Palestine Journal of Mathematics

Vol 14(3)(2025) , 1262–1272

ON SOME SPECTRAL BOUNDS OF COMPLETE MULTIPARTITE GRAPHS

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Communicated by Ambat Vijayakumar

MSC 2010 Classifications: Primary 05C05; Secondary 05C38.

Keywords and phrases: Sectral radius, Principal eigenvector, Walks, Minimum vertex cover, Clique number..

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract This article presents some spectral bounds on the spectral radius $\tau_1(G)$, energy E_G, and extremarkarke components y_{max} and y_{min} of the principal eigenvector $Y_1 = (y_{1_1}, y_{1_2}, \cdots, y_{1_n})^T$ of a complete multipartite graph G. Here we emphasize implementing parameters such as the sum of the degrees of the vertices contained in a minimum vertex cover, the total number of walks of odd length, cardinalities, the number of partite set, etc., contained in a complete multipartite graph G while establishing the bounds. Additionally, by considering some complete multipartite graphs as examples, we examine how the newly constructed bounds behave in terms of the parameters mentioned above.

1 Introduction

Let G be a simple, connected, and undirected graph of order n and size e with vertex set $U(G)$=$\{u_1, u_2, u_3, \cdots, u_n\}$ and edge set $E(G)$. For any $u_i \in U(G)$ $(i, j = 1, 2, \cdots, n)$, the number of edges $u_j u_i \in E(G)$ incident to u_i is the degree $d(u_i)$ of u_i. A vertex cover S of graph G is a subset of $U(G)$ such that each edge of G is incident to at least one of the vertices contained in S. The cardinality of a minimum vertex cover of graph G is known as the minimum vertex cover number, denoted by ρ. If the vertices of a minimum vertex cover are mutually non-adjacent, then the vertex cover is known as an independent minimum vertex cover. An induced subgraph H of G of order h is said to be an h-clique if all of the vertices contained in $U(H)$ are mutually adjacent. The order of a maximum clique contained in G is called the clique number ω of G. A set S of vertices in G is called an independent set if the vertices in S are mutually non-adjacent. Again, the cardinality of the maximum independent set contained in G is the independence number ψ of G. An l-length walk $u_{i_1} u_{i_2} u_{i_3} \cdots u_{i_l} u_{i_l+1}$ $(i = 1, 2, \cdots, n)$ in G is a sequence of $l + 1$ vertices such that $u_{i_j} u_{i_j+1} \in E(G)$ $(j = 1, 2, \cdots, l)$. We denote the total number of l-length walks in G by $W_l(G)$.

A graph $K_{m_1,m_2,\cdots,m_p}$ is said to be a complete multipartite graph if the vertex set $U(K_{m_1,m_2,\cdots,m_p})$ can be partitioned into p-partite sets P_i, such that $|P_i| = m_i$ and the vertices of any two distinct partite sets are adjacent to each other. A complete graph K_n of order n is a complete multipartite graph in which each partite contains exactly one vertex. Similarly, a star graph $K_{1,n-1}$ of order n is a complete multipartite graph composed of 2-partite sets, such that one of them contains exactly one vertex, while the other contains the rest of the $n-1$ vertices. Again, the adjacency matrix $A(G) = [a_{ij}]$ of graph G is a matrix of order n with $a_{ij} = 1$, if and only if $u_i u_j \in E(G)$ or zero, otherwise. The eigenvalues $\tau_i(G)$ $(\tau_1(G) \geq \tau_2(G) \geq \cdots \geq \tau_n(G))$ of the adjacency matrix $A(G)$ corresponding to a graph G of order n form the spectrum of G. And in general, the eigenvalues $\tau_i(G)$ $(i = 1, 2, \cdots, n)$ of $A(G)$ are called the eigenvalues of G. Again, the energy E_G of G is defined as $E_G = \sum_{i=1}^{n} |\tau_i|$. It is important to note that $\tau_1(G)$ is known as the spectral radius of G. The unique eigenvector $Y_1=(y_{1_1}, y_{1_2}, \cdots, y_{1_n})^T$ of unit length corresponding to $\tau_1(G)$, such that $y_{1_i} > 0$ for all $i = 1, 2, \cdots, n$ is the principal eigenvector of G. The maximal and minimal components of the principal eigenvector Y_1 of G are denoted here by y_{max} and y_{min}, respectively. For basic results and definitions of graphs, we refer to [18].

Estimating the eigenvalues, especially the spectral radius of a graph, is one of the widely studied topics and has a significant influence on spectral graph theory. Mathematicians such as Collatz and Sinogowitz [17], Nosal [13], Brigham and Dutton [2], Friedland [9], Hong [20], Aimei Yu et al. [19], Nikiforov [11], Bollobás and Nikiforov [1], and Feng, Li, and Zhang [8], among others, have enriched this area with their notable contributions. Since 2000, scholars have started to estimate the components of the principal eigenvector and the ratios of these components, as they reveal a variety of properties of a graph. The ratios of the components of the principal eigenvector of a graph are considered to measure the graph's regularity. Britta Papendieck and Peter Recht [14] were pioneers in estimating the values of the components of the principal eigenvector of a graph. Later, following their work, Zhao and Hong [22], Zhang [21], Cioaba and Gregory [3], Nikiforov [12], Goldberg [10], etc. also developed several bounds on the components as well as the ratios of the components of the principal eigenvector of a graph. It is worth noting that establishing bounds in terms of various parameters not only aids in estimating the precise values of τ_1, y_{max}, and y_{min} but also provides insight into their interrelationships. On the other hand, the spectrum and eigenvectors of complete multipartite graphs exhibit several noteworthy properties that have drawn the interest of mathematicians to this class of graphs. John Smith made some of the first contributions to the study of the spectral properties of complete multipartite graphs. In 1970, he showed in [15] that a graph contains precisely one positive eigenvalue if and only if it is a complete multipartite graph. Later, Friedrich Esser and Frank Harary [7], C. Delorme [5], Dragan Stevanović, Ivan Gutman, et al. [16] contributed well-researched materials along this line. Thus, we have made an effort to make some contribution to this field by providing estimations for the spectral radius, as well as the extreme components and the ratio of the extreme components of the principal eigenvector of a complete multipartite graph.

Section 2 of this article includes some lemmas that assist our findings, which are introduced in Sections 3 and 4. Section 3 is entirely dedicated to some bounds on the spectral radius and energy of a complete multipartite graph, with an in-depth discussion of these results. On the other hand, Section 4 contains bounds on y_{max} and y_{min} of the principal eigenvector Y_1 of a complete multipartite graph.

2 Preliminaries

Lemma 2.1. *Let G be a graph of order n and size e, and let ρ be its minimum vertex cover number. If $M = \{u_1{}^*, u_2{}^*, \cdots, u_\rho{}^*\}$ is a minimum vertex cover of G, then*

$$e \leq \sum_{i=1}^{\rho} d(u_i{}^*). \tag{2.1}$$

Equality holds in (2.1) if and only if G contains an independent minimum vertex cover.

Proof. The proof is obