Palestine Journal of Mathematics

Vol 14(3)(2025) , 1236–1244

Complementing permutations of bipartite almost self-complementary 3-uniform hypergraphs

Lata Kamble, Charusheela Deshpande and Bhagyashri Athawale

Communicated by Ambat Vijayakumar

MSC 2010 Classifications: Primary 05C65.

Keywords and phrases: 3-uniform hypergraph, bipartite almost self-complementary, complementing permutation.

Corresponding Author: Lata Kamble

Abstract A bipartite 3-uniform hypergraph $H(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is almost self-complementary if it is isomorphic with its complement $\bar{H}(V_1, V_2)$ with respect to $K^3_{(m,n)} - e$. Every permutation σ of the set $V = V_1 \cup V_2$ such that $\sigma(e)$ is an edge of $\bar{H}$ if and only if e is an edge in H is called complementing permutation. In the present paper, we study the cycle structure of complementing permutation of bipartite almost self-complementary 3-uniform hypergraphs.

1 Introduction

Ringel [8] and Sachs [9] studied self-complementary graphs (2-hypergraphs). They characterized the cycle structure of complementing permutations of self-complementary graphs. The cycle structure of complementing permutation of self-complementary 3-uniform hypergraphs is analyzed by Kocay [7]. He proved that σ is a complementing permutation of a self-complementary 3-uniform hypergraph if and only if either (i) every cycle of σ has even length or (ii) σ has 1 or 2 fixed points and all the other cycles of σ have length a multiple of 4. A. Symański, A.P. Wojda ([10],[11],[12]) and S. Gosselin [3], independently studied k-uniform self-complementary hypergraphs of order n and gave the structure of corresponding complementing permutations.

T. Gangopadhyay and S.P. Rao Hebbare [2] studied structural properties of r-partite complementing permutations. In [5], a bipartite self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ is studied and the cycle structure of complementing permutation of bipartite self-complementary 3-uniform hypergraphs is analyzed.

Almost self-complementary 2-uniform hypergraphs i.e almost self-complementary graphs are introduced by Clapham in [1]. In [4], almost self-complementary 3-uniform hypergraphs are studied and the cycle structure of corresponding complementing permutations is analyzed.

Bipartite almost self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ is introduced in [6] by L.N. Kamble and et.al.

In this paper, we analyze the cycle structure of complementing permutations of a bipartite almost self-complementary 3-uniform hypergraph.

2 Preliminary definitions and results

Let V be a finite set with n elements. By $\binom{V}{k}$ we denote the set of all k-subsets of V.

Definition 2.1. **(k-uniform hypergraph)** A k-uniform hypergraph is a pair $H = (V; E)$, where $E \subseteq \binom{V}{k}$. V is called a vertex set, and E an edge set of H.

Definition 2.2. **(Complete k-uniform hypergraph)** A complete k-uniform hypergraph is the k-uniform hypergraph, K^k_n on n vertices whose edge set is $\binom{V}{k}$.

Definition 2.3. (Complement of a k-uniform hypergraph) The complement $\bar{H}$ of a k-uniform hypergraph $H(V, E)$ is the hypergraph with vertex set V and edge set $\bar{E}$=$\binom{V}{k}\setminus E$.

We say that $\bar{H}$ is the complement of H with respect to K_n^k.

Definition 2.4. Two k-uniform hypergraphs $H = (V; E)$ and $H' = (V'; E')$ are said to be isomorphic if there is a bijection $\sigma : V(H) \to V(H')$ which induces a bijection from $E(H)$ to $E(H')$.

Definition 2.5. (Self-complementary k-uniform hypergraph) A k-uniform hypergraph H is called self-complementary if it is isomorphic to its complement $\bar{H}$.

Thus H is self-complementary if and only if there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\bar{H}$. Such a σ is called a **complementing permutation**.

In the following example, we illustrate a self-complementary 3-uniform hypergraph.

Example 2.6. Consider a 3-uniform hypergraph $H(V, E)$ where the vertex set $V = \{u_1, u_2, u_3, u_4, u_5\}$ and the edge set $E = \{\{u_1, u_2, u_3\}, \{u_1, u_3, u_4\}, \{u_1, u_2, u_5\}, \{u_3, u_4, u_5\}, \{u_1, u_3, u_5\}\}$. The complement of H is given by $\bar{H}$ with vertex set V and the edge set $\bar{E} = \{\{u_2, u_3, u_4\}, \{u_1, u_2, u_4\}, \{u_2, u_3, u_5\}, \{u_1, u_4, u_5\}, \{u_2, u_4, u_5\}\}$.
Consider a bijection $\sigma : V \to V$ defined as $\sigma = (u_1\ u_2\ u_3\ u_4)(u_5)$. Observe that e is an edge in H if and only if $\sigma(e)$ is an edge in $\bar{H}$. H is isomorphic to its complement $\bar{H}$ with complementing permutation $\sigma = (u_1\ u_2\ u_3\ u_4)(u_5)$. Hence H is self-complementary.

Definition 2.7. (Almost complete k-uniform hypergraph) The hypergraph $\tilde{K_n^k} = K_n^k - e$, is called an almost complete 3-uniform hypergraph.

That is an almost complete k-uniform hypergraph is defined by deleting any one edge from K_n^k. The deleted edge is denoted by e and called it as the missing edge and the corresponding vertices of e the special vertices.

