Review of μ-lifting modules and related concepts

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Abstract *A module N is said to be μ-lifting if for each submodule G of N, there is T satisfies that $N = T \oplus T'$ and $G \cap T' \ll_\mu T'$. In this review, we present some generalizations of μ-lifting modules property that the author have studied and developed over the last 6 years. To this end, we provide results, examples and open problems μ-lifting modules and related concepts. Our concepts make it clear how much of the μ-lifting condition is maintained in various closure operations.*

1 Introduction

All rings are associative and all modules are unital. A submodule G of an R-module N is called small in N (briefly $G \ll N$) if whenever $N = G + K, K \leqslant N, N = K$, [11], [8] and [14]. Some generalizations of small submodules are introduced, some of them are [12] and [10]. Kamil and khalid [13], Kamil [6] defined μ-small subonodules by adding a condition $\frac{N}{K}$ is cosingular. A module N is called μ-hollow if each proper submodule of N is μ - small in N. This condition was studied by kamil and khalid as μ-supplemented and some thier related concepts, [2]. A submodule G of a module N is μ-supplement, to K if $K \cap N \ll_\mu N$ and $N = G + K$, if each submodule of N has μ-supplement, N is called μ-supplemented . Some of related concepts of μ-supplement was studied in[2] . In this review, we focus on varies concepts that the authors studied and developed over the last 6 years. We present examples, results and open problems

The concept of μ-lifting and some of their generalizations that present in this review are.

(i) *A module N is called μ-lifting module if for each submadule G of N there is a summand T of N satisfies $G \cap T' \ll_\mu T', N = T \oplus T'$.*

(ii) *A module N is called $H - \mu$-supplemented when it is provided that for each submodule G of N, there is a summand T of N with $\frac{G+T}{G} \ll_\mu \frac{N}{G}$ and $\frac{G+T}{T} \ll_\mu \frac{N}{T}$*

(iii) *A module N is contacted fully invariant μ-lifting if for each fully invariant submodule G of N, the is a summand T of N with $N = T \oplus T', G \cap T' \ll_\mu T$*

(iv) *A module N is called μ-supplemented when all submadules of N have μ-supplement in N.*

(v) *If each submodules of N has μ-supplement direct summand N is called $\oplus - \mu$ supplemented.*

The following diagram summarize these conditions from stronger to weaker.
μ-hollow $\Rightarrow$ μ-Lifting $\Rightarrow$ H - μ-supplemented.
$\Downarrow$
$\oplus - \mu$-supplemented $\Rightarrow$ Fully invariant μ-lifting.
The reverse implications of these conceptsare false, in general, as follows exclaiming.

(a) *μ-lifting $\nRightarrow$ μ-hollow: $\mathbb{Z}_6$ as $\mathbb{Z}$-module.*

(b) *H - μ-supplemented $\nRightarrow$ μ-lifting: $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module.*

(c) $\oplus - \mu$-supplemented $\not\Rightarrow$ μ-Lifting: : $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module.

This review is not exhaustive list of finding results, examples for μ-lifting modules and generalizations. It satisfies to arouse more interest in the various generalizations of this types of modules. We recommend that readers to read the papers in the references list which are carry more details overview of these topics

Let R be any ring and N be R-module. If $G \leqslant N$, then $G \ll N, G \trianglelefteq N$ denotes G is a submodule of N, G small in N and G is a fully invariant in N, respectively.

For a ring $R, \mathbb{Z}$ and $\mathbb{Z}_n$ denote ring of integers and the quotient $\frac{\mathbb{Z}}{\mathbb{Z}_n}$ respectively.

2 μ-Hollow modules

In [13], the authors introduced the new generalization condition of the hollow modules, to understand this concept, we need the next definition and results.

Definition 2.1. [13]. Let G be a submodule of a module N, G is μ-small submodule of N (briefly $G \ll_\mu N$), if $N = G + P, \frac{N}{P}$ is cosingular implies $N = P$.

The following example shows that μ-small submodule is a proper generalization of small submodule.

Example 2.2. One Can observe that $\mathbb{Z}_6$ as $\mathbb{Z}$-module is noncosingular, hence every submodule of $\mathbb{Z}_6$ is noncosingular. So, $\langle \overline{2} \rangle$ and $\langle \overline{3} \rangle$ are μ-small in $\mathbb{Z}_6$ which are not small in $\mathbb{Z}_6$.

Proposition 2.3. *[13]. When N is cosingular, the smallness and μ-smallness are equivalent.*

Corollary 2.4. *[13]. For a small module N, each μ-small submodule of N is small in N.*

Corollary 2.5. *[13]. Taking N be a module, if N is not contains any maximal submodule, then μ - smalthess and smallness conditions are equivalent.*

Next theorem gives the fundamental features of the μ-small submodules.

