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HIGHER-ORDER REGULARITY AND CONTINUOUS DATA ASSIMILATION FOR THE VELOCITY-VORTICITY MODEL OF THE g-NAVIER-STOKES EQUATIONS

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Abstract In this paper, we aim to improve the known results of the weak and strong solutions of the two-dimensional velocity-vorticity model of the g-Navier Stokes equations (g-NSEs). Using energy estimation methods, we obtain the higher-order regularity of the solution of this model. Then we introduce an algorithm for continuous data assimilation for the velocity–vorticity model of the g-NSEs in the two-dimensional case. We give conditions on the spatial resolution of the observed data which are sufficient to show that the solution of the algorithm approaches, at an exponential rate, the unique exact unknown solution of this model.

1 Introduction

The velocity-vorticity formulation of the Navier-Stokes equations expressing flow movements has been addressed by many researchers for different problems [1, 2, 3, 4, 5, 6]. Velocity-vorticity equations are important in expressing strong rotating flows. The advantages of the velocity-vorticity forms of considered equation systems are expressed numerically [2, 3, 4, 7]. Velocity-vorticity models enable us to reach solutions closer to the physics of fluids, thanks to the same physical properties of fluids. These types of problems are considered in [5, 8, 9]. We inspire them in our study. In [1], we proved the existence of the weak and strong solutions of the velocity-vorticity model of the g-NSEs.

Numerical computations regarding mathematical models of nonlinear evolutionary systems require precision in the initial data. Nevertheless, in most instances, the initial data can be calculated discretely, and it is frequently impossible to attain a sufficient resolution. The continuous data assimilation method, introduced by Charney et al., [10] refers to the process of completing the resolution of the initial condition. This can be attained by adding the observational data into a model and as it is integrated in time, an approximate solution converging to the exact solution is obtained. Foias and Prodi introduced the theory of determining modes [11] and Olson and Titi revealed the connection between continuous data assimilation and the theory of determining modes [12, 13]. Azouani, Olson, and Titi presented a novel algorithm in [14] that is also called the AOT algorithm: continuous data assimilation (CDA). Their approach, although with the addition of a spatial interpolation operator, revived the "nudging" approaches of the 1970s (see, for example, [15, 16]). The CDA algorithm applied to the 2D Navier-Stokes equations in [14]. This stimulated a large amount of recent research on the CDA algorithm; see, e.g., [6, 14, 17, 18, 19, 20, 21, 22, 23, 24]. In this study we consider the following velocity-vorticity model of the g-NSEs [1] in two-dimensions;

$$\frac{\partial u}{\partial t}-\nu\Delta_g u+\nu\frac{1}{g}(\nabla g\cdot\nabla)u+w\times u+\nabla P=f, \tag{1.1}$$

$$\frac{\partial w}{\partial t}-\nu\Delta_g w+\nu\frac{1}{g}(\nabla g\cdot\nabla)w+(u\cdot\nabla)\,w=\nabla\times f+w\left(\frac{\nabla g}{g}\cdot u\right), \tag{1.2}$$

$$\nabla\cdot(gu)=0, \qquad \nabla\cdot(gw)=0 \tag{1.3}$$

with the initial condition

$$u(x,0)=u_0, \qquad w(x,0)=w_0 \tag{1.4}$$

and under the periodic boundary condition. Here $u(x,t)=(u_1,u_2)$ is the fluid velocity, $\nu\geq 0$ is the viscosity, the pressure p and $P=p+\frac{1}{2}\left|u\right|^2$, $w=u_{2x_1}-u_{1x_2}$ is the scalar vorticity, f represents a body force, g is a suitable smooth real-valued function. We assume that

$$(i)\ \ g(x_1,x_2)\in C^\infty_{per}(\Omega)$$

$$(ii)\ \ 0<m_0\leq g(x_1,x_2)\leq M_0 \ \text{ for all } (x_1,x_2)\in\Omega,$$

$$(iii)\quad \left\|\nabla\partial^s g\right\|_\infty=\sup_{(x_1,x_2)\in\Omega}\left|\nabla\,\partial^s g(x_1,x_2)\right|<\infty$$

here m_0 and M_0 are two constants and s is a nonnegative integer. g-Laplacian operator be defined as in the same [25], in the following

$$-\Delta_g u:=-\frac{1}{g}(\nabla\cdot g\nabla)u=-\Delta u-\frac{1}{g}(\nabla g\cdot\nabla)u.$$

In this study, a general positive constant will be represented by the letter c. It may vary from one line to the next. This paper is organized as follows: In section 2, we present the mathematical spaces and introduce certain notations. In section 3, we prove this system's higher-order regularity. In section 4, the conditions for the convergence of the approximate solutions to the reference solutions of our models are also provided.

2 Preliminaries and Functional Setting

In this section, we introduce some notations and the definitions of some spaces that are used in this paper (see e.g in [25, 26]). The standard notations used throughout this work are provided here. $L^2(\Omega,g)$ denotes the Hilbert space with the inner product $(u,v)_g=\int\limits_\Omega u\cdot vgdx$, $u,v\in L^2(\Omega,g)$ and the norm $\|u\|^2_{L^2(\Omega,g)}=(u,u)_g$. The norm and inner product of H_g are the same as those of $L^2(\Omega,g)$. The norm in $H^1(\Omega,g)$ is defined as

$$\|u\|^2_{H^1(\Omega,g)}=\left[(u,u)_g+\sum_{i=1}^{2}(D_iu,D_iu)_g\right]^{\frac{1}{2}}$$

where $D_i=\frac{\partial}{\partial x_i}$. The norm of $H^1(\Omega,g)$ is the same in V_g. Since the following inequalities are satisfied, the two spaces $L^2(\Omega)$ and $L^2(\Omega,g)$ have equivalent norms.

$$m_0\left\|u\right\|^2_{L^2(\Omega)}\leq\left\|u\right\|^2_{L^2(\Omega,g)}\leq M_0\left\|u\right\|^2_{L^2(\Omega)}$$

We define

$$V_1=\left\{u\in\left(C^\infty_{per}(\Omega)\right)^2:\nabla\cdot gu=0,\int\limits_\Omega udx=0\right\}.$$

Denote by $\mathbf{H}_g$ the closure of V_1 in $\left(L^2(\Omega, g)\right)^2$and by $\mathbf{V}_g$ the closure of V_1 in $\left(H^1(\Omega, g)\right)^2$ in two-dimensions. Vorticity is considered as a scalar, we define vorticity space as

$$V_2 = \left\{ u \in C^\infty_{per}(\Omega) : \nabla \cdot gu = 0, \int_\Omega u dx = 0 \right\}.$$

Denote by H_g the closure of V_2 in $L^2(\Omega, g)$ and by V_g the closure of V_2 in $H^1(\Omega, g)$,

$$H_{gcurl} = \left\{ f \in H_g : \nabla \times f \in L^2(\Omega, g) \right\}$$

Also, we consider the orthogonal projection P_g as $P_g : L^2_{per}(\Omega, g) \longrightarrow H_g$. Now we rewrite the g-Stokes operator as in the following

$$A_g u = P_g \left[-\frac{1}{g} \left(\nabla \cdot g \nabla u \right) \right].$$

A_g has sufficient countable eigenvalues as shown below;

$$0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

where $\lambda_g = \frac{4\pi^2 m_0}{M_0}$. The Poincaré inequality

$$\sqrt{\lambda_g} \left\| \phi \right\|_{L^2} \leq \left\| \nabla \phi \right\|_{L^2},$$

satisfy for all $\phi \in V_g$. Since the operators A_g and P_g are self-adjoint, by integration we get

$$\langle A_g u, u \rangle_g = \int_\Omega \left(\nabla u, \nabla u \right) g dx = \langle \nabla u, \nabla u \rangle_g = \left\| \nabla u \right\|_{L^2}.$$

We denote the bilinear operator B_g as $B_g : V_g \times V_g \to V_g'$, $B_g(u, v) = P_g(u \cdot \nabla)v$. The trilinear form b_g is now defined by $\langle B_g(u, v), w \rangle_{V_g'}$ and

$$b_g(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx.$$

It is easy to check that if $u, v, w \in V_g$, then

$$i)\ b_g(u, v, w) = -b_g(u, w, v)$$
$$ii)\ b_g(u, v, v) = 0$$

$C_g u$ is defined by $C_g u = P_g \left[\frac{1}{g} \left(\nabla g \cdot \nabla \right) u \right]$ and the inner product of $v \in V_g$, we get

$$\langle C_g u, v \rangle_g = \left\langle \frac{1}{g} \left(\nabla g \cdot \nabla \right) u, v \right\rangle_g = b_g(\frac{\nabla g}{g}, u, v).$$

The following lemma in [26].