Example 2.8. Consider a complete 3-uniform hypergraph on 5 vertices, K_5^3 with the vertex set $V = \{u_1, u_2, u_3, u_4, u_5\}$ and the edge set $E = \{\{u_1, u_2, u_3\}, \{u_1, u_3, u_4\}, \{u_1, u_2, u_5\}, \{u_3, u_4, u_5\}, \{u_1, u_3, u_5\}, \{u_2, u_3, u_4\}, \{u_1, u_2, u_4\}, \{u_2, u_3, u_5\}, \{u_1, u_4, u_5\}, \{u_2, u_4, u_5\}\}$.
We delete an edge $\{u_1, u_2, u_3\}$ from K_5^3. The remaining edges form an edge set of an almost complete 3-uniform hypergraph on 5 vertices which is denoted by $\tilde{K_5^3}$. Edge set of $\tilde{K_5^3}$ is $\tilde{E} = \{\{u_1, u_3, u_4\}, \{u_1, u_2, u_5\}, \{u_3, u_4, u_5\}, \{u_1, u_3, u_5\}, \{u_2, u_3, u_4\}, \{u_1, u_2, u_4\}, \{u_2, u_3, u_5\}, \{u_1, u_4, u_5\}, \{u_2, u_4, u_5\}\}$.

Definition 2.9. (Almost self-complementary k-uniform hypergraph) A k-uniform hypergraph H with n vertices is almost self-complementary if it is isomorphic with its complement $\bar{H}$ with respect to $\tilde{K_n^k}$.

This means that a 3-uniform hypergraph H with n vertices is almost self-complementary if $\tilde{K_n^3}$ can be decomposed into two isomorphic factors with H as one of the factor.
Almost self-complementary k-uniform hypergraph of order n may be called self-complementary in $K_n^k - e$.

In the following example, we illustrate an almost self-complementary 3-uniform hypergraph.

Example 2.10. Let $V = \{u_1, u_2, u_3, u_4, x, y, z\}$. Clearly, K_7^3 contains an odd number of edges. Therefore after deleting one edge from K_7^3, the remaining edges are even in number. Let us delete $e = \{x, y, z\}$ from K_7^3. Let $\tilde{K_7^3} = K_7^3 - e$.
We can factorize $\tilde{K_7^3}$ into two isomorphic factors H and $\bar{H}$.
Let H be the 3-uniform hypergraph on V with edge set
$E = \{\{u_1, u_2, u_3\}, \{u_1, u_2, u_4\}, \{u_1, u_2, x\}, \{u_3, u_4, x\}, \{u_1, u_4, x\}, \{u_1, u_3, x\}, \{u_1, u_4, y\}, \{u_2, u_4, y\}, \{u_1, u_3, z\}, \{u_1, u_2, z\}, \{u_2, u_3, z\}, \{u_3, x, y\}, \{u_4, x, y\}, \{u_3, x, z\}, \{u_4, y, z\}, \{u_3, y, z\}, \{u_2, y, z\}\}$.
Complement of H in $\tilde{K_7^3}$ is given by $\bar{H}$ having edge set

$\bar{E} = \{\{u_2, x, u_4\}, \{u_2, x, y\}, \{u_2, x, u_3\}, \{u_4, y, u_3\}, \{u_2, y, u_3\}, \{u_2, u_4, u_3\}, \{u_2, y, u_1\}, \{x, y, u_1\}, \{u_2, u_4, z\}, \{u_2, x, z\}, \{x, u_4, z\}, \{u_4, u_3, u_1\}, \{y, u_3, u_1\}, \{u_4, u_3, z\}, \{y, u_1, z\}, \{u_4, u_1, z\}, \{x, u_1, z\}\}$.
Define a permutation $\sigma : V \to V$ by $\sigma(x) = u_3, \sigma(y) = u_1, \sigma(z) = z$, $\sigma(u_1) = u_2$, $\sigma(u_4) = y$, $\sigma(u_2) = x$, $\sigma(u_3) = u_4$. That is $\sigma = (x\ u_3\ u_4\ y\ u_1\ u_2)(z)$.
Clearly H is isomorphic to its complement $\bar{H}$ with complementing permutation σ. Hence H is almost self-complementary 3-uniform hypergraph.

Definition 2.11. (Bipartite Hypergraph) A hypergraph H with vertex set V and edge set E is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$ for any $e \in E$.

Furthermore if $|e| = k$ for every $e \in E$ then we call H, a bipartite k-uniform hypergraph, and denote it as $H^k(V_1, V_2)$. If $|V_1| = m$ and $|V_2| = n$ then $H^k(V_1, V_2) = H^k_{(m,n)}$.
If $H^3(V_1, V_2)$ is a bipartite 3-uniform hypergraph then every edge of $H^3(V_1, V_2)$ contains one vertex from one part and two vertices from the other part of the partition V_1 and V_2 of V. Thus any triple of vertices $\{x, y, z\}$ such that x, y, z belong to a single part of the partition of V is not an edge of $H^3(V_1, V_2)$.
The following example illustrates a bipartite 3-uniform hypergraph.

Example 2.12. Consider a hypergraph $H(V; E)$ with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $E = \{\{v_1, v_5, v_6\}, \{v_1, v_2, v_6\}, \{v_3, v_4, v_5\}, \{v_1, v_3, v_6\}\}$. H is a bipartite 3-uniform hypergraph with partition $P = (V_1, V_2)$ where $V_1 = \{v_1, v_3, v_4\}$ and $V_2 = \{v_2, v_5, v_6\}$ of vertex set V of H.

Definition 2.13. (Complete Bipartite 3-uniform Hypergraph) A 3-uniform hypergraph H with the vertex set $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$ and the edge set $E = \{e : e \subset V, |e| = 3 \text{ and } e \cap V_i \neq \emptyset, \text{ for } i = 1, 2\}$ is called the complete bipartite 3-uniform hypergraph. It is denoted as $K^3(V_1, V_2)$ or $K^3_{(m,n)}$.

Definition 2.14. (Complement of bipartite 3-uniform hypergraph) Given a bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$, we define its bipartite complement to be the 3-uniform hypergraph $\bar{H} = \bar{H}^3(V_1, V_2)$ where $V(\bar{H}) = V(H)$ and $E(\bar{H}) = E(K^3(V_1, V_2)) - E(H)$.

Definition 2.15. (Bipartite self-complementary 3-uniform hypergraph) A bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$ is said to be self-complementary if it is isomorphic to its bipartite complement $\bar{H} = \bar{H}^3(V_1, V_2)$, that is there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\bar{H}$.

That is there exists a bijection $\sigma : V \to V$ such that $e = \{x, y, z\}$ is an edge in H if and only if $\sigma(e) = \{\sigma(x), \sigma(y), \sigma(z)\}$ is an edge in $\bar{H}$. Such a σ is called a complementing permutation.

In the following example, we illustrate bipartite self-complementary 3-uniform hypergraph.

Example 2.16. Let $V = \{u_1, u_2, v_1, v_2\}$ such that $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2\}$.
i) Consider $H_1(V_1, V_2)$ with edge set $\{\{u_1, u_2, v_1\}, \{u_1, u_2, v_2\}\}$ and its complement $\bar{H}_1(V_1, V_2)$ with edge set $\{\{v_1, v_2, u_1\}, \{v_1, v_2, u_2\}\}$.
Clearly $H_1(V_1, V_2)$ is a bipartite self-complementary with complementing permutation $\sigma_1 = (u_1\ v_1)(u_2\ v_2)$ or $\sigma_2 = (u_1\ v_1\ u_2\ v_2)$.
ii) Consider $H_2(V_1, V_2)$ with edge set $\{\{u_1, u_2, v_1\}, \{v_1, v_2, u_1\}\}$ and its complement $\bar{H}_2(V_1, V_2)$ with edge set $\{\{u_2, v_1, v_2\}, \{u_1, u_2, v_2\}\}$.
Clearly $H_2(V_1, V_2)$ is a bipartite self-complementary with complementing permutation $\sigma_3 = (u_1\ u_2\ v_1\ v_2)$ or $\sigma_4 = (u_1\ u_2)(v_1\ v_2)$.

In [5], the cycle structure of complementing permutation of bipartite self-complementary 3-uniform hypergraphs is analyzed.

Definition 2.17. (Almost complete bipartite 3-uniform hypergraph) The hypergraph $\tilde{K}^3_{(m,n)} = K^3_{(m,n)} - e$ is called an almost complete bipartite 3-uniform hypergraph.

The hypergraph $\tilde{K}^3_{(m,n)}$ is a bipartite 3-uniform hypergraph obtained by deleting any edge e from $K^3_{(m,n)}$. We always denote by e the edge deleted from $K^3_{(m,n)}$, call it as the missing edge, and the corresponding vertices of e the special vertices.

Definition 2.18. (Bipartite almost self-complementary 3-uniform hypergraph) A 3-uniform bipartite hypergraph $H(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is almost self-complementary if it is isomorphic with its complement $\bar{H}(V_1, V_2)$ with respect to $\tilde{K}^3_{(m,n)} = K^3_{(m,n)} - e$.

This means that, a 3-uniform hypergraph $H(V_1, V_2)$ is almost self complementary if $\tilde{K}^3_{(m,n)}$ can be decomposed into two isomorphic factors with $H(V_1, V_2)$ as one of the factors.

We use the shortform "bipasc" for bipartite almost self-complementary 3-uniform hypergraph.

In the following example, we illustrate an isomorphic factorization of $K^3_{(2,3)}$ after deleting one edge.

Example 2.19. Consider $K^3_{(2,3)}$. Let $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$ be a partition of vertex set $V = \{u_1, u_2, v_1, v_2, v_3\}$. The edge set of $K^3_{(2,3)}$ is $E = \{\{u_1, u_2, v_1\}, \{u_1, u_2, v_2\}, \{u_1, u_2, v_3\}, \{v_1, v_2, u_1\}, \{v_1, v_2, u_2\}, \{v_1, v_3, u_1\}, \{v_1, v_3, u_2\}, \{v_2, v_3, u_1\}, \{v_2, v_3, u_2\}\}$.
We note that the edge set of $K^3_{(2,3)}$ contains an odd number of edges. We delete one edge from $K^3_{(2,3)}$ say $\{u_1, u_2, v_3\}$. Thus the edge set of almost complete bipartite 3-uniform hypergraph, $\tilde{K}^3_{(m,n)}$ is then $\tilde{E} = \{\{u_1, u_2, v_1\}, \{u_1, u_2, v_2\}, \{v_1, v_2, u_1\}, \{v_1, v_2, u_2\}, \{v_1, v_3, u_1\}, \{v_1, v_3, u_2\}, \{v_2, v_3, u_1\}, \{v_2, v_3, u_2\}\}$. Consider the bipartite 3-uniform hypergraph $H(V_1, V_2)$ with the edge set $E = \{\{u_1, u_2, v_1\}, \{v_1, v_2, u_1\}, \{v_1, v_3, u_1\}, \{v_1, v_3, u_2\}\}$ and its complement $\bar{H}$ in $\tilde{K}^3_{(m,n)}$ with the edge set $\bar{E} = \{\{u_1, u_2, v_2\}, \{v_1, v_2, u_2\}, \{v_2, v_3, u_2\}, \{v_2, v_3, u_1\}\}$.
H is isomorphic to $\bar{H}$ with complementing permutation $\sigma = (u_1\ u_2)(v_1\ v_2)(v_3)$. Hence H is a bipartite almost self-complementary 3-uniform hypergraph where $e = \{u_1, u_2, v_3\}$ is the missing edge.

The following theorem gives necessary and sufficient conditions on the order of bipartite 3-uniform hypergraph $H^3_{(m,n)}$ to be almost self-complementary which is proved in [6].

Theorem 2.20. *There exists a bipasc 3-uniform hypergraph $H^3_{(m,n)}$ with partition (V_1, V_2) of vertex set V where $|V_1| = m$ and $|V_2| = n$ if and only if $m \neq n$ and either*
(i) one is congruent to 1 modulo 4 and the other is congruent to 2 or 3 modulo 4, or
(ii) one is congruent to 2 modulo 4 and the other is congruent to 1 or 3 modulo 4.

In the next section, we analyze cycle structure of complementing permutations of bipartite almost self-complementary 3-uniform hypergraph.

3 Complementing permutations of bipartite almost self-complementary 3-uniform hypergraph

Given a bipasc H with partition (V_1, V_2) of the vertex set V and edge set E, let the edges of H be colored red and the remaining edges of $\tilde{K}^3_{(m,n)}$ be colored green. Since the 2 factors are isomorphic, there is a permutation σ of the vertices of $\tilde{K}^3_{(m,n)}$ that induces a mapping of the red edges onto the green edges. Let e denote the missing edge. We consider σ as a permutation of the vertices of $K^3_{(m,n)}$, and denote by σ' the corresponding mapping induced on the set of edges of $K^3_{(m,n)}$. Thus σ' maps each red edge onto a green edge. However, the mapping σ' need not necessarily map each green edge onto a red edge. This would be so if σ' mapped e onto itself, but it may be that σ' maps e onto a red edge and some green edge onto e. Such a σ (which, for definiteness, we shall always assume induces a mapping from red to green) will (as for s.c. 3-uniform hypergraphs) be called a complementing permutation. It will be useful to consider the cycles of the induced mapping σ'.

Any edge $\{u, v, w\}$ in $K^3_{m,n}$ gives rise to a cycle $\{u, v, w\}, \sigma'\{u, v, w\}, \sigma'^2\{u, v, w\}, \cdots$ etc. These must be alternately edges of H and $\bar{H}$. Hence if this cycle does not include the missing edge e then the length of this cycle must be even. And if it includes the missing edge e then its length must be odd. Moreover, it is the only cycle of σ' of odd length.

The following remarks regarding the cycles of induced mapping σ' will be used to prove a number of results about the structure of complementing permutation σ.

Remark 3.1. A cycle of σ' that does not include e must be of even length, consisting of edges alternately red and green i.e edges of H and $\bar{H}$ alternately.

Remark 3.2. The cycle of σ' that includes e has an odd length, consisting of e followed by red and green edges alternately. This must be the only one cycle of σ' having odd. Further, this length equals 1 when σ' maps e onto itself.

Remark 3.3. For any edge $\{u, v, w\}$ in $K^3_{m,n}$ the length of the cycle of σ' containing $\{u, v, w\}$ depends on the length of the cycles of σ containing u, v, w. There are three cases according to the appearance of u, v, w in the cycles of σ.

I) Suppose u, v and w are all in the same cycle of σ, of length L, say. Let $(v_1, v_2, \cdots, v_L)$ represent the cycle. Consider a cycle $\{u, v, w\}, \sigma'\{u, v, w\}, \sigma'^2\{u, v, w\}, \cdots$. This cycle will also have length L unless $L = 3r$ and u, v and w are v_1, v_{1+r} and v_{1+2r}. The length of this cycle will then be r, so that r must be even.

II) Suppose the vertices u and v belong to the same cycle C_1 of length L_1 and the vertex w belongs to the cycle C_2 of length L_2. Clearly L_1 is even. We consider the following cases.
Case (i). If $L_1 = 2$ then the length of the cycle of σ' containing the edge $\{u, v, w\}$ is L_2.
Case (ii). If $L_1 = 2h, h > 1$ such that $\sigma^h(u) = v$ then the length of the cycle of σ' containing the edge $\{u, v, w\}$ is the least common multiple of h and L_2. If $\sigma^h(u) \neq v$ then the length of the cycle of σ' containing the edge $\{u, v, w\}$ is the least common multiple of L_1 and L_2.

III) If u is in a cycle C_1 of length L_1, v is in a cycle C_2 of length L_2 and w is in a cycle C_3 of length L_3 where C_1, C_2, C_3 are distinct cycles of the permutation σ then the length of the cycle of σ' containing the edge $\{u, v, w\}$ is the least common multiple of L_1, L_2 and L_3. Not all of u, v, and w can be fixed points of σ unless $\{u, v, w\}$ is the missing edge.

In the following example, we illustrate complementing permutations of a bipasc $H(V_1, V_2)$.

Example 3.4. (a) Consider $K^3_{(1,3)}$. Let $V_1 = \{u_1\}$ and $V_2 = \{v_1, v_2, v_3\}$ be a partition of the vertex set $V = \{u_1, v_1, v_2, v_3\}$.
The edge set of $K^3_{(1,3)}$ is $E = \{\{u_1, v_1, v_2\}, \{u_1, v_1, v_3\}, \{u_1, v_2, v_3\}\}$. We delete the edge $e = \{u_1, v_2, v_3\}$ from $K^3_{(1,3)}$.
Consider H with edge set $E_1 = \{\{u_1, v_1, v_2\}\}$. Then $\bar{H}$ has edge set $E_2 = \{\{u_1, v_1, v_3\}\}$. Clearly, H is isomorphic to $\bar{H}$ with complementing permutation $\sigma = (u_1)(v_1)(v_2\ v_3)$ or $\sigma = (u_1)(v_1\ v_3\ v_2)$ with the missing edge $\{u_1, v_2, v_3\}$.
Consider the cycles of the corresponding induced mapping σ'.
$\sigma' = (\{u_1, v_1, v_2\}, \{u_1, v_1, v_3\})(\{u_1, v_2, v_3\})$ or $\sigma' = (\{u_1, v_1, v_2\}, \{u_1, v_3, v_1\}, \{u_1, v_2, v_3\})$. We observe that there is exactly one odd cycle of σ' containing the missing edge $\{u_1, v_2, v_3\}$ while all other cycles of σ' are even.