Theorem 2.6. *[13]. Take N be a module, the next properties are hold.*

(i) Let $G \leqslant K \leqslant N$, then $K \ll_\mu M$ if and only if $G \ll_\mu N$ and $\frac{K}{G} \ll_\mu \frac{N}{G}$.

(ii) Taking G, K be submodules of N, then $G + K \ll_\mu N$ if and only if G and K are μ small in N. Furthermore, if $G_1, G_2, \ldots, G_n$ are μ-small in $N, \forall i = 1, 2, 3, \ldots, n$, then $\sum_{i=1}^n G_i \ll_\mu N$.

(iii) Let $G \leqslant K \leqslant N$. If $G \ll_\mu K$, then $G \ll_\mu N$.

(iv) For a homomorphism $\psi : N \to N'$, if $G \ll_\mu N$, then $\psi(G) \ll_\mu N'$.

(v) Let $N = N_1 \oplus N_2$ be a module with $G_1 \leqslant N_1$ and $G_2 \leqslant N_2$, then $G_1 \oplus G_2 \ll_\mu N_1 \oplus N_2$ if and only if $G_1 \ll_\mu N_1$, and $G_2 \ll_\mu N_2$.

The following example shows that the inverse image of μ-small submodule not necessary again μ-small.

Example 2.7. [13]. Let $\alpha : \mathbb{Z} \to \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$ be the canonical, observe that 0 is μ-small in $\mathbb{Z}_2$ while $\alpha^{-1}(0) = 2\mathbb{Z}$ is not μ-small in $\alpha^{-1}(\mathbb{Z}_2) = \mathbb{Z}$.

Proposition 2.8. *[13]. Let $G \leqslant K$ be submodules of a module N, if K is a direct summand of N and $G \ll_\mu N$, then $G \ll_\mu K$.*

Proposition 2.9. *[13]. Let G, K, L be submodules of a module N with $G \leqslant K \leqslant L \leqslant N$. If $K \ll_\mu L$, then $G \ll_\mu N$.*

Remark 2.10. The converse of proposition is false in general. For example, in $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module note that $0 \leqslant \langle \overline{4} \rangle \leqslant \langle \overline{2} \rangle \leqslant \mathbb{Z}_{12}$. Obvouis that $0 \ll_\mu \mathbb{Z}_{12}$, however, $\langle \overline{4} \rangle$ is not μ-small in $\langle \overline{2} \rangle$.

Proposition 2.11. *[13]. Let G, K, L be submodules of a module N such that $G \leqslant K \leqslant L \leqslant N$, then $\frac{L}{G} \ll_\mu \frac{N}{G}$ if and only if $\frac{L}{K} \ll_\mu \frac{N}{K}$ and $\frac{K}{G} \ll_\mu \frac{N}{G}$*

Theorem 2.12. *[13]. Let $N = G + K$ be a module such that $\frac{N}{K}$ is cosingular, let $K \leqslant L$ and $\frac{L}{K} \ll_\mu \frac{N}{K}$. Then $\frac{(G \cap L)}{(G \cap K)} \ll_\mu \frac{N}{(G \cap K)}$.*

Definition 2.13. [13]. Let G be a submodule of a module N, G is said to be μ-coclosed submodule of N (briefly $\leqslant_{\mu cc} N$) if whenever $\frac{G}{S} \ll_\mu \frac{N}{S}$, then $S = $ G.

Examples and Remarks 2.14.

- (i) In $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module, $\langle \overline{3} \rangle$ is μ-coclosed submodule of $\mathbb{Z}_{12}$.

- (ii) In $\mathbb{Z}_8$ as $\mathbb{Z}$-module $\langle \overline{2} \rangle$ is not μ-coclosed submodule.

- (iii) Each coclosed submodule in a module N is μ-coclesed in N. When N is cosingular, they are equivalent.

- (iv) Each direct summand of a module N is μ-coclosed.

Proposition 2.15. *[13]. Let N be a module and G be a submodule of N, if $S \leqslant G \leqslant N$ and $S \ll_\mu N$ then $S \ll_\mu G$.*

The next proposition introduced the basic features of μ-coclosed submodules

Proposition 2.16. *[13].For a module N with $G \leqslant K \leqslant N$, then.*

- (i) *If K is μ-coclosed in N, then $\frac{K}{G}$ is μ-coclosed in $\frac{N}{G}$.*

- (ii) *If $G \leqslant_{\mu cc} N$, then $G \leqslant_{\mu cc} K$, when $K \ll_\mu N$, the reverse implication is true.*

- (iii) *Let $L \leqslant_{\mu cc} N$, then $\frac{K}{G} \ll_\mu \frac{N}{G}$ if and only if $\frac{K}{G} \ll_\mu \frac{L}{G}$, where $G \leqslant K \leqslant L$.*

Definition 2.17. [13]. Let N be a nonzero R-module, we say that N is μ - Hollow module if every proper submodule is μ-small in N.

The next theorem gives a condition that equivalent to μ-Hollow modules.

Theorem 2.18. *[13]. A module N is μ-Hollow if and only if each proper submodule G of N with $\frac{N}{G}$ is cosingular is small in N.*

Proposition 2.19. *[13].μ-hollow modules are closed under epimorphisms.*

Proposition 2.20. *[13].A nonzero μ-hollow submadule of a module N is either μ-small in N or it is μ-coclosed in N but not both.*

Proposition 2.21. *[13]. Each nonzero μ-coclosed submodule of μ-hollow module is again μ-hollow.*

Theorem 2.22. *[13]. Each nonzero module over a V-ring is μ-hollow module.*

3 μ-Supplemented modules with related concepts

We examine the class of μ-supplemented condition, in this section. Additionally, this section presents some additional concepts linked to μ-supplemented conceptions.

Definition 3.1. [2]. For a submodules G and K of a module N, we say that K is μ -supplement to G in N if $= G + K$ and $G \cap K \ll_\mu K$. In case each submedule of N has μ supplement in N, then N is said to be μ-supplemented module.

Examples and Remarks 3.2.

(i) Each μ-hollow is μ-supplemented, $\mathbb{Z}_4$ as $\mathbb{Z}$-module shows that the converse implication is false in general.

(ii) μ-supplement relation is not symmetric. in general, $\mathbb{Z}_4$ is μ-supplement of $\langle \overline{2} \rangle$ but not conversely.

(iii) when $N = G \oplus K$, then G and K are μ-supplement to each other.

Proposition 3.3. *[2]. A μ-hollow submodule G of a module N is μ-supplement of every proper submodule K of N with $N = G + K$.*

    The next theorem presents an equivalent condition to the condition of μ-supplement submodule.

Theorem 3.4. *[2]. For submodules G and K of a module N, the following conditions are equivalent.*

(i) *K is μ-supplement of G in N.*

(i) *$N = G + K$ and for any proper submodule S of K with $\frac{K}{S}$ is cosingular, then $N \neq G + S$.*

    Next, we present some basic features of μ-supplements.

Proposition 3.5. *[2]. "Let N be a module" and G, K are submodules of N, if K is μ supplement of G in N, then.*

(i) *When $N = S + K$, for some submodule S of G, then K is μ-supplement of S in N.*

(ii) *If $Y \ll_\mu N$, then K is μ-supplement of $G + Y$.*

(iii) *$\frac{K+Y}{Y}$ is μ-supplement of $\frac{G}{Y}$ in $\frac{N}{Y}$ for every submodule Y of G.*

Proposition 3.6. *[2]. For submodules G, K and Y of a module N, then.*

(i) *For $N = N_1 \oplus N_2$, if G is μ-supplem of G' in N_1 and K is μ-supplement of K' in N_2, then $G \oplus K$ is μ-supplement of $G' \oplus K'$ in N.*

(ii) *If G is μ-supplement of K and K is μ-supplement of Y in N, then K is μ-supplement of G in N.*

Proposition 3.7. *[2]. Any finite sum of μ-supplemented modules again μ-supplemented.*

Proposition 3.8. *[2]. μ-Supplemented modules are closed under epimorphisms and factors.*

Definition 3.9. [2]. A module N is called amply μ-supplemented if it is provided that for any submodules G and K of N such that $N = G + K$, there is a μ-supplement S of G contained in K.

Proposition 3.10. *[2]. Amply μ-supplemented modules are closed under homomorphisms and factors.*

    A module N is called π-projective module if it is provided, for any submodules G and K of N such that $N = G + k$, there is $\phi \in \mathrm{End}(N)$ with $\mathrm{Im}\,\phi \leqslant G$ and $\mathrm{Im}(I - \phi) \leqslant K$, view [11].