Lemma 2.1. *[26] If n=2 , then*

$$|b_g(u, v, w)| \leq \begin{cases} c \left\| u \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla u \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla v \right\|_{L^2} \left\| w \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla w \right\|^{\frac{1}{2}}_{L^2}, & \forall\, u, v, w \in V_g, \\ c \left\| u \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla u \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla v \right\|^{\frac{1}{2}}_{L^2} \left\| Av \right\|^{\frac{1}{2}}_{L^2} \left\| w \right\|_{L^2}, & \forall\, u \in V_g, v \in D(A), w \in H_g, \\ c \left\| u \right\|^{\frac{1}{2}}_{L^2} \left\| Au \right\|^{\frac{1}{2}}_{L^2} \left\| \nabla v \right\|_{L^2} \left\| w \right\|_{L^2}, & \forall\, u \in D(A), v \in V_g, w \in H_g. \end{cases} \tag{2.1}$$

The following lemma in [27].

Lemma 2.2. *[27] Let $T > 0$ be fixed. Suppose that $Y(t)$ is a continuous function that is locally integrable and that it satisfies the following inequalities:*

$$\frac{\partial Y}{\partial t} + \alpha(t)Y \le 0,$$

$$\liminf_{t\to\infty} \int_t^{t+T} \alpha(s)ds \ge c,$$

$$\limsup_{t\to\infty} \int_t^{t+T} \alpha^{-}(s)ds < \infty$$

for some $c > 0$, where $\alpha^{-} = \max\{-\alpha, 0\}$. Then, $Y(t) \to 0$ at an exponential rate, as $t \to \infty$.

Before giving the main results we begin using the Helmholtz-Leray projector P_g on the system (1.1)-(1.4) to produce the functional differential equation shown below:

$$\frac{du}{dt} + \nu A_g u + \nu C_g u + P_g (w \times u) = P_g f, \tag{2.2}$$

$$\frac{dw}{dt} + \nu A_g w + \nu C_g w + B_g(u, w) = P_g (\nabla \times f) + P_g \left(w \left(\frac{\nabla g}{g} \cdot u \right) \right), \tag{2.3}$$

$$u(0) = u_0, \qquad w(0) = w_0. \tag{2.4}$$

Theorem 2.3. *[1] Suppose $u_0 \in \mathbf{H}_g$ and $w_0 \in H_g$ and $f \in L^2(0,T;H_{gcurl})$. Then the system (2.2)-(2.4) has a unique weak solution (u, w) such that $u \in C(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$, $u_t \in L^2(0,T;\mathbf{V}'_g)$ and $w \in C(0,T;H_g) \cap L^2(0,T;V_g)$, $w_t \in L^2(0,T;V'_g)$.*

Theorem 2.4. *[1] Let the initial data $u_0 \in \mathbf{V}_g$ and $w_0 \in V_g$ and $f \in L^2(0,T;H_{gcurl})$. Then system (2.2)-(2.4) has a unique strong solution (u, w) such that $u \in L^2(0,T;D(\mathbf{A}_g)) \cap L^\infty(0,T;\mathbf{V}_g)$ and $w \in L^2(0,T;D(A_g)) \cap L^\infty(0,T;V_g)$.*

Theorem 2.5. *[28] The system (2.2)-(2.4) possesses the existence of global attractor A that is compact in $\mathbf{H}_g \times H_g$ and is maximal among all bounded invariant sets.*

3 Higher-Order Regularity

In [1], we proved the existence and uniqueness of strong solutions (1.1)-(1.4). The theorem we will give below is related to the higher-order regularity of the solution in addition to the proof of the global existence and uniqueness of the strong solution for the (1.1)-(1.4) system.

$\mathbf{H}^1_g$ and H^1_g bounds were given in [1]. Now we provide $\mathbf{H}^2_g$ and H^2_g a priori estimates for u and w respectively. Then we obtain the $\mathbf{H}^s_g$ bounds on u and the H^s_g bounds on w for $s \ge 3$.

Theorem 3.1. *For the initial data $u_0 \in \mathbf{H}^s_g, w_0 \in H^s_g$ and $f \in L^2(0,T;H^{s-1}_{gcurl})$ for $s \ge 1, s \in N$ then there exist the solution $u \in L^\infty(0,T;\mathbf{H}^s_g \cap \mathbf{V}_g) \cap L^2(0,T;\mathbf{H}^{s+1}_g \cap \mathbf{V}_g)$ and, $w \in L^\infty(0,T;H^s_g \cap V_g) \cap L^2(0,T;H^{s+1}_g \cap V_g)$.*

Proof. We proved that $u \in C(0,T;\mathbf{H}_g) \cap L^2(0,T;\mathbf{V}_g)$ and $w \in C(0,T;H_g) \cap L^2(0,T;V_g)$ due to the weak solutions of our model for u and w [1]. So we have

$$\sup_{(x_1,x_2)\in\Omega} \|u(s)\|^2_{L^2} \le K_1, \qquad \sup_{(x_1,x_2)\in\Omega} \|w(s)\|^2_{L^2} \le K_2$$

and

$$\int_0^T \|\nabla u(t)\|^2_{L^2}\, dt \le K_3, \qquad \int_0^T \|\nabla w(t)\|^2_{L^2}\, dt \le K_4.$$

Similarly, we proved that $u \in L^2(0,T;D(\mathbf{A}_g)) \cap L^\infty(0,T;\mathbf{V}_g)$ and $w \in L^2(0,T;D(A_g)) \cap L^\infty(0,T;V_g)$ due to the strong solutions of our model for u and w [1]. So we have

$$\sup_{(x_1,x_2)\in\Omega} \|\nabla u(s)\|_{L^2}^2 \le K_5, \qquad \sup_{(x_1,x_2)\in\Omega} \|\nabla w(s)\|_{L^2}^2 \le K_6$$

and

$$\int_0^T \|\Delta u(t)\|_{L^2}^2 \, dt \le K_7, \qquad \int_0^T \|\Delta w(t)\|_{L^2}^2 \, dt \le K_8.$$

Now we provide $\mathbf{H}_g^2$ and H_g^2 a priori estimates for u and w respectively. We begin multiplying (2.2) by $-\Delta^2 u$, integrating by parts over Ω, and we get

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + 2\nu\|\nabla\Delta u\|_{L^2}^2 \le 2\nu\left|b_g\left(\frac{\nabla g}{g}, u, \Delta^2 u\right)\right| + 2\left|\left(P_g\left(w\times u\right), \Delta^2 u\right)_g\right|$$
$$+ 2\left|\left(P_g f, \Delta^2 u\right)_g\right|. \tag{3.1}$$

In the last inequality, the first integral on the right side is bounded by

$$2\nu\left|b_g\left(\frac{\nabla g}{g}, u, \Delta^2 u\right)\right| \le 2\nu\left|b_g\left(\frac{\Delta g}{g}, u, \nabla\Delta u\right)\right| + 2\nu\left|b_g\left(\frac{\nabla g}{g}, \nabla u, \nabla\Delta u\right)\right|$$
$$\le \frac{\nu}{2}\|\nabla\Delta u\|_{L^2}^2 + \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla u\|_{L^2}^2 + \frac{4\nu\|\nabla g\|_\infty^2}{m_0^2}\|\Delta u\|_{L^2}^2$$

where we used Cauchy-Schwarz and Young inequalities. Here the second term is bounded by

$$2\left|\left(P_g\left(w\times u\right), \Delta^2 u\right)_g\right| \le 2\int_\Omega |\nabla w|\,|u|\,|\nabla\Delta u|\, g dx + 2\int_\Omega |w|\,|\nabla u|\,|\nabla\Delta u|\, g dx$$
$$\le \frac{\nu}{2}\|\nabla\Delta u\|_{L^2}^2 + \frac{4c^2}{\nu\lambda_g}\|\Delta w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{4c^2}{\nu\lambda_g}\|\nabla w\|_{L^2}^2\|\Delta u\|_{L^2}^2$$

where we used (2.1) and Young and Poincaré inequalities. By Cauchy-Schwarz and Young inequalities, the third term is bounded by

$$2\left|\left(P_g f, \Delta^2 u\right)_g\right| \le \frac{\nu}{4}\|\nabla\Delta u\|_{L^2}^2 + \frac{4}{\nu}\|\nabla f\|_{L^2}^2.$$

Using together all of the estimates in the equation (3.1), we write

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \frac{3\nu}{4}\|\nabla\Delta u\|_{L^2}^2 \le \left(\frac{4\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{4c^2}{\nu\lambda_g}\|\nabla w\|_{L^2}^2\right)\|\Delta u\|_{L^2}^2$$
$$+ \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla u\|_{L^2}^2 + \frac{4c^2}{\nu\lambda_g}\|\Delta w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{4}{\nu}\|\nabla f\|_{L^2}^2 \tag{3.2}$$

and

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 \le M_1\|\Delta u\|_{L^2}^2 + M_2 \tag{3.3}$$

where $M_1 = \frac{4\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{4c^2}{\nu\lambda_g}\|\nabla w\|_{L^2}^2$, $M_2 = \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla u\|_{L^2}^2 + \frac{4c^2}{\nu\lambda_g}\|\Delta w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{4}{\nu}\|\nabla f\|_{L^2}^2$.

$$\int_0^T M_1(\tau)d\tau \le \frac{4\nu T\|\nabla g\|_\infty^2}{m_0^2} + \frac{4c^2}{\nu\lambda_g}K_6 \le K_9$$

and

$$\int_0^T M_2(\tau)d\tau \leq \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}K_3 + \frac{4c^2}{\nu\lambda_g}K_8K_3 + \frac{4}{\nu}\|\nabla f\|_{L^2(0,T;L^2)}^2 \leq K_{10}.$$

Hence, applying Grönwall's inequality in (3.3), we obtain

$$\sup_{t\in[0,T]}\|\Delta u(t)\|_{L^2}^2 \leq e^{K_9}\|\Delta u_0\|_{L^2}^2 + e^{K_9}K_{10} \leq K_{11}$$

where the constant K_9 depends on the $\mathbf{H}_g^1$ norms w and K_{10} depends on the $\mathbf{H}_g^1$ and H_g^1 norms u and w respectively. Thus, $u \in L^\infty(0,T;\mathbf{H}_g^2)$. Similarly, we begin multiplying (2.3) by $-\Delta^2 w$, integrating by parts over Ω, and we obtain

$$\frac{d}{dt}\|\Delta w\|_{L^2}^2 + 2\nu\|\nabla\Delta w\|_{L^2}^2 \leq 2\nu\left|b_g\left(\frac{\nabla g}{g}, w, \Delta^2 w\right)\right| + 2\left|b_g\left(u, w, \Delta^2 w\right)\right|$$

$$+2\left|\left(P_g\left(\nabla\times f\right), \Delta^2 w\right)_g\right| + 2\left|\left(w\left(\frac{\nabla g}{g}\cdot u\right), \Delta^2 w\right)_g\right|. \quad (3.4)$$

In the inequality above, the first integral on the right side is bounded by

$$2\nu\left|b_g\left(\frac{\nabla g}{g}, w, \Delta^2 w\right)\right| \leq 2\nu\left|b_g\left(\frac{\Delta g}{g}, w, \nabla\Delta w\right)\right| + 2\nu\left|b_g\left(\frac{\nabla g}{g}, \nabla w, \nabla\Delta w\right)\right|$$

$$\leq \frac{\nu}{2}\|\nabla\Delta w\|_{L^2}^2 + \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla w\|_{L^2}^2 + \frac{4\nu\|\nabla g\|_\infty^2}{m_0^2}\|\Delta w\|_{L^2}^2$$

where we used Cauchy-Schwarz and Young inequalities. We bounded the second term by

$$2\left|b_g\left(u, w, \Delta^2 w\right)\right| \leq 2\left|b_g\left(\nabla u, w, \nabla\Delta w\right)\right| + 2\left|b_g\left(u, \nabla w, \nabla\Delta w\right)\right|$$

$$\leq \frac{3\nu}{4}\|\nabla\Delta w\|_{L^2}^2 + \frac{2c}{\nu\lambda_g}\|\Delta u\|_{L^2}^2\|\Delta w\|_{L^2}^2$$

$$+\frac{27c}{2\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\Delta w\|_{L^2}^2$$

where we used (2.1) and Young and Poincaré inequalities. By Cauchy-Schwarz and Young inequalities, we bound the third term by

$$2\left|\left(P_g\left(\nabla\times f\right), \Delta^2 w\right)_g\right| \leq \frac{\nu}{4}\|\nabla\Delta w\|_{L^2}^2 + \frac{4}{\nu}\|\nabla\left(\nabla\times f\right)\|_{L^2}^2.$$

The fourth integral on the right side is bounded by

$$2\left|\left(w\left(\frac{\nabla g}{g}\cdot u\right), \Delta^2 w\right)_g\right| \leq \frac{\nu}{2}\|\nabla\Delta w\|_{L^2}^2 + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\Delta w\|_{L^2}^2\|\nabla u\|_{L^2}^2$$

$$+\frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\nabla w\|_{L^2}^2\|\Delta u\|_{L^2}^2$$

where we used (2.1) and Young and Poincaré inequalities. We use all the above inequalities in (3.4), we get

$$\frac{d}{dt}\|\Delta w\|_{L^2}^2 \leq \left(\frac{4\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{2c}{\nu\lambda_g}\|\Delta u\|_{L^2}^2 + \frac{27c}{2\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\right.$$

$$\left. + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\nabla u\|_{L^2}^2\right)\|\nabla w\|_{L^2}^2 + \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla w\|_{L^2}^2$$

$$+ \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\nabla w\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \frac{4}{\nu}\|\nabla(\nabla\times f)\|_{L^2}^2 \tag{3.5}$$

and

$$\frac{d}{dt}\|\Delta w\|_{L^2}^2 \leq M_3\|\nabla w\|_{L^2}^2 + M_4 \tag{3.6}$$

where

$$M_3 = \frac{4\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{2c}{\nu\lambda_g}\|\Delta u\|_{L^2}^2 + \frac{27c}{2\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\nabla u\|_{L^2}^2,$$

$$M_4 = \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}\|\nabla w\|_{L^2}^2 + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\nabla w\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \frac{4}{\nu}\|\nabla(\nabla\times f)\|_{L^2}^2.$$

$$\int_0^T M_3(\tau)d\tau \leq \frac{4\nu T\|\nabla g\|_\infty^2}{m_0^2} + \frac{2c}{\nu\lambda_g}K_{11} + \frac{27c}{2\nu^3}K_1K_3 + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}K_3 \leq K_{12}$$

and

$$\int_0^T M_4(\tau)d\tau \leq \frac{4\nu\|\Delta g\|_\infty^2}{m_0^2}K_4 + \frac{4c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}K_4K_{11} + \frac{4}{\nu}\|f\|_{L^2(0,T;H^1_{gcurl})}^2 \leq K_{13}.$$

Now, using Grönwall's inequality in (3.6), we obtain

$$\sup_{t\in[0,T]}\|\Delta w(t)\|_{L^2}^2 \leq e^{K_{12}}\|\Delta w_0\|_{L^2}^2 + e^{K_{12}}K_{13} \leq K_{14}$$

where the constant K_{12} depends on $\mathbf{H}_g^1$ and H_g^1 norms u and w respectively and K_{13} depends on the $\mathbf{H}_g^2$ norms u. Therefore $w \in L^\infty(0,T;H_g^2)$. Now we will obtain the estimate for $s \geq 3$ to complete the proof of Theorem 3.1.

Next, we begin multiplying (2.2) by $\partial^\alpha u$ after applying the operator ∂^α, where is a multi-index with $|\alpha| = 2$, and integrating by parts over Ω, and we obtain

$$\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 + 2\nu\|\nabla\partial^\alpha u\|_{L^2}^2 \leq -2\nu\frac{1}{M_0}\left(\partial^\alpha(\nabla g\cdot\nabla)u, \partial^\alpha u\right)_g + 2\left(\partial^\alpha(w\times u), \partial^\alpha u\right)_g$$

$$+ 2\left(\partial^\alpha f, \partial^\alpha u\right)_g$$

and

$$\frac{d}{dt}\left\|\partial^{\alpha}u\right\|_{L^2}^2+2\nu\left\|\nabla\partial^{\alpha}u\right\|_{L^2}^2\leq-2\nu\frac{1}{M_0}\sum_{0<\mu\leq\alpha}\binom{\alpha}{\mu}\int\limits_{\Omega}\left(\nabla\partial^{\mu}g\cdot\nabla\right)\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx$$

$$-2\sum_{0<\mu\leq\alpha}\binom{\alpha}{\mu}\int\limits_{\Omega}\partial^{\mu}w\cdot\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx$$

$$+2\int\limits_{\Omega}\partial^{\mu}f\cdot\partial\partial^{\alpha}u\cdot gdx:=I_1+I_2+I_3$$

where we used $\nabla\cdot gu=0$ and μ is also a multi-index and $\mu\leq\alpha$ indicates that $|\mu|\leq|\alpha|$ and $\mu_i\leq\alpha_i$ for $i=1,2$. Then, we estimate the I_1 for the next three cases. For $\mu\leq\alpha$ and $|\mu|=0$, it is bounded by

$$2\nu\frac{1}{M_0}\left|\sum_{|\mu|=0}\binom{\alpha}{\mu}\int\limits_{\Omega}\left(\nabla\partial^{\mu}g\cdot\nabla\right)\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx\right|\leq2c\frac{\nu}{M_0}\left\|\nabla\partial^{\mu}g\right\|_{\infty}\left\|\nabla\partial^{\alpha-\mu}u\right\|_{L^2}\left\|\partial^{\alpha}u\right\|_{L^2}$$

$$\leq\frac{\nu}{4}\left\|\nabla\partial^{\alpha}u\right\|_{L^2}^2+\frac{4c^2\nu}{M_0^2}\left\|\nabla g\right\|_{\infty}^2\left\|u\right\|_{\mathbf{H}_g^{|\alpha|}}^2$$

where we used Cauchy Schwarz and Young inequalities. For $\mu\leq\alpha$ and $|\mu|=1$, it is bounded by

$$2\nu\frac{1}{M_0}\left|\sum_{|\mu|=1}\binom{\alpha}{\mu}\int\limits_{\Omega}\left(\nabla\partial^{\mu}g\cdot\nabla\right)\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx\right|\leq2c\frac{\nu}{M_0}\left\|\nabla\partial^{\mu}g\right\|_{\infty}\left\|\nabla\partial^{\alpha-\mu}u\right\|_{L^2}\left\|\partial^{\alpha}u\right\|_{L^2}$$

$$\leq\frac{2c\nu}{M_0}\left\|\nabla\partial g\right\|_{\infty}\left\|u\right\|_{\mathbf{H}_g^{|\alpha|}}^2$$

where we used Cauchy Schwarz inequality. For $\mu\leq\alpha$ and $|\mu|=2$, it is bounded by

$$2\nu\frac{1}{M_0}\left|\sum_{|\mu|=2}\binom{\alpha}{\mu}\int\limits_{\Omega}\left(\nabla\partial^{\mu}g\cdot\nabla\right)\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx\right|\leq2c\frac{\nu}{M_0}\left\|\nabla\partial^{\mu}g\right\|_{\infty}\left\|\nabla\partial^{\alpha-\mu}u\right\|_{L^2}\left\|\partial^{\alpha}u\right\|_{L^2}$$

$$\leq\frac{\nu}{4}\left\|u\right\|_{\mathbf{H}_g^{|\alpha|}}^2+\frac{4c^2\nu}{M_0^2}\left\|\nabla\partial^2g\right\|_{\infty}^2\left\|u\right\|_{\mathbf{H}_g^1}^2$$

where we used Cauchy-Schwarz and Young inequalities. In the next three cases, the estimations for I_2 proceed in a similar manner. Using integration by part, for $\mu\leq\alpha$ and $|\mu|=0$

$$\left|2\sum_{|\mu|=0}\binom{\alpha}{\mu}\int\limits_{\Omega}\partial^{\mu}w\cdot\partial^{\alpha-\mu}u\cdot\partial^{\alpha}u\cdot gdx\right|\leq2c\left\|\partial^{\mu}w\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\partial^{\mu}w\right\|_{L^2}^{\frac{1}{2}}\left\|\partial^{\alpha}u\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\partial^{\alpha}u\right\|_{L^2}^{\frac{1}{2}}\left\|\partial^{\alpha-\mu}u\right\|_{L^2}$$

$$\leq\frac{\nu}{4}\left\|\nabla\partial^{\alpha}u\right\|_{L^2}^2+\frac{4c^2}{\nu\lambda_g}\left\|w\right\|_{H_g^1}^2\left\|u\right\|_{\mathbf{H}_g^{|\alpha|}}^2$$

where (2.1), Poincaré and Young inequalities were applied. For $\mu \le \alpha$ and $|\mu| = 1$, it is bounded by

$$\left| 2 \sum_{|\mu|=1} \binom{\alpha}{\mu} \int_{\Omega} \partial^{\mu} w \cdot \partial^{\alpha-\mu} u \cdot \partial^{\alpha} u \cdot g dx \right| \le 2c \left\| \partial^{\mu} w \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \partial^{\mu} w \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^{\alpha} u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \partial^{\alpha} u \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^{\alpha-\mu} u \right\|_{L^2}$$

$$\le \frac{\nu}{4} \left\| \nabla \partial^{\alpha} u \right\|_{L^2}^2 + \frac{4c^2}{\nu \lambda_g} \left\| w \right\|_{H_g^2}^2 \left\| u \right\|_{\mathbf{H}_g^1}^2$$

where (2.1), Poincaré and Young inequalities were applied. For $\mu \le \alpha$ and $|\mu| = 2$, similarly we used (2.1), Poincaré and Young inequalities

$$\left| 2 \sum_{|\mu|=2} \binom{\alpha}{\mu} \int_{\Omega} \partial^{\mu} w \cdot \partial^{\alpha-\mu} u \cdot \partial^{\alpha} u \cdot g dx \right| \le 2c \left\| \partial^{\mu} w \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \partial^{\mu} w \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^{\alpha} u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \partial^{\alpha} u \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^{\alpha-\mu} u \right\|_{L^2}$$

$$\le \frac{\nu}{4} \left\| \nabla \partial^{\alpha} w \right\|_{L^2}^2 + \frac{4c^2}{\nu \lambda_g} \left\| \nabla \partial^{\alpha} u \right\|_{L^2}^2 \left\| u \right\|_{L^2}^2 .$$

As for I_3, we integrate by parts and estimate as

$$\left| 2 \int_{\Omega} \partial^{\alpha} f \cdot \partial^{\alpha} u \cdot g dx \right| \le 2 \sum_{|\mu|=2} \binom{\alpha}{\mu} \int_{\Omega} \partial^{\mu} f \cdot \partial \partial^{\alpha} u \cdot g dx \le \frac{\nu}{4} \left\| \nabla \partial^{\alpha} u \right\|_{L^2}^2 + \frac{4c^2}{\nu} \left\| f \right\|_{H_g^2}^2 .$$

Similarly we begin multiply (2.3) by $\partial^{\alpha} w$ after applying the operator ∂^{α}, where is a multi-index with $|\alpha| = 2$, and integrating by parts over Ω, and we obtain

$$\frac{d}{dt} \left\| \partial^{\alpha} w \right\|_{L^2}^2 + 2\nu \left\| \nabla \partial^{\alpha} w \right\|_{L^2}^2 \le -2\nu \frac{1}{M_0} \left(\partial^{\alpha} \left(\nabla g \cdot \nabla \right) w, \partial^{\alpha} w \right)_g - 2 \left(\partial^{\alpha} \left(u \cdot \nabla \right) w, \partial^{\alpha} w \right)_g$$

$$+ 2 \left(\partial^{\alpha} \left(\nabla \times f \right), \partial^{\alpha} w \right)_g + 2 \left(\partial^{\alpha} w \left(\frac{\nabla g}{g} \cdot u \right), \partial^{\alpha} w \right)_g$$

and

$$\frac{d}{dt} \left\| \partial^{\alpha} w \right\|_{L^2}^2 + 2\nu \left\| \nabla \partial^{\alpha} w \right\|_{L^2}^2 \le -2\nu \frac{1}{M_0} \sum_{0<\mu\le\alpha} \binom{\alpha}{\mu} \int_{\Omega} \left(\nabla \partial^{\mu} g \cdot \nabla \right) \partial^{\alpha-\mu} w \cdot \partial^{\alpha} w \cdot g dx$$

$$- 2 \sum_{0<\mu\le\alpha} \binom{\alpha}{\mu} \int_{\Omega} \left(\partial^{\mu} u \cdot \nabla \right) \cdot \partial^{\alpha-\mu} w \cdot \partial^{\alpha} w \cdot g dx$$

$$+ 2 \int_{\Omega} \partial^{\mu} \left(\nabla \times f \right) \cdot \partial \partial^{\alpha} w \cdot g dx$$

$$+ 2 \sum_{0<\mu\le\alpha} \binom{\alpha}{\mu} \int_{\Omega} \partial^{\alpha-\mu} w \cdot \partial^{\mu} \left(\nabla g \cdot u \right) \cdot \partial^{\alpha} w \cdot g dx$$

$$:= J_1 + J_2 + J_3 + J_4$$

where we used $\nabla \cdot gw = 0$ and μ is also a multi-index and $\mu \leq \alpha$ indicates that $|\mu| \leq |\alpha|$ and $\mu_i \leq \alpha_i$ for $i = 1, 2$. Then, we estimate the J_1 in the following three cases for μ. For $\mu \leq \alpha$ and $|\mu| = 0$, it is bounded by

$$2\nu \frac{1}{M_0} \left| \sum_{|\mu|=0} \binom{\alpha}{\mu} \int_\Omega (\nabla \partial^\mu g \cdot \nabla) \partial^{\alpha-\mu} w \cdot \partial^\alpha w \cdot g dx \right| \leq 2c \frac{\nu}{M_0} \|\nabla \partial^\mu g\|_\infty \left\|\nabla \partial^{\alpha-\mu} w\right\|_{L^2} \|\partial^\alpha w\|_{L^2}$$

$$\leq \frac{\nu}{4} \|\nabla \partial^\alpha w\|_{L^2}^2 + \frac{4c^2 \nu}{M_0^2} \|\nabla g\|_\infty^2 \|w\|_{H_g^{|\alpha|}}^2$$

where we used Cauchy Schwarz and Young inequality. For $\mu \leq \alpha$ and $|\mu| = 1$, it is bounded by

$$2\nu \frac{1}{M_0} \left| \sum_{|\mu|=1} \binom{\alpha}{\mu} \int_\Omega (\nabla \partial^\mu g \cdot \nabla) \partial^{\alpha-\mu} w \cdot \partial^\alpha w \cdot g dx \right| \leq 2c \frac{\nu}{M_0} \|\nabla \partial^\mu g\|_\infty \left\|\nabla \partial^{\alpha-\mu} w\right\|_{L^2} \|\partial^\alpha w\|_{L^2}$$

$$\leq \frac{2c\nu}{M_0} \|\nabla \partial g\|_\infty \|w\|_{H_g^{|\alpha|}}^2$$

where we used Cauchy Schwarz inequality. For $\mu \leq \alpha$ and $|\mu| = 2$, it is bounded by

$$2\nu \frac{1}{M_0} \left| \sum_{|\mu|=2} \binom{\alpha}{\mu} \int_\Omega (\nabla \partial^\mu g \cdot \nabla) \partial^{\alpha-\mu} w \cdot \partial^\alpha w \cdot g dx \right| \leq 2c \frac{\nu}{M_0} \|\nabla \partial^\mu g\|_\infty \left\|\nabla \partial^{\alpha-\mu} w\right\|_{L^2} \|\partial^\alpha w\|_{L^2}$$

$$\leq \frac{\nu}{4} \|w\|_{H_g^{|\alpha|}}^2 + \frac{4c^2 \nu}{M_0^2} \left\|\nabla \partial^2 g\right\|_\infty^2 \|w\|_{H_g^1}^2$$

where we used Cauchy-Schwarz and Young inequalities. The estimates for J_2 follow similarly in the following two cases. For $\mu \leq \alpha$ and $|\mu| = 1$, we integrate by parts and bound it by

$$\left| 2 \sum_{|\mu|=1} \binom{\alpha}{\mu} \int_\Omega (\partial^\mu u \cdot \nabla) \cdot \partial^{\alpha-\mu} w \cdot \partial^\alpha w \cdot g dx \right|$$

$$\leq 2c \|\partial^\mu u\|_{L^2}^{\frac{1}{2}} \|\nabla \partial^\mu u\|_{L^2}^{\frac{1}{2}} \|\partial^\alpha w\|_{L^2}^{\frac{1}{2}} \|\nabla \partial^\alpha w\|_{L^2}^{\frac{1}{2}} \left\|\nabla \partial^{\alpha-\mu} w\right\|_{L^2}$$

$$\leq \frac{\nu}{4} \|\nabla \partial^\alpha w\|_{L^2}^2 + \frac{4c^2}{\nu \lambda_g} \|u\|_{\mathbf{H}_g^2}^2 \|w\|_{H_g^{|\alpha|}}^2$$

where (2.1), Poincaré and Young inequalities were applied. For $\mu \leq \alpha$ and $|\mu| = 2$, it is bounded by

$$\left| 2 \sum_{|\mu|=2} \binom{\alpha}{\mu} \int_\Omega (\partial^\mu u \cdot \nabla) \cdot \partial^{\alpha-\mu} w \cdot \partial^\alpha w \cdot g dx \right|$$

$$\leq 2c \|\partial^\mu u\|_{L^2}^{\frac{1}{2}} \|\nabla \partial^\mu u\|_{L^2}^{\frac{1}{2}} \|\partial^\alpha w\|_{L^2}^{\frac{1}{2}} \|\nabla \partial^\alpha w\|_{L^2}^{\frac{1}{2}} \left\|\nabla \partial^{\alpha-\mu} w\right\|_{L^2}$$

$$\leq \frac{\nu}{4} \|\nabla \partial^\alpha w\|_{L^2}^2 + \frac{4c^2}{\nu \lambda_g} \|w\|_{H_g^1}^2 \|\nabla \partial^\alpha u\|_{L^2}^2$$

where (2.1), Poincaré and Young inequalities were applied. As for J_3 is similar and we obtain

$$\left|2\int_\Omega \partial^\alpha (\nabla\times f)\cdot\partial^\alpha w\cdot g dx\right| \le 2\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_\Omega \partial^\mu (\nabla\times f)\cdot\partial\partial^\alpha w\cdot g dx$$

$$\le \frac{\nu}{4}\|\nabla\partial^\alpha w\|_{L^2}^2 + \frac{4c^2}{\nu}\|f\|_{H^2_{gcurl}}^2.$$

In the following three cases follow similarly for J_4. l is also a multi-index and $l\le\mu$ indicates that $|l|\le|\mu|$ and $l_i\le\mu_i$ for $i=1,2$. For $\mu\le\alpha$, $l\le\mu$ and $|\mu|=0$ and $|l|=0$, we integrate by parts using Cauchy Schwarz and Young inequalities by

$$\frac{2}{m_0}\left|\sum_{|\mu|=0}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\cdot\partial^\mu(\nabla g\cdot u)\cdot\partial^\alpha w\cdot g dx\right|$$

$$\le \frac{2}{m_0}\sum_{|\mu|=0}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\left(\sum_{|l|=0}\binom{\mu}{l}\int_\Omega \partial^{\mu-l}\nabla g\cdot\partial^l u\right)\cdot\partial^\alpha w\cdot g dx$$

$$\le \frac{2c}{m_0}\|\nabla g\|_\infty\|\nabla\partial^\alpha w\|_{L^2}\|\partial^\alpha w\|_{L^2}\|u\|_{L^2}$$

$$\le \frac{\nu}{4}\|\nabla\partial^\alpha w\|_{L^2}^2 + \frac{4c^2}{\nu m_0^2}\|\nabla g\|_\infty^2\|w\|_{H_g^{|\alpha|}}^2\|u\|_{L^2}^2.$$

For $\mu\le\alpha$, $l\le\mu$ and $|\mu|=1$ and $|l|=0$, it is bounded by

$$\frac{2}{m_0}\left|\sum_{|\mu|=1}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\cdot\partial^\mu(\nabla g\cdot u)\cdot\partial^\alpha w\cdot g dx\right|$$

$$\le \frac{2}{m_0}\sum_{|\mu|=1}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\left(\sum_{|l|=0}\binom{\mu}{l}\int_\Omega \partial^{\mu-l}\nabla g\cdot\partial^l u\right)\cdot\partial^\alpha w\cdot g dx$$

$$\le \frac{2c}{m_0}\|\nabla\partial g\|_\infty\|\partial w\|_{L^2}^{\frac12}\|\nabla\partial w\|_{L^2}^{\frac12}\|u\|_{L^2}\|\partial^\alpha w\|_{L^2}^{\frac12}\|\nabla\partial^\alpha w\|_{L^2}^{\frac12}$$

$$\le \frac{\nu}{4}\|\nabla\partial^\alpha w\|_{L^2}^2 + \frac{4c^2}{\nu\lambda_g m_0^2}\|\nabla\partial g\|_\infty^2\|w\|_{H_g^2}^2\|u\|_{L^2}^2$$

where (2.1), Poincaré and Young inequalities were applied. For $\mu\le\alpha$, $l\le\mu$ and $|\mu|=1$ and

$|l| = 1$, it is bounded by

$$\frac{2}{m_0}\left|\sum_{|\mu|=1}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\cdot\partial^{\mu}\left(\nabla g\cdot u\right)\cdot\partial^{\alpha}w\cdot gdx\right|$$

$$\leq \frac{2}{m_0}\sum_{|\mu|=1}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\left(\sum_{|l|=1}\binom{\mu}{l}\int_{\Omega}\partial^{\mu-l}\nabla g\cdot\partial^{l}u\right)\cdot\partial^{\alpha}w\cdot gdx$$

$$\leq \frac{2c}{m_0}\left\|\nabla g\right\|_{\infty}\left\|\partial w\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\partial w\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla u\right\|_{L^2}\left\|\partial^{\alpha}w\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\partial^{\alpha}w\right\|_{L^2}^{\frac{1}{2}}$$

$$\leq \frac{\nu}{4}\left\|\nabla\partial^{\alpha}w\right\|_{L^2}^{2}+\frac{4c^2}{\nu\lambda_g m_0^2}\left\|\nabla\partial g\right\|_{\infty}^{2}\left\|w\right\|_{H_g^2}^{2}\left\|u\right\|_{\mathbf{H}_g^1}^{2}$$

where (2.1), Poincaré and Young inequalities were applied. Similarly for $\mu \leq \alpha$, $l \leq \mu$ and $|\mu| = 2$ and $|l| = 0$, it is bounded by

$$\frac{2}{m_0}\left|\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\cdot\partial^{\mu}\left(\nabla g\cdot u\right)\cdot\partial^{\alpha}w\cdot gdx\right|$$

$$\leq \frac{2}{m_0}\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\left(\sum_{|l|=0}\binom{\mu}{l}\int_{\Omega}\partial^{\mu-l}\nabla g\cdot\partial^{l}u\right)\cdot\partial^{\alpha}w\cdot gdx$$

$$\leq \frac{\nu}{4}\left\|\nabla\partial^{\alpha}w\right\|_{L^2}^{2}+\frac{4c^2}{\nu\lambda_g m_0^2}\left\|\nabla\partial^2 g\right\|_{\infty}^{2}\left\|w\right\|_{H_g^1}^{2}\left\|u\right\|_{L^2}^{2}$$

where (2.1), Poincaré and Young inequalities were applied. Similarly for $\mu \leq \alpha$, $l \leq \mu$ and $|\mu| = 2$ and $|l| = 1$, it is bounded by

$$\frac{2}{m_0}\left|\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\cdot\partial^{\mu}\left(\nabla g\cdot u\right)\cdot\partial^{\alpha}w\cdot gdx\right|$$

$$\leq \frac{2}{m_0}\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_{\Omega}\partial^{\alpha-\mu}w\left(\sum_{|l|=1}\binom{\mu}{l}\int_{\Omega}\partial^{\mu-l}\nabla g\cdot\partial^{l}u\right)\cdot\partial^{\alpha}w\cdot gdx$$

$$\leq \frac{\nu}{4}\left\|\nabla\partial^{\alpha}w\right\|_{L^2}^{2}+\frac{4c^2}{\nu\lambda_g m_0^2}\left\|\nabla\partial g\right\|_{\infty}^{2}\left\|w\right\|_{H_g^1}^{2}\left\|u\right\|_{\mathbf{H}_g^1}^{2}$$

where (2.1), Poincaré and Young inequalities were applied. Similarly for $\mu \leq \alpha$, $l \leq \mu$ and

$|\mu|=2$ and $|l|=2$, it is bounded by

$$\frac{2}{m_0}\left|\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\cdot\partial^{\mu}\left(\nabla g\cdot u\right)\cdot\partial^\alpha w\cdot gdx\right|$$

$$\leq \frac{2}{m_0}\sum_{|\mu|=2}\binom{\alpha}{\mu}\int_\Omega \partial^{\alpha-\mu}w\left(\sum_{|l|=2}\binom{\mu}{l}\int_\Omega \partial^{\mu-l}\nabla g\cdot\partial^l u\right)\cdot\partial^\alpha w\cdot gdx$$

$$\leq \frac{\nu}{4}\left\|\nabla\partial^\alpha w\right\|_{L^2}^2+\frac{4c^2}{\nu\lambda_g m_0^2}\left\|\nabla g\right\|_\infty^2\left\|w\right\|_{H_g^1}^2\left\|u\right\|_{\mathbf{H}_g^2}^2$$

where (2.1), Poincaré and Young inequalities were applied. Summing up all the above estimates, we have

$$\frac{d}{dt}\left(\|\partial^\alpha u\|_{L^2}^2+\|\partial^\alpha w\|_{L^2}^2\right)+M_5\left(\|\nabla\partial^\alpha u\|_{L^2}^2+\|\nabla\partial^\alpha w\|_{L^2}^2\right)\leq M_6\left(\|\partial^\alpha u\|_{L^2}^2+\|\partial^\alpha w\|_{L^2}^2\right)+M_7$$

where the constants M_5, M_6 and M_7 depend on $\mathbf{H}_g^2$ and H_g^2 norm u and w respectively, while M_7 also depends on the H^2_{gcurl} norm of f. Therefore, applying Grönwall's inequality then gives $u\in L^\infty\left(0,T;\mathbf{H}_g^2\cap\mathbf{V}_g\right)\cap L^2\left(0,T;\mathbf{H}_g^3\cap\mathbf{V}_g\right)$ and $w\in L^\infty\left(0,T;H_g^2\cap V_g\right)\cap L^2\left(H_g^3\cap V_g\right)$. Consequently, by inductively restating the same arguments as before, we obtain $\mathbf{H}_g^s$ uniform bound on u and H_g^s uniform bound on w for all integers $s\geq 3$. Proof of Theorem 3.1 is thus complete. □

4 Data Assimilation Algorithm For The Velocity-Vorticity Model Of The g-Navier-Stokes Equations

The main idea for the data assimilation algorithm for a general evolutionary equation is given in [22]. Using this main idea the data assimilation algorithm is analyzed for different problems [6, 12, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24]. We will consider a linear operator $I_h: H_g^1\to L^2$ satisfying

$$\left\|u-I_h(u)\right\|_{L^2}\leq ch\left\|u\right\|_{H_g} \tag{4.1}$$

for every $u\in H_g$ where $c>0$ is a constant and h is the spatial resolution size. This implies that

$$\left\|P_g\left(u-I_h(u)\right)\right\|_{L^2}\leq ch\left\|u\right\|_{V_g}$$

for every $u\in V_g$ [22]. In this study we consider the following velocity-vorticity model of the g-NSEs will be used to obtain the reference solution (1.1)-(1.4) using observational data:

$$\frac{\partial z}{\partial t}-\nu\Delta_g z+\nu\frac{1}{g}(\nabla g\cdot\nabla)z+\eta\times z+\nabla P=f-\mu_1 P_g\left(I_h(z)-I_h(u)\right), \tag{4.2}$$

$$\frac{\partial\eta}{\partial t}-\nu\Delta_g\eta+\nu\frac{1}{g}(\nabla g\cdot\nabla)\eta+(z\cdot\nabla)\,\eta=\nabla\times f+\eta\left(\frac{\nabla g}{g}\cdot z\right)-\mu_2 P_g\left(I_h(\eta)-I_h(w)\right), \tag{4.3}$$

$$\nabla\cdot(gz)=0,\qquad\qquad \nabla\cdot(g\eta)=0, \tag{4.4}$$

$$z(x,0)=z_0,\qquad\qquad \eta(x,0)=\eta_0 \tag{4.5}$$

here μ_1, $\mu_2>0$ is a relaxation parameter. In functional form, the system (4.2)-(4.5) reads as:

$$\frac{dz}{dt}+\nu A_g z+\nu C_g z+P_g\left(\eta\times z\right)=P_g f-\mu_1 P_g\left(I_h(z)-I_h(u)\right), \tag{4.6}$$

$$\frac{d\eta}{dt}+\nu A_g\eta+\nu C_g\eta+B_g(z,\eta)=P_g\left(\nabla\times f\right)+P_g\left(\eta\left(\frac{\nabla g}{g}\cdot z\right)\right)-\mu_2 P_g\left(I_h(\eta)-I_h(w)\right), \tag{4.7}$$

$$z(0) = z_0, \qquad \eta(0) = \eta_0. \tag{4.8}$$

The global regularity of our system was established in [1] and the existence of global attractor was given in [28]. Following the techniques that were introduced to prove the existence and uniqueness of solutions for the g-Navier-Stokes equations and the velocity-vorticity model of the g-NSEs (see for example, [1, 25]) we can show the existence of the solution (z, η) of the system (4.2)-(4.5).

Theorem 4.1. *Let I_h satisfy (4.1) and $(u(x,t), w(x,t))$ be a strong solution in the global attractor of (1.1)-(1.4). Let μ_1, $\mu_2 > 0$ be arbitrary and $h \ll 1$ be chosen such that $\mu_1 ch^2 \leq \nu$ and $\mu_2 ch^2 \leq \nu$ then, (4.6)-(4.8) has a unique strong solution (z, η) that satisfies;*

$$z \in C(0,T;\mathbf{V}_g) \cap L^2(0,T;D(\mathbf{A}_g)), \quad \frac{dz}{dt} \in L^2(0,T;\mathbf{H}_g),$$

$$\eta \in C(0,T;V_g) \cap L^2(0,T;D(A_g)), \quad \frac{d\eta}{dt} \in L^2(0,T;H_g).$$

Moreover, the strong solution (z, η) depends continuously on the initial data in the $\mathbf{H}_g \times H_g$ norm. If we choose μ_1, μ_2 large enough such that

$$\mu_1 > \frac{6\nu \|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2}{\nu\lambda_g} K_3 + \frac{6c^2 \|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2} K_4, \tag{4.9}$$

$$\mu_2 > \frac{6\nu \|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2 \|\nabla g\|_\infty^2}{\nu m_0^2} K_1, \tag{4.10}$$

and $h > 0$ small enough such that $\mu_1 ch^2 \leq \nu$ and $\mu_2 ch^2 \leq \nu$, where the positive constants K_i are given in the proof of Theorem 2.3, Theorem 2.4 then,

$$\|u(t) - z(t)\|_{\mathbf{H}_g^2} + \|w(t) - \eta(t)\|_{H_g^2} \to 0$$

at an exponential rate as $t \to \infty$.

Proof. Define $\xi = u - z$ and $\gamma = w - \eta$. Then,

$$\frac{d\xi}{dt} + \nu A_g \xi + \nu C_g \xi + P_g(\gamma \times u) + P_g(\eta \times \xi) = -\mu_1 P_g(I_h(\xi)), \tag{4.11}$$

$$\frac{d\gamma}{dt} + \nu A_g \gamma + \nu C_g \gamma + B_g(u,\gamma) + B_g(\xi,\eta) = P_g\left(\gamma\left(\frac{\nabla g}{g} \cdot u\right)\right)$$

$$+ P_g\left(\eta\left(\frac{\nabla g}{g} \cdot \xi\right)\right) - \mu_2 P_g(I_h(\gamma)), \tag{4.12}$$

where $\xi(x,0) = u_0(x) - z_0(x)$, $\gamma(x,0) = w_0(x) - \eta_0(x)$. By taking $L^2(\Omega, g)$ inner product of (4.11) and (4.12) with ξ and γ respectively, we obtain,

$$\frac{d}{dt}\|\xi\|_{\mathbf{H}_g}^2 + 2\nu\|\xi\|_{\mathbf{V}_g}^2 + 2\nu b_g\left(\frac{\nabla g}{g}, \xi, \xi\right) + 2(\gamma \times u, \xi)_{\mathbf{H}_g} + 2(\eta \times \xi, \xi)_{\mathbf{H}_g}$$

$$= -2\mu_1 (P_g(I_h(\xi)), \xi)_{\mathbf{H}_g},$$

$$\frac{d}{dt}\|\gamma\|_{H_g}^2 + 2\nu\|\gamma\|_{V_g}^2 + 2\nu b_g\left(\frac{\nabla g}{g}, \gamma, \gamma\right) + 2b_g(\xi, \eta, \gamma)$$

$$= 2\left\langle P_g\left(\gamma\left(\frac{\nabla g}{g} \cdot u\right)\right), \gamma\right\rangle_{H_g} + 2\left\langle P_g\left(\eta\left(\frac{\nabla g}{g} \cdot \xi\right)\right), \gamma\right\rangle_{H_g} - 2\mu_2 (P_g(I_h(\gamma)), \gamma)_{H_g}.$$

By (2.1), Cauchy-Schwarz, Young and Poincaré inequalities, we have:

$$2\nu\left|b_g\left(\frac{\nabla g}{g},\xi,\xi\right)\right| \leq \frac{\nu}{6}\|\xi\|_{\mathbf{V}_g}^2 + \frac{6\nu\|\nabla g\|_\infty^2}{m_0^2}\|\xi\|_{\mathbf{H}_g}^2,$$

$$2\left|(\gamma\times u,\xi)_{\mathbf{H}_g}\right| \leq \frac{\nu}{6}\|\gamma\|_{V_g}^2 + \frac{6c^2}{\nu\lambda_g}\|\xi\|_{\mathbf{V}_g}^2\|u\|_{\mathbf{V}_g}^2,$$

$$\begin{aligned}-2\mu_1\left(P_g\left(I_h(\xi)\right),\xi\right)_{\mathbf{H}_g} &= -2\mu_1\left(\left(I_h(\xi)-\xi\right),\xi\right)_{\mathbf{H}_g} - 2\mu_1(\xi,\xi)_{\mathbf{H}_g}\\ &\leq 2\mu_1\|I_h(\xi)-\xi\|_{\mathbf{H}_g}\|\xi\|_{\mathbf{H}_g} - 2\mu_1\|\xi\|_{\mathbf{H}_g}^2\\ &\leq 2\mu_1 ch\|\xi\|_{\mathbf{V}_g}\|\xi\|_{\mathbf{H}_g} - 2\mu_1\|\xi\|_{\mathbf{H}_g}^2\\ &\leq \mu_1c^2h^2\|\xi\|_{\mathbf{V}_g}^2 - 2\mu_1\|\xi\|_{\mathbf{H}_g}^2 \leq \nu\|\xi\|_{\mathbf{V}_g}^2 - \mu_1\|\xi\|_{\mathbf{H}_g}^2,\end{aligned}$$

$$2\nu\left|b_g\left(\frac{\nabla g}{g},\gamma,\gamma\right)\right| \leq \frac{\nu}{6}\|\gamma\|_{V_g}^2 + \frac{6\nu\|\nabla g\|_\infty^2}{m_0^2}\|\gamma\|_{H_g}^2,$$

$$2\nu\left|b_g(\xi,\eta,\gamma)\right| \leq \frac{\nu}{6}\|\gamma\|_{V_g}^2 + \frac{6c^2}{\nu\lambda_g}\|\xi\|_{\mathbf{V}_g}^2\|\eta\|_{H_g}^2,$$

$$2\left|\left\langle P_g\left(\gamma\left(\frac{\nabla g}{g}\cdot u\right)\right),\gamma\right\rangle_{H_g}\right| \leq \frac{\nu}{6}\|\gamma\|_{V_g}^2 + \frac{6c^2\|\nabla g\|_\infty^2}{\nu m_0^2}\|\gamma\|_{H_g}^2\|u\|_{\mathbf{H}_g}^2,$$

$$2\left|\left\langle P_g\left(\eta\left(\frac{\nabla g}{g}\cdot \xi\right)\right),\gamma\right\rangle_{H_g}\right| \leq \frac{\nu}{6}\|\gamma\|_{V_g}^2 + \frac{6c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\eta\|_{V_g}^2\|\xi\|_{\mathbf{H}_g}^2,$$

and

$$\begin{aligned}-2\mu_2\left(P_g\left(I_h(\gamma)\right),\gamma\right)_{H_g} &= -2\mu_2\left(\left(I_h(\gamma)-\gamma\right),\gamma\right)_{H_g} - 2\mu_2(\gamma,\gamma)_{H_g}\\ &\leq 2\mu_2\|I_h(\gamma)-\gamma\|_{H_g}\|\gamma\|_{H_g} - 2\mu_2\|\gamma\|_{H_g}^2\\ &\leq 2\mu_2 ch\|\gamma\|_{V_g}\|\gamma\|_{H_g} - 2\mu_2\|\gamma\|_{H_g}^2\\ &\leq \mu_2c^2h^2\|\gamma\|_{V_g}^2 - 2\mu_2\|\gamma\|_{H_g}^2 \leq \nu\|\gamma\|_{V_g}^2 - \mu_2\|\gamma\|_{H_g}^2.\end{aligned}$$

Thus, it follows from these estimates that,

$$\begin{aligned}&\frac{d}{dt}\left(\|\xi\|_{\mathbf{H}_g}^2 + \|\gamma\|_{H_g}^2\right) + \left(\frac{5\nu}{6} - \frac{6c^2}{\nu\lambda_g}\|\eta\|_{V_g}^2\right)\|\xi\|_{\mathbf{V}_g}^2 + \frac{\nu}{6}\|\gamma\|_{V_g}^2\\ &\leq \left(\frac{6\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2}{\nu\lambda_g}\|u\|_{\mathbf{V}_g}^2 - \mu_1 + \frac{6c^2\|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\eta\|_{V_g}^2\right)\|\xi\|_{\mathbf{H}_g}^2\\ &+ \left(\frac{6\nu\|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2\|\nabla g\|_\infty^2}{\nu m_0^2}\|u\|_{\mathbf{H}_g}^2 - \mu_2\right)\|\gamma\|_{H_g}^2.\end{aligned}$$

Let,

$$\alpha(t) = \frac{6\nu \|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2}{\nu\lambda_g}\|u\|_{\mathbf{V}_g}^2 - \mu_1 + \frac{6c^2 \|\nabla g\|_\infty^2}{\nu\lambda_g m_0^2}\|\eta\|_{V_g}^2,$$

$$\beta(t) = \frac{6\nu \|\nabla g\|_\infty^2}{m_0^2} + \frac{6c^2 \|\nabla g\|_\infty^2}{\nu m_0^2}\|u\|_{\mathbf{H}_g}^2 - \mu_2.$$

Then, we have

$$\frac{d}{dt}\left(\|\xi\|_{\mathbf{H}_g}^2 + \|\gamma\|_{H_g}^2\right) + \min\left\{\lambda_g\left(\frac{5\nu}{6} - \frac{6c^2}{\nu\lambda_g}\|\eta\|_{V_g}^2\right), \frac{\nu\lambda_g}{6}, \alpha(t), \beta(t)\right\}\left(\|\xi\|_{\mathbf{H}_g}^2 + \|\gamma\|_{H_g}^2\right)$$

$$\leq 0$$

since

$$\int_t^{t+1} \alpha(s)ds \geq \frac{\mu_1}{2} > 0 \quad \text{and} \quad \int_t^{t+1} \alpha(s)ds \leq \frac{3\mu_1}{2} < \infty$$

$$\int_t^{t+1} \beta(s)ds \geq \frac{\mu_2}{2} > 0 \quad \text{and} \quad \int_t^{t+1} \beta(s)ds \leq \frac{3\mu_2}{2} < \infty.$$

By the uniform Grönwall's lemma, Lemma 2.2, it follows that

$$\|\xi\|_{\mathbf{H}_g}^2 + \|\gamma\|_{H_g}^2 \to 0$$

at an exponential rate, as $t \to \infty$. □

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