(b) Consider $K^3_{2,3}$. Let $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$ be a partition of the vertex set $V = \{u_1, u_2, v_1, v_2, v_3\}$. The edge set of $K^3_{(2,3)}$ is
$E = \{\{u_1, v_1, v_2\}, \{u_1, v_1, v_3\}, \{u_1, v_2, v_3\}, \{u_2, v_1, v_2\}, \{u_2, v_1, v_3\}, \{u_2, v_2, v_3\}, \{u_1, u_2, v_1\}, \{u_1, u_2, v_2\}, \{u_1, u_2, v_3\}\}$.
We delete $\{u_1, u_2, v_3\}$ from $K^3_{(2,3)}$.
Consider H with edge set $E_1 = \{\{u_1, u_2, v_1\}, \{v_1, v_2, u_2\}, \{u_1, v_1, v_3\}, \{u_1, v_2, v_3\}\}$. Then $\bar{H}$ has the edge set $E_2 = \{\{u_1, u_2, v_2\}, \{u_1, v_1, v_2\}, \{u_2, v_2, v_3\}, \{u_2, v_1, v_3\}\}$. Clearly, H is isomorphic to $\bar{H}$ with complementing permutation $\sigma_1 = (u_1\ u_2)(v_1\ v_2)(v_3)$ or $\sigma_2 = (u_1\ u_2)(v_1\ v_2\ v_3)$ with the missing edge $\{u_1, u_2, v_3\}$.

Consider the cycles of the corresponding induced mapping σ'.
$\sigma' = (\{u_1, v_1, v_2\}, \{u_1, v_1, v_3\})(\{u_1, v_2, v_3\})$ or $\sigma' = (\{u_1, v_1, v_2\}, \{u_1, v_3, v_1\}, \{u_1, v_2, v_3\})$. We observe that there is exactly one odd cycle of σ' containing the missing edge $\{u_1, v_2, v_3\}$ while all other cycles of σ' are even.

(c) Consider $K^3_{1,7}$. Let $V_1 = \{u_1\}$ and $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a partition of the vertex set $V = \{u_1, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. The edge set of $K^3_{(1,7)}$ is $E = \{\{u_1, v_1, v_2\}, \{u_1, v_1, v_3\}, \{u_1, v_1, v_4\}, \{u_1, v_1, v_5\}, \{u_1, v_1, v_6\}, \{u_1, v_1, v_7\}, \{u_1, v_2, v_3\}, \{u_1, v_2, v_4\}, \{u_1, v_2, v_5\}, \{u_1, v_2, v_6\}, \{u_1, v_2, v_7\}, \{u_1, v_3, v_4\}, \{u_1, v_3, v_5\}, \{u_1, v_3, v_6\}, \{u_1, v_3, v_7\}, \{u_1, v_4, v_5\}, \{u_1, v_4, v_6\}, \{u_1, v_4, v_7\}, \{u_1, v_5, v_6\}, \{u_1, v_5, v_7\}, \{u_1, v_6, v_7\}\}$.
We delete the edge $\{u_1, v_1, v_4\}$ from $K^3_{(1,7)}$.
Consider H with edge set $E_1 = \{\{u_1, v_2, v_5\}, \{u_1, v_1, v_2\}, \{u_1, v_3, v_4\}, \{u_1, v_5, v_6\}, \{u_1, v_1, v_3\}, \{u_1, v_3, v_5\}, \{u_1, v_5, v_1\}, \{u_1, v_1, v_7\}, \{u_1, v_3, v_7\}, \{u_1, v_5, v_7\}\}$.
Then $\bar{H}$ has the edge set $E_2 = \{\{u_1, v_3, v_6\}, \{u_1, v_2, v_3\}, \{u_1, v_4, v_5\}, \{u_1, v_6, v_1\}, \{u_1, v_2, v_4\}, \{u_1, v_4, v_6\}, \{u_1, v_6, v_2\}, \{u_1, v_2, v_7\}, \{u_1, v_4, v_7\}, \{u_1, v_6, v_7\}\}$.
Clearly, H is isomorphic to $\bar{H}$ with complementing permutation $\sigma = (u_1)(v_1\ v_2\ v_3\ v_4\ v_5\ v_6)(v_7)$ with the missing edge $\{u_1, v_1, v_4\}$.
We observe that there is exactly one odd cycle of σ' containing the missing edge $\{u_1, v_1, v_4\}$ while all other cycles of σ' are even.
Here the special vertices v_1 and v_4 both belong to the same cycle of σ of length 6 such that $\sigma^3(v_1) = v_4$ and the special vertex u_1 is fixed. Hence the length of the cycle of σ' containing the missing edge $\{u_1, v_1, v_4\}$ is 3 which is the least common multiple of the length of the cycle containing u_1 and half of the length of the cycle containing v_1 and v_4. It is the only odd cycle of σ' containing the missing edge $\{u_1, v_1, v_4\}$. And all other cycles of σ' are of even length.