Proposition 3.11. *[2]. "Let N be a π-Projective module" N, N is amply μ-supplemented if and only if it is μ-supplemented.*

Corollary 3.12. *[2]. "For a projective module" N, N is μ-supplemented if and only if it is amply μ-supplemented.*

Proposition 3.13. *[2]. When every submodule of a module H is μ supplemented, N is amply μ-supplemented.*

Proposition 3.14. *[2]. When every submodule of a module N is μ-supplemented, N is amply μ-supplemented.*

Corollary 3.15. *[2]. For a ring R, the following criteria are identical.*

(i) *Each R-module is μ-supplemented.*

(ii) *Each R-module is amply μ-supplemented.*

Definition 3.16. [2]. A module N is called weakly μ-supplemented if for each submodule G of N, there is a submodele K of N with $N = G + K$ and $G \cap K \ll_\mu N$

Remark 3.17. [2]. Weakly μ-supplemented generalize the concept of μ-supplemented module, Q as $\mathbb{Z}$ module shows that this generalization is proper.

Proposition 3.18. *[2]. Consider the following conditions, for a submodule G of a module N.*

(i) *G is μ-supplement in N.*

(ii) *G is μ-coclosed in N.*

(iii) *When S $\ll_\mu N$, then $S \ll_\mu G$, for any submodule S of G.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$, when N is weakly μ-supplemented.

Definition 3.19. [2]. A module N is said to be cofinitely μ-supplemented (for short cof- μ supplemented), when every cofinite submodule of N has μ-supplement in N.

Remark 3.20. [2]. The $\mathbb{Z}$ module Q shows that cof- μ-supplemented is a proper generalization of μ supplemented module.

Under certain condition cof- μ-supplemented and μ-supplemented conditions are equivalents, as the following proposition shows.

Proposition 3.21. *[2]. A finitely generated module is μ-supplemented if and only if it is cof- μ supplemented.*

Proposition 3.22. *[2]. Any sum of cof μ-supplemented modules again cof- μ-supplemented.*

Proposition 3.23. *[2]. Cof- μ-supplemented modules are closed under homomorphisms, quotients, and direct summands.*

Definition 3.24. [2]. When each submodule of a module N has a μ-supplement direct summand, N is called $\oplus - \mu$-supplemented.

A module N has a property (D3), if N_1 and N_2 are direct summands of N. With $N = N_1 + N_2$ implies that $N_1 \cap N_2$ a gain direct summand of , see [9].

Proposition 3.25. *[2]. If N has (D3) $\oplus - \mu$-supplemented module, then each direct summand of N has also $\oplus - \mu$-supplemented property.*

A submodule G of module N is said to be fully invariant if for all endomorphisms f on N, G contains $f(G)$,N is duo module if each of its submodules are fully invariant, [7] and [1].

Proposition 3.26. *[2]. A fully invariant direct summand of a $\oplus - \mu$-supplemented module has also $\oplus - \mu$-supplemented property.*

Theorem 3.27. *[2]. A direct sum of $\oplus - \mu$-supplemented modules is a gain $\oplus - \mu$ supplemented*

Theorem 3.28. *[2]. Put $N = \bigoplus_{i=1}^{n} N_i$ be a direct sum of relatively projective modules, then N is $\oplus$ - μ-supplemented if and only if each N_i is $\oplus - \mu$-supplemented.*

Proposition 3.29. *[2]. For a fully invariant submodule G of a $\oplus - \mu$-supplemented module N, we have the factor $\frac{N}{G}$ is $\oplus$ - μ-supplemented.*

Corollary 3.30. *[2]. For a duo module $\oplus - \mu$-supplemented module N, every factor module of N again has $\oplus$ - μ-supplemented property.*

Theorem 3.31. *[2]. Taking $N = N_1 \oplus N_2, N_2$ is $\oplus$ - μ-supplemented if and only if $N = G + K$, for some direct summand K of N with K submodule of $N_2, N = G + K$ and $G \cap K \ll_\mu K$, for all $\frac{G}{N_1}$ submodule of $\frac{N}{N_1}$.*

Proposition 3.32. *[2]. Take N be a $\oplus - \mu$ - supplemented, we have $N_1 = N_1 \oplus N_2$, with $Z^*(N_1) \ll_\mu N_1$ and $Z^*(N_2) = N_2$.*

Theorem 3.33. *[2]. For a module N satisfing D3 condition, the next conditions are equivalents.*

(i) *Each direct summand of N is $\oplus$ - μ-supplemented.*

(ii) *N has $\oplus$ - μ-supplemented property.*

(iii) *For some $\oplus - \mu$-supplemented modules N_1 and N_2 with $Z^*(N_1) \ll_\mu N_1$ and $Z^*(N_2) = N_2, N = N_1 \oplus N_2$.*

4 μ-Lifting modules

In [3], the authors defined μ-coessential submodules and present their features, this requirement is very helpful for the design of μ-lifting modules.

Definition 4.1. [3]. Let S and G be submodules of a module N with $S \leqslant G \leqslant N$, we say that S is μ-coessential submodule of G in N when $\frac{G}{S} \ll_\mu \frac{N}{S}$, for short, we denote ($S \leqslant_{\mu ce} G$ in N).

Examples and Remarks 4.2.

(i) In $\mathbb{Z}_8$ as $\mathbb{Z}$-module, clearly that $\langle \overline{4} \rangle \leqslant \mu ce \langle \overline{2} \rangle$ in $\mathbb{Z}_8$.

(ii) $\{\overline{0}\}$ is not μ-coessential submodule of $\langle \overline{3} \rangle$ in $\mathbb{Z}_6$ as $\mathbb{Z}$-module.

(iii) $\{\overline{0}\}$ is μ-coessential submodule of $\langle \overline{3} \rangle$ in $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module, this example shows that μ-coessential submedule is a proper generalization of coessential submodule. When N is cosingular, they are equivalent.

Proposition 4.3. *[3]. Taking $K \leqslant G \leqslant N$ and $\frac{N}{K}$ is cosingular, then K is μ-coessential submodule of G in N if and only if for every submodule S of N, we have $N = G + S$ gives $N = K + S$*

The next results present the basic features of μ-coessential submodules.

Proposition 4.4. *[3]. Put $S \leqslant K \leqslant G \leqslant N, K \leqslant_{\mu ce} G$ in N if and only if $\frac{K}{S} \leqslant_{\mu ce} \frac{G}{S}$ in $\frac{N}{S}$.*

Proposition 4.5. *[3]. For $S \leqslant K \leqslant G \leqslant N$, then $S \leqslant_{\mu ce} G$ in N if and only if $S \leqslant_{\mu ce} K$ and $K \leqslant_{\mu ce} G$ in N.*

Proposition 4.6. *[3]. For a module N, if $K \leqslant_{\mu ce} G$ in N and $S \leqslant_{\mu ce} L$, then $K + S \leqslant_{\mu ce} G + L$ in N.*

Proposition 4.7. *[3]. Taking $K \leqslant_{\mu ce} G$ in N and S be any submodule of N, then $K + S \leqslant_{\mu ce} G + S$ in N, the reverse implication is true when $S \ll_\mu N$.*

Proposition 4.8. *[3]. Let N be a module and let $S \ll_\mu N$, if $K \leqslant_{\mu ce} G$ in N, then $K \leqslant_{\mu ce} G + S$ in N.*

Proposition 4.9. *[3]. Let $\psi : N \to N'$ be a homomorphism. If $K \leqslant_{\mu ce} G$ $f(K) \leqslant_{\mu ce} f(G)$ in N'.*

Proposition 4.10. *[3]. For submodules S, K and G of a module N, the next conditions are equivalent.*

(i) *If $S \leqslant_{\mu ce} S + K$ in N, then $S \cap K \leqslant_{\mu ce} K$ in N.*

(ii) *If $S \leqslant_{\mu ce} K$ in N and $L \leqslant N$, then $S \cap L \leqslant_{\mu ce} K \cap L$ in N.*

(iii) *If $S \leqslant_{\mu ce} K$ in N and $L \leqslant_{\mu ce} G$ in N, then $S \cap L \leqslant_{\mu ce} K \cap G$ in N.*

Proposition 4.11. *[3]. Let $K \leqslant G \leqslant N$, if $G = K + L$ and $L \leqslant_{\mu ce} N$ in N, then $K \leqslant_{\mu ce} G$ in N.*

Theorem 4.12. *[3]. Let $N = K + G$ be a module such that $\frac{N}{G}$ is cosingular, if $G \leqslant L$ and $G \leqslant_{\mu ce} L$ in N, then $K \cap G \leqslant_{\mu ce} K \cap L$ in N.*

Proposition 4.13. *[3]. For submodules S, K and G of a module N. If S is μ-supplement of K in N and K is μ-supplement of G in N such that $S \leqslant G$, then $S \leqslant_{\mu ce} G$ in N.*

Definition 4.14. [3]. A module N is called μ-lifting if for each submodule G of N, there is a submodule T of N with $N = T \oplus T', G \cap T' \ll_{\mu} T'$

Examples and Remarks 4.15.

(i) μ-Lifting modules are proper generalization of lifting modules, view [3] In case of cosingular modules they are equivalent.

(ii) Clearly that μ-lifting modules are generalizations of μ-hollow modules, $\mathbb{Z}_6$ as $\mathbb{Z}$ module shows that this generalization is proper.

(iii) Each μ-lifting is $\oplus$ - μ-supplemented but not conversely, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is $\oplus - \mu$ supplemented $\mathbb{Z}$-module while it is not μ-lifting.

The next results introduce some equivalent condition to μ-lifting property.

Proposition 4.16. *[3]. A module N is μ-lifting if and only if for each submodule G of N, there is a submodule T of N satisfing $N = T \oplus T', T' \leqslant N$ and $G \cap T' \ll_{\mu} N$.*

Proposition 4.17. *[3]. For a module, the next conditions are equivalent.*

(i) *N has μ-lifting property.*

(ii) *Each submodule G of N can be written as $G = T \oplus K$, where $K \ll_{\mu} N$ and T is a direct summand of N.*

(iii) *There exists a direct summand T of N with $T \leqslant G$ and $T \leqslant_{\mu ce} G$ in N, for every submodule G of N.*

Theorem 4.18. *[3]. The next conditions are equivalent for a module N.*

(i) *N satisfies the μ-lifting property.*

(ii) *For any submodule G of N, there is μ-supplement K of G in N such that $G \cap K$ is a direct summand of G.*

Proposition 4.19. *[3]. The module N has μ-lifting property if and only if for all submodule G of N, there is idempotent $\phi \in End\,(N)$ with $\varphi(N) \leqslant G$ and $(I - \varphi)(G) \ll_{\mu} (I - \phi)(N)$.*

Proposition 4.20. *[3]. μ-hollow and μ-lifting modules are equivalent when it is an indecomposable module.*

Proposition 4.21. *[3]. μ-lifting modules are closed under direct summands.*

A module N is said to have distributive property when $G \cap (K + S) = (G \cap K) + (G \cap S)$, for all G, K, S submodules of N,[7] and [11].

It is not necessary that the quotient of μ-lifting module a gain μ-lifting. Observe that $N = \mathbb{Z}_8 \oplus \mathbb{Z}_8$ is μ-lifting $\mathbb{Z}$-module while $\frac{N}{ket(\psi+I)} \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not μ-lifting, where $\psi \oplus I : \mathbb{Z}_8 \oplus \mathbb{Z}_8 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2$ is an epimorphism, $\psi : N \to \frac{N}{\mathbb{Z}_8}$ is the canonical.

The next result shows that under certain conditions, the quotient of μ-lifting module also μ-lifting.

Proposition 4.22. *[3].Let G be a submodule of a module $N, \frac{N}{G}$ is again μ-lifting in each of the following cases.*

(i) *For all direct summands T of $N \frac{T+G}{G}$ is a direct summand of $\frac{N}{G}$.*

(ii) *N holds distributive condition.*

Proposition 4.23. *[3]. Let G be a fully invariant submodule of a μ-lifting module N, then $\frac{N}{G}$ is μ-lifting.*

Proposition 4.24. *For a μ-lifting module, each cosingular μ-coclosed submodule of N is a direct summand.*

Remark 4.25. [3]. In general, direct summands of μ-lifting modules is not μ-lifting. The $\mathbb{Z}$-module $N = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is an example shows that.

Proposition 4.26. *[3]. Let N_1 and N_2 be modules that hold μ-lifting condition, then $N = N_1 \oplus N_2$ again has μ-lifting property if any of the next cases satisfies.*

(i) $\operatorname{ann}(N_1) + \operatorname{ann}(N_2) = R.$

(ii) *N is distributive.*

(iii) *N is duo module.*

(iv) *N_1 is N_2-projective.*

5 H-μ-Supplemented modules

As a generalization of H-supplemented that introduced in [14], the class of modules that meet the H- μ-supplemented condition is the focus of this section; this idea represents a new appropriate generalization of all $\mathrm{H} - \mu$ supplemented and μ-lifting modules, [4] and [5].The relation μ, which is very helpful for the H - μ-supplemented situation, must be presented in this part.

Definition 5.1. [5] For a module N, define the relation μ on the set of submocules as following. $G\mu K$ when $\frac{G+K}{K} \ll_\mu \frac{N}{K}$ and $\frac{G+K}{G} \ll_\mu \frac{N}{G}$

Lemma 5.2. *[5] The relation μ is an equivalence relation.*

Examples and Remarks 5.3.

(i) Taking $G \leqslant K$ be any two submodules of a module N, $G\mu K$ if and only if $\mathrm{G} \leqslant_{\mu ce} K$

(ii) It is simple to prove that $6\mathbb{Z}$ is not μ equivalent to $4\mathbb{Z}$ in $\mathbb{Z}$ as $\mathbb{Z}$-module.

