum_{j=k}^{\infty} b_{nj} v_{jk},$$

respectively. Thus we obtain $H^{(n)} = (h_{mk}^{(n)}) \in (\lambda, c)$ and $H = (h_{nk}) \in (\lambda, \mu)$ if and only if $B = (b_{nk}) \in (\lambda_U, \mu)$ (see Theorem 4.1 of [33]).

The following conditions are now listed:

$$\sup_{m\in\mathbb{N}}\sum_{k=1}^{m}\left|\sum_{j=k}^{m}(-1)^{j+k}\frac{S_k}{P_j}b_{nj}\right|^r<\infty, \tag{3.6}$$

$$\lim_{m\to\infty}\sum_{j=k}^{m}(-1)^{j+k}\frac{S_k}{P_j}b_{nj}=\tilde{b}_{nk}, \qquad \forall n,k\in\mathbb{N}, \tag{3.7}$$

$$\lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{j=k}^{m}(-1)^{j+k}\frac{S_k}{P_j}b_{nj}\right|=\sum_{k}|\tilde{b}_{nk}| \qquad \forall n\in\mathbb{N}, \tag{3.8}$$

$$\sup_{m\in\mathbb{N}}\sum_{k}|\tilde{b}_{nk}|^r<\infty, \tag{3.9}$$

$$\sup_{N\in F}\sum_{k}\left|\sum_{n\in\mathbb{N}}\tilde{b}_{nk}\right|^r<\infty, \tag{3.10}$$

$$\lim_{n\to\infty}\tilde{b}_{nk}=\tilde{\alpha}_k; \quad k\in\mathbb{N}, \tag{3.11}$$

$$\lim_{n\to\infty}\sum_{k}|\tilde{b}_{nk}|=\sum_{k}|\tilde{\alpha}_k|, \tag{3.12}$$

$$\lim_{n\to\infty}\sum_{k}\tilde{b}_{nk}=0, \tag{3.13}$$

$$\sup_{n,k\in\mathbb{N}}|\tilde{b}_{nk}|<\infty, \tag{3.14}$$

$$\sup_{k,m\in\mathbb{N}}\left|\sum_{j=k}^{m}(-1)^{j+k}\frac{S_k}{P_j}b_{nj}\right|<\infty, \tag{3.15}$$

$$\sup_{k\in\mathbb{N}}\sum_{n}|\tilde{b}_{nk}|<\infty, \tag{3.16}$$

$$\sup_{N,K\in F}\left|\sum_{n\in N}\sum_{k\in K}\tilde{b}_{nk}\right|<\infty. \tag{3.17}$$

Thus, utilizing Lemma 3.8 and the findings in [41], we may deduce the following results from the given conditions.

Theorem 3.9. *a)* $B=(b_{nk})\in\left(\ell_1(\widehat{P}),\ell_\infty\right)\Leftrightarrow$ (3.7), (3.14) *and* (3.15) *hold.*

b) $B=(b_{nk})\in\left(\ell_1(\widehat{P}),c\right)\Leftrightarrow$ (3.7), (3.11), (3.14) *and* (3.15) *hold.*

c) $B=(b_{nk})\in\left(\ell_1(\widehat{P}),c_0\right)\Leftrightarrow$ (3.7), *with* $\tilde{\alpha}_k=0$, (3.11), (3.14) *and* (3.15) *hold.*

d) $B=(b_{nk})\in\left(\ell_1(\widehat{P}),\ell_1\right)\Leftrightarrow$ (3.7), (3.15) *and* (3.16) *hold.*

Theorem 3.10. *For* $1<p<\infty$,

a) $B=(b_{nk})\in\left(\ell_p(\widehat{P}),\ell_\infty\right)\Leftrightarrow$ (3.6), (3.7) *and* (3.9) *hold.*

b) $B=(b_{nk})\in(\ell_p(\tilde{P}),c)\Leftrightarrow$ (3.6), (3.7), (3.9) *and* (3.11) *hold.*

c) $B=(b_{nk})\in\left(\ell_p(\widehat{P}),c_0\right)\Leftrightarrow$ (3.6), (3.7), (3.9) *and with* $\widehat{\alpha}_k=0$ (3.11) *hold.*

d) $B=(b_{nk})\in\left(\ell_p(\widehat{P}),\ell_1\right)\Leftrightarrow$ (3.6), (3.7) *and* (3.10) *hold.*

Theorem 3.11. *a)* $B=(b_{nk})\in\left(\ell_\infty(\widehat{P}),\ell_\infty\right)\Leftrightarrow$ (3.7), (3.8) *and in case* $r=1$ (3.9) *hold.*

b) $B = (b_{nk}) \in \left(\ell_\infty(\widehat{P}), c\right) \Leftrightarrow$ (3.7), (3.8), (3.11) *and* (3.12) *hold.*

c) $B = (b_{nk}) \in \left(\ell_\infty(\widehat{P}), c_0\right) \Leftrightarrow$ (3.7), (3.8) *and* (3.13) *hold.*

d) $B = (b_{nk}) \in \left(\ell_\infty(\widehat{P}), \ell_1\right) \Leftrightarrow$ (3.7), (3.8) *and* (3.17) *hold.*

3.2 Certain geometric properties of the $\ell_p(\widehat{P})$

One of the most significant properties in functional analysis is the geometric property of Banach spaces. We look at [8, 13, 12, 26, 34, 23, 39, 21] for more details. If every bounded sequence (b_n) in U enables a subsequence (s_n) such that the sequence $\{t_k(s)\}$ is convergent in the norm in U, then U is said to satisfy the Banach-Saks property (see [23]), where

$$\{t_k(s)\} = \frac{1}{k+1}(s_0 + s_1 + \ldots + s_k) \quad (k \in \mathbb{N}). \tag{3.18}$$

A Banach space U has the weak Banach-Saks property for given any weakly null sequence $(b_n) \subset U$ if there exists a subsequence (s_n) of (b_n) such that the $\{t_k(s)\}$ is strongly convergent to zero.

According to García-Falset in [12], the coefficient is as follows:

$$R(U) = \sup\left\{\liminf_{n\to\infty} \|b_n - b\| : (b_n) \subset B(U), b_n \overset{w}{\to} b, b \in B(U)\right\}, \tag{3.19}$$

where the unit ball of U is indicated by $B(U)$.

Remark 3.12. A Banach space U possesses the weak fixed point property for $R(U) < 2$ [13].

For $\forall n \in \mathbb{N}$, some $M > 0$ and $1 < p < \infty$, if every weakly null sequence (b_k) possesses a subsequence (b_{k_l}) such that

$$\left\|\sum_{l=0}^{n} b_{k_l}\right\| < M(n+1)^{1/p}, \tag{3.20}$$

a Banach space possesses the Banach-Saks type p or the property $(BS)_p$ (see [34]).

With $1 < p < \infty$, we can now get the following results from the geometric properties of the space $\ell_p(\widehat{P})$.

Theorem 3.13. *The* $\ell_p(\widehat{P})$ $(1 < p < \infty)$ *possesses the Banach-Saks type* p.

Proof. We take (ε_n) sequence such that $(\varepsilon_n) > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{1}{2}$, and moreover we take a weakly null sequence (b_n) in $B(\ell_p(\widehat{P}))$. Set $s_0 = b_0 = 0$ and $s_1 = b_{n_1} = b_1$. After, there is a $u_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=u_1+1}^{\infty} s_1(i)e^{(i)}\right\|_{\ell_p(\widehat{P})} < \varepsilon_1. \tag{3.21}$$

There is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=0}^{u_1} b_n(i)e^{(i)}\right\|_{\ell_p(\widehat{P})} < \varepsilon_1 \tag{3.22}$$

when $n \geq n_2$, because (b_n) is a weakly null sequence implies $b_n \to 0$ coordinatewise. Set $s_2 = b_{n_2}$. Then there is an $u_2 > u_1$ such that

$$\left\|\sum_{i=u_2+1}^{\infty} s_2(i)e^{(i)}\right\|_{\ell_p(\widehat{P})} < \varepsilon_2. \tag{3.23}$$

Considering that $b_n \to 0$ coordinatwise, there is an such that $n_3 > n_2$

$$\left\| \sum_{i=0}^{h_2} b_n(i) e^{(i)} \right\|_{\ell_p(\widehat{P})} < \varepsilon_2, \tag{3.24}$$

when $n \geq n_3$.

Two increasing subsequences, (u_i) and (n_i), could be obtained when we continue in this way, such that

$$\left\| \sum_{i=0}^{u_j} b_n(i) e^{(i)} \right\|_{\ell_p(\widehat{P})} < \varepsilon_j, \tag{3.25}$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=u_j+1}^{\infty} s_j(i) e^{(i)} \right\|_{\ell_p(\widehat{P})} < \varepsilon_j. \tag{3.26}$$

where $s_j = b_{n_j}$. Thus,

$$\begin{aligned} \left\| \sum_{j=0}^{n} s_j \right\|_{\ell_p(\widehat{P})} &= \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{u_{j-1}} s_j(i) e^{(i)} + \sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} + \sum_{i=u_j+1}^{\infty} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{P})} \\ &\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{P})} + 2 \sum_{j=0}^{n} \varepsilon_j. \end{aligned}$$

Alternatively, we can see that $\|x\|_{\ell_p(\widehat{P})} \leq 1$. Hence, we have that

$$\begin{aligned} &\left\| \sum_{j=0}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{P})}^p = \\ &= \sum_{j=0}^{n} \sum_{i=u_{j-1}+1}^{u_j} \left| \sum_{k=2}^{i} \sum_{j=0}^{k} \frac{P_j}{S_k} s_j(k) \right|^p \\ &\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left| \sum_{k=2}^{i} \sum_{j=0}^{k} \frac{P_j}{S_k} s_j(k) \right|^p \leq (n+1). \end{aligned}$$

Thus, it may be obtained that

$$\left\| \sum_{j=0}^{n} \left(\sum_{i=u_{j-1}+1}^{u_j} s_j(i) e^{(i)} \right) \right\|_{\ell_p(\widehat{P})} \leq (n+1)^{\frac{1}{p}}.$$

Making use of the knowledge that $1 \leq (n+1)^{\frac{1}{p}}$ for all $n \in \mathbb{N}$ and $1 < p < \infty$, we possess

$$\left\| \sum_{j=0}^{n} s_j \right\|_{\ell_p(\widehat{P})} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}.$$

As a consequence, $\ell_p(\widehat{P})$ possesses the Banach-Saks type p. This ends the proof. □

Remark 3.14. Because $\ell_p(\widehat{P})$ is linearly isomorphic to ℓ_p, $R(\ell_p(\widehat{P})) = R(\ell_p) = 2^{\frac{1}{p}}$.

Remarks 3.12 and Remarks 3.14 lead us to the following theorem.

Theorem 3.15. *The space $\ell_p(\widehat{P})$ $(1 < p < \infty)$ possesses the weak fixed point property.*

4 Conclusion

It has an important place in the theory of summablty with special matrices such as Cesàro, Abel, Borel, Fibonacci and Pascal. With this study, we have added a new one to these special matrices. Pell numbers is utilized in this article to define the sequence spaces $c_0(\widehat{P})$, $c(\widehat{P})$, $\ell_\infty(\widehat{P})$ and $\ell_p(\widehat{P})$ $(1 \leq p < \infty)$. Then, we looked at the topological and certain geometric properties of the $\ell_p(\widehat{P})$ space. The Pell numbers, which play a significant role in algebra, were moved to the area of sequence spaces and summability, which is an invention.

Declarations

The authors state that their interests do not conflict with each other.

The authors state that the manuscript has not contain additional data.

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