Given a bipasc H with partition (V_1, V_2) of vertex set V and edge set E, let σ be a permutation of the vertices of $K^3_{(m,n)}$ and σ' be the corresponding mapping induced on the set of edges of $\tilde{K}^3_{(m,n)}$. It will be useful to consider the cycles of the induced mapping σ'.
In further sections we will denote the missing edge by $e = \{x, y, z\}$ and $x, y \in V_1$ and $z \in V_2$ will be called the special vertices.

Let $\xi(H, (V_1, V_2))$ be the set of all complementing permutations of the bipasc H. A cycle of a complementing permutation is said to be **pure** if it permutes only the vertices belonging to a single set of the partition (V_1, V_2) of the vertex set V and is said to be **mixed** otherwise.
We define two subclasses of $\xi(H, (V_1, V_2))$ as follows
$\xi_P(H, (V_1, V_2)) = \{\sigma \in \xi(H, (V_1, V_2))$, all cycles of σ are pure $\}$
$\xi_M(H, (V_1, V_2)) = \{\sigma \in \xi(H, (V_1, V_2))$, all cycles of σ are mixed $\}$.
If $\sigma \in \xi_P(H, (V_1, V_2))$ then σ is said to be pure and if $\sigma \in \xi_M(H, (V_1, V_2))$ then σ is said to be mixed.

Theorem 3.5. *If $\sigma \in \xi(H, (V_1, V_2))$ then all the cycles of σ are either pure or mixed, that is* $\xi(H, (V_1, V_2)) = \xi_P(H, (V_1, V_2)) \cup \xi_M(H, (V_1, V_2))$.

Proof. Suppose σ contains a pure cycle C_1 and a mixed cycle C_2. Let $C_1 = (v_1\ v_2\ ...\ v_k)$ where $v_1, v_2, ..., v_k \in V_1$. Because C_2 is mixed there exists at least one $a \in V_1$ and one $b \in V_2$ such that $a, b \in C_2$ and $\sigma(b) = a$. Now the edge $\{b, v_1, v_2\} \in K^3(V_1, V_2)$ is either in H or $\bar{H}$ but $\sigma(\{b, v_1, v_2\}) = \{a, v_2, v_3\}$ is neither in $\bar{H}$ nor in H being an invalid triple.

□

From the above Theorem 3.5 we get that if σ is any complementing permutation, then it is either pure or mixed. Further, if σ contains a fixed vertex u that is $\sigma(u) = u$, then it must be pure.
We now show that $\xi_M(H, (V_1, V_2)) = \emptyset$ and then give the structure of any $\sigma \in \xi_P(H, (V_1, V_2))$.

3.1 Cycle structure of mixed complementing permutations

In this section, we analyze the cycle structure of mixed complementing permutations. Let, if possible, $\sigma \in \xi_M(H, (V_1, V_2))$. Then σ has no fixed vertex. Further, every cycle of σ has length

≥ 2.

Example 3.4(a) illustrates all possible complementing permutations for a bipasc on 4 vertices. We note that none of these belongs to $\xi_M(H,(V_1,V_2))$. In order to analyze the cycle structure of mixed complementing permutation, in the remaining part of this section we consider bipasc with at least 5 vertices.

Theorem 3.6. *If σ is a complementing permutation of a bipasc H then σ can not be mixed.*

Proof. Suppose σ is a mixed complementing permutation of a bipasc H such that σ has a cycle of length L containing all the special vertices $x,y \in V_1,\ z \in V_2$. Then L must be odd since the length of the cycle of σ' containing e is odd.

If $L > 3$ then it must contain at least one vertex other than x,y and z. This means there is a valid edge say e_1 other than e containing vertices from C. The length of the cycle of σ' containing e_1 must be even. Hence L is even, a contradiction.

Thus only possibility is $L = 3$. In this case C is either $C = (x\ y\ z)$ or $C = (x\ z\ y)$. Let $C = (x\ y\ z)$ and let C' be another cycle of σ. As C' is mixed there are vertices u,v in C such that $u \in V_1$, $v \in V_2$ and $\sigma(v) = u$. Consider a triple $\{x,z,v\}$ which is a valid edge but $\sigma(\{x,z,v\}) = \{y,x,u\}$ is not a valid edge, a contradiction. We get a similar contradiction if $C = (x\ z\ y)$.
Hence, σ can not be mixed. □

In the next section, we give the cycle structure of pure complementing permutations.

3.2 Cycle structure of pure complementing permutations

If σ is a pure complementing permutation of a bipasc $H(V_1,V_2)$, then σ can be written as $\sigma = \sigma_1\sigma_2$, where σ_1 permutes the vertices of V_1 and σ_2 permutes the vertices of V_2. As we are considering pure complementing permutation, all the special vertices can not occur in the same cycle of σ.
The following theorem gives cycle structure of pure complementing permutations.

Proposition 3.7. *Suppose $H^3_{(m,n)}$ is a bipasc with $\sigma = \sigma_1\sigma_2 \in \xi_P(H,(V_1,V_2))$ and $e = \{x,y,z\}$ is the missing edge with special vertices $x,y \in V_1$ and $z \in V_2$. Then either*
(i) x,y,z are fixed and all the other cycles of σ_1 and σ_2 are of length a multiple of 4, or
(ii) σ_1 has a cycle of length 3 containing x and y, z is fixed in σ_2 and all the other cycles of σ_1 and σ_2 are of length a multiple of 4, or
(iii) σ_1 has a cycle of length $4h+2, h \geq 0$ containing x and y such that $\sigma^{2h+1}(x) = y$, one fixed vertex in σ_1, z is fixed in σ_2 and all the other cycles of σ_1 and σ_2 are of length a multiple of 4, or
(iv) σ_1 has a cycle of length $4h+2, h > 0$ containing x and y such that $\sigma^{2h+1}(x) = y$, all the other cycles of σ_1 are of length a multiple of 4, z is fixed in σ_2 and all the other cycles of σ_2 are of even length, or
(v) σ_1 has a cycle of length 2 containing x and y, all the other cycles of σ_1 are of length a multiple of 4, σ_2 has a cycle of odd length $L > 1$ containing z and all the other cycles of σ_2 are of even length.