(iii) It is obvouis that $G\mu 0$ if and only if $G \ll_\mu M$, for any submodule G of N.

    The next results introduce characterization of the relation μ.

Theorem 5.4. *[5]. Taking G and K be any submodules of a module N, the next conditions are equivalent.*

(i) $G\mu K$.

(ii) $G \leqslant_{\mu ce} G + K$ in N and $K \leqslant_{\mu ce} \mathrm{G} + K$ in N.

(iii) *For any $S \leqslant N$ satisfying $N = G + K + S$, with $\frac{N}{S}$ is cosingular, we have $N = G + S$ and $N = K + S$.*

(iv) *For any submodule L of N, satisfying $N = L + G$, $\frac{N}{L}$ is cosingular implies $N = L + K$, and $N = K + P$, for any submodule P of N with $\frac{N}{P}$ is cosingular, implies $N = K + P$.*

Corollary 5.5. *[5]. For submodules G and K of a module N, if $G \leqslant K + L, K \leqslant G + P$, with $\mathrm{L} \ll_\mu N, P \ll_\mu N$, then $G\mu K$.*

Proposition 5.6. *[5]. Let G, K and L be submodutes of a module N. If $G\mu k$, then $\mathrm{G} \ll_\mu N$ if and only if $K \ll_\mu N$.*

Proposition 5.7. *[5]. Let $N = T \oplus T'$, and let G, K be submodules of T. Then $G\mu K$ in N if and only if $G\mu K$ in T.*

Proposition 5.8. *[5]. Let G and K be submodules of a module N, then $G\mu K$ if and only if $\frac{G}{L}$ $\mu\frac{K}{L}$, for every submodule L of N contained in $G \cap K$.*

Proposition 5.9. *[5]. For a module N take G_1, G_2, K_1 and K_2 be submodules of N, if $G_1\mu K_1$ and $G_2\mu K_2$, then $(G_1 + G_2)\,\mu\,(K_1 + \mathrm{K})$.*

Corollary 5.10. *[5]. Let $G, K_1, K_2, \ldots, K_n$ be submocules of a module N, if $G\mu K_i$ for each $i = 1, 2, \ldots, n$, then $G\mu \sum_{i=1}^{n} K_i$*

Corollary 5.11. *[5]. Let G, K and L are submodules of a module N, if $G\mu K$, then $(G+L)\mu(\mathrm{K}+L)$, when $L \ll_\mu N$, the converse implication is true.*

Proposition 5.12. *[5] The following statements are hold for an epimorphism $\psi : N \to N'$.*

 (i) Let G and K be submodules of N, with $G\mu k$, then $\psi(G)\mu\psi(K)$.

 (ii) Let G' and K' be submodies of N' with $G, \mu K'$, then $\psi^{-1}(G)\mu\psi^{-1}(K)$.

Proposition 5.13. *[5]. Let $N = N_1 \oplus N_2$ be a module, $G \leqslant N_1$ and $K \leqslant N_2$, then $G\mu N_1$ and $K\mu N_2$ if and only if $(G \oplus K)\mu\,(N_1 \oplus N_2)$*

Definition 5.14. [5]. A module N is said to be H - μ-supplemented if there is a direct summand T of N such that $G\mu T$, for any submodule G of N.

Examples and Remarks 5.15.

 (i) H - μ-supplemented is a proper generalization of μ-lifting, see example 3.3, in [5].

 (ii) H - μ-supplemented is a proper generalization of H -supplemented modules, See, example (3.17) in [5].

 (iii) It is obvious that $\mathbb{Q}$ is not $H - \mu$ supplemented $\mathbb{Z}$-module.

Proposition 5.16. *[5]. If each submodule of a module N has unique μ-coclosure, then N is H - μ-supplemented if and only if it is μ-lifting.*

Proposition 5.17. *[5]. Taking N be any module, the following conditions are equivalent.*

 (i) N is H - μ - supplemented.

 (ii) There is a direct summand T of N with $N = T \oplus T'$ Satisfies $(G + T) \cap T' \ll_\mu T'$, for every submodule G of N

 (iii) For all submodule G of N there is direct summand T of N such that $G \oplus T = T + X, X \ll_\mu N$.

Proposition 5.18. *[5]. put N be an $H - \mu$-supplemented, for each submodule G of N, there is a direct summand T of N with $N = T \oplus T'$ satisfies $G \cap T' \ll_\mu T'$.*

Proposition 5.19. *[5]. A module N is $H - \mu$-supplemented if and only if for all submodule G of N, there is a direct summand T of N and a submedule P of N satisfies $G \leqslant_{\mu ce} P$ in N and $T \leqslant_{\mu ce} P$ in N.*

Proposition 5.20. *[5]. Let G be a submodule of an H- μ-supplemented, then $\frac{N}{G}$ is again $H - \mu$ supplemented if any one of the next conditions hold.*

 (i) $\frac{T+G}{G}$ is direct summand of $\frac{N}{G}$, for all direct summand T of N.