Proof. We prove the theorem by considering the occurrence of special vertices $x,y \in V_1$ and $z \in V_2$ in σ. We observe that there are only two possibilities either x and y belong to the same cycle of σ_1 or x and y belong to different cycles of σ_1. Note that z can not belong to an even cycle. Otherwise, the cycle of σ' containing the edge e will be of even length which will be a contradiction.

Case (A). Suppose x,y belong to different cycles C_1 and C_2 of length L_1 and L_2 respectively and z belongs to an odd cycle C_3 of length L_3. The length of the cycle of σ' containing the edge e, which is the least common multiple of L_1, L_2 and L_3, must be odd. Hence L_1 and L_2 are odd.

If $L_1 \geq 3$ then we get a cycle of σ' of odd length not containing the edge e, a contradiction. Hence $L_1 = 1$. Similarly $L_2 = L_3 = 1$.
Further, if $C_1' = (a_1\ a_2\ \cdots\ a_r)$ and $C_2' = (b_1\ b_2\ \cdots\ b_s)$ are any other cycles in σ_1 and σ_2

respectively then both C_1' and C_2' must be even. Since, if C_1' is odd then length of the cycle of σ' containing the edge $\{a_1, x, z\}$ is odd and this cycle does not contain e, a contradiction. Similarly, if C_2' is odd, then the length of the cycle of σ' containing the edge $\{b_1, x, z\}$ is odd and this cycle does not contain e, a contradiction. Hence both C_1' and C_2' are even. In this case the length of the cycle of σ' containing the edge $\{a_i, a_j, z\}$ is either r or $\frac{r}{2}$ when $a_j = a_{(i+\frac{r}{2})}$. This will be even only if r is a multiple of 4. Similarly, we can prove that C_2' must be of length a multiple of 4. This proves (i) of the theorem.

Case (B). Suppose x, y belong to cycle C_1 of length L_1 and z belongs to cycle C_2 of length L_2, which is odd.
(1) If L_1 is odd then
(a) $L_2 = 1$. This is because, if $L_2 \geq 3$ then for any vertex v in C_2 such that $v \neq z$, a cycle of σ' containing the edge $\{z, v, x\}$ is of odd length and it does not contain e, a contradiction. Therefore $L_2 = 1$. In this case if $L_1 > 3$ then $C_1 = (u_1\ u_2\ u_3\ \cdots u_{L_1})$ where $u_i = x$ and $u_j = y$ for some $i \neq j$. Then we can find an odd cycle of σ' not containing e, which is a contradiction. Hence $L_1 = 3$.
(b) If $C_1' = (a_1\ a_2\ \cdots\ a_r)$ is any other cycle of σ_1, then $r \neq 1$. If $r = 1$ then $C_1' = (a_1)$ and the cycle of σ' containing the edge $\{x, z, a_1\}$ will be of odd length and it does not contain e, a contradiction. Therefore $r > 1$. In this case the length of the cycle of σ' containing the edge $\{a_i, a_j, z\}$ is either r or $\frac{r}{2}$, when $a_j = a_{(i+\frac{r}{2})}$. This will be even only if r is a multiple of 4.
(c) Similarly, if $C_2' = (b_1\ b_2\ \cdots\ b_s)$ is any other cycle of σ_2, then $s \neq 1$. For, if $s = 1$ then $C_1' = (b_1)$ and hence the cycle of σ' containing the edge $\{x, z, b_1\}$ is of odd length and it does not contain e, a contradiction. Therefore $s > 1$. In this case the cycle of σ' containing the edge $\{b_i, b_j, u\}$ for any u in C_1, will have the length either least common multiple of L_1 and s or the least common multiple of L_1 and $\frac{s}{2}$, when $b_j = b_{(i+\frac{s}{2})}$, which must be even. Thus s must be a multiple of 4. This proves (ii) of the theorem.
(2) Let L_1 be even, and let $C_1 = (u_1\ u_2\ \cdots\ u_{L_1})$. Then for any v in C_2, the length of the cycle of σ' containing the edge $\{u_i, u_j, v\}$ is either the least common multiple of L_1 and L_2 or the least common multiple of $\frac{L_1}{2}$ and L_2 when $u_j = u_{(i+\frac{L_1}{2})}$. Moreover, if $u_i = x, u_j = y$ and $v = z$ then this length must be odd. Hence the only possibility is that $L_1 = 4h + 2$ with $\sigma^{2h+1}(x) = y$. Further,
(a) If $h > 0$ then $L_2 = 1$, for if $L_2 > 1$ then for any vertex $d \in C_2, d \neq z$, length of the cycle of σ' containing the edge $\{x, z, d\}$ is odd which does not contain e, a contradiction.
Further, let $C_1' = (a_1\ a_2\ \cdots\ a_r)$ be any cycle in σ_1.
(i) If r is odd then $r = 1$. For if $r > 1$ then for any a_i, a_j in C_1' the length of the cycle of σ' containing the edge $\{a_i, a_j, z\}$ is odd and it does not contain e, a contradiction. Hence $r = 1$ which implies that $C_1' = (a_1)$. In this case, if $C_1'' = (a_1'\ a_2'\ \cdots\ a_{r'}')$, $(C_1'' \neq C_1')$ is any other cycle in σ_1, then r' must be even this is because for any vertex a_i' in C_1', the length of the cycle in σ' containing the edge $\{a_1, a_i', z\}$ is r', which must be even. In this case for any a_i', a_j' in C_1'', the length of the cycle of σ' containing the edge $\{a_i', a_j', z\}$ is the least common multiple of r' and $\frac{r'}{2}$, when $a_j' = a_{(i+\frac{r'}{2})}$. This implies that r' must be a multiple of 4.
Further, let $C_2' = (b_1\ b_2\ \cdots\ b_s)$ be any other cycle in σ_2. Then C_2' can not be an odd cycle. This is because if C_2' is an odd cycle then for any b in C_2', the length of the cycle of σ' containing the edge $\{x, y, b\}$ will be of odd length which does not contain e, a contradiction. Hence C_2' must be an even cycle. In this case for any b_i, b_j in C_2', the length of the cycle of σ' containing the edge $\{a_1, b_i, b_j\}$ is either s or $\frac{s}{2}$, which must be even. Hence s must be a multiple of 4.
(ii) Now, if r is even, then for any a_i, a_j in C_1' the length of the cycle of σ' containing the edge $\{a_i, a_j, z\}$ is either $\frac{r}{2}$ or r and both must be even. Hence r must be a multiple of 4. And if $C_2' = (b_1\ b_2\ \cdots\ b_s)$ is any other cycle in σ_2, then arguing as above, s must be even.
This proves (iv) of the theorem.
(b) Suppose $h = 0$. That is $C_1 = (x\ y)$. Let $C_1' = (a_1\ a_2\ \cdots\ a_r)$ be any cycle in σ_1.
(i) Suppose $L_2 = 1$.
Now if r is odd then $r = 1$ this is because if $r > 1$ for any a_i, a_j in C_1' the length of the cycle of σ' containing the edge $\{a_i, a_j, z\}$ is odd not containing e, a contradiction. Further it is the only cycle of odd length in σ_1 otherwise we get a cycle of σ' of odd length not containing e, a contradiction.

If r is even then the length of the cycle in σ' containing the edge $\{a_i, a_j, z\}$ is either r or $\frac{r}{2}$, which does not contain e. It must be even. Hence r must be a multiple of 4.
Similarly we can prove that, if $C_2' = (b_1\ b_2\ \cdots\ b_s)$ is any other cycle in σ_2 then s must be even provided $r \neq 1$ otherwise if $r = 1$ then s must be a multiple of 4.
Above argument together with $(2)(a)(i)$ proves (iii) of the theorem.
(ii) Now, suppose $L_2 > 1$. Then r must be even if not then for any v_i, v_j in C_2 and a_k in C_1', the length of the cycle in σ' containing the edge $\{v_i, v_j, a_k\}$ is odd, not containing the edge e, a contradiction. Further for any vertices a_i, a_j in C_1', length of the cycle in σ' containing the edge $\{a_i, a_j, z\}$ is either the least common multiple of r and L_2 or the least common multiple of $\frac{r}{2}$ or L_2, which must be even. As L_2 is odd, r must be a multiple of 4. This proves (v) of the theorem. □

Acknowledgment: Authors wish to thank Professor N.S. Bhave for fruitful discussions.

References

[1] C. R. J. Clapham, *Graphs self-Complementary in $K_n - e$*, Discrete Math. **81** (1990), 229–235.

[2] T. Gangopadhyay and S.P. Rao Hebbare, *Structural properties of r-partite complementing permutations*, Tech. Report No. 19/77, I.S.I, Calcutta.

[3] S. Gosselin, *Generating self-complementary uniform hypergraphs*, Discrete Math., **310** (2010), 1366–1372.

[4] L.N. Kamble, C.M. Deshpande, B.Y. Bam, *Almost self-complementary 3-uniform hypergraphs*, Discuss. Math. Graph Theory, **37** (2017), 131–140.

[5] L.N. Kamble, C.M. Deshpande, B.P. Athawale, *On self-complementary bipartite 3-uniform hypergraph*, Ars. Combin. **146** (2019), 293–305.

[6] L. N. Kamble, C. M. Deshpande, B. Y. Bam, *T*he existance of bipartite almost self-complementary 3-uniform hypergraphs, Opuscula Math. **43(5)** (2023), 663–673.

[7] W. Kocay, *Reconstructing graphs as subsumed graphs of hypergraphs, and some self-complementary triple systems*, Graphs Combin. **8**(1992), 259–276.

[8] G. Ringel, *Über selbstkomplementäre graphen*, Arch. Math. **14** (1963), 354–358.

[9] H. Sachs, *Über selbstkomplementäre graphen*, Publ. Math. Drecen, **9** (1962), 270–288.

[10] A. Szymański, A.P. Wojda, *A note on k-uniform self-complementary hypergraphs of given order*, Discuss. Math. Graph Theory **29** (2009), 199–202.

[11] A. Szymański, A.P. Wojda, *Self-complementing permutations of k-uniform hypergraphs*, Discrete Math. and Theoretical Computer Science **11** (2009), 117–124.

[12] A.P. Wojda, *Self complementary hypergraphs*, Discuss. Math. Graph Theory **26** (2006), 217–224.

Author information

Lata Kamble, Department of Mathematics, Abasaheb Garware College, Pune-411004, India.
E-mail: `lata7429@gmail.com`

Charusheela Deshpande, Department of Mathematics, College of Engineering, Pune-411006, India.
E-mail: `dcm.math@coep.ac.in`

Bhagyashri Athawale, Department of Mathematics, College of Engineering, Pune-411006, India.
E-mail: `bhagyashriathawale@gmail.com`

Received: 2024-08-23
Accepted: 2024-10-05.