 (ii) N has distributive condition.

Proposition 5.21. *[5]. Let G be a fully invarand submodule of $H - \mu$-supplemented module N, then $\frac{N}{G}$ is $H - \mu$-supplemented.*

Proposition 5.22. *[5]. Let G be a submodule of an $H - \mu$-supplemented module N, if for every $\varphi^2 = \varphi \in \mathrm{End}(N)$, there is $\psi^2 = \psi \in \mathrm{End}(N)$ Such that $\frac{P}{G} \leqslant_{\mu ce} \frac{G+\varphi(N)}{G}$ in $\frac{N}{G}$, where $\mathrm{Im}\,\psi = \frac{P}{G}$, then N is $H - \mu$-supplemented.*

6 Fully invariant μ-lifting modules.

An important component of the μ-lifting condition is that it supplied a μ-lifting condition for specific submodule kinds. Given that a large number of the most significant submodule types are fully invariant appears that many of the advantages of the μ-lifting condition may be attained as the definition F.T- μ-lifting condition.

Definition 6.1. [4]. A module N is said to be fully invariant μ-lifting module (For short FI-μ lifting) if for any fully invariant submodule G of N, there is a summand T of N such that $N = T \oplus T'$ and $G \cap \mathrm{T}' \ll_\mu T'$.

Examples and Remarks 6.2.

(i) $\mathbb{Z}$ as $\mathbb{Z}$-module is not fully invarion μ-lifting.

(ii) $N = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$ module show that a fully invariant μ-lifting modules are proper generalization of μ-lifting.

The next results present some conditions that are equivalent to fully invariant μ-lifting condition.

Proposition 6.3. *[4]. A module N is fully invariant μ-lifting if and only if for each fully invariant submodule G of N, there is a summand T of N satisfies $G \cap T' \ll_\mu N, N = T \oplus T'$.*

Proposition 6.4. *[4]. The next conditions are equivalent for a module N.*

(i) *N has fully invariant μ-lifting condition.*

(ii) *Each fully invariant submodule G of N can be written as $G = T \oplus X$, where T is a summand of N and X is μ-Small in N.*

(iii) *There is a decomposition $N = T \oplus T'$ satisfies $T \leqslant_{\mu ce} G$ in N, for every fully invariant submodule G of N.*

Theorem 6.5. *[4]. The following properties are equivalent for a module N.*

(i) *N satisfies fully invariant μ-lifting condition.*

(ii) *Each fully invariant G in N has μ-supplement K in N such that $G \cap K$ is a direct summand of G.*

Proposition 6.6. *[4]. The following satatements are equivalent for a module N.*

(i) *N is fully invariant μ-lifting.*

(ii) *Each fully invariant in N has μ-supplement direct summand of N.*

(iii) *There is a supplement T of N and a direct summand T' of T satisfies $T \leqslant_{\mu ce} G$ in N and all homomorphisms $\psi : N \to \frac{N}{T \cap T'}$ can be lifted to endomorphism $\varphi : N \to N, \varphi(n) + (T \cap T') = \psi(n), \forall n \in N$.*

Proposition 6.7. *[4]. Consider the next conditions for a module N.*

(i) *N is μ lifting.*

(ii) *N is $\oplus - \mu$ supplemented.*

(iii) *N is fully invariant μ-lifting.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

[4].

Proposition 6.8. *If each direct summand of an H-supplemented module is cosingular then it is fully invariant μ-lifting.*

Proposition 6.9. *[4]. Each fully invariant direct summand of fully invariant μ-lifting module is again fully invariant μ-lifting.*

Theorem 6.10. *[4]. Fully invariant μ-lifting modules are closed under direst sums.*

Proposition 6.11. *[4]. Taking $N = N_1 \oplus N_2, N_2$ is F.I- μ-lifting if and only if for each fully invariant submodule $\frac{G}{N_1}$ of $\frac{N}{N_1}$ there is a summand T of N satisfies $N = T + G, G \cap T \ll_\mu T$ and $T \leqslant N_2$.*

Open problems.

(i) Determine the prerequisites and requirements needed to characterize direct sums of H - μ-supplemented modules.

(ii) Determine necessary and / or sufficient conditions when a submodule of $H - \mu$ supplemented is H - μ-supplemented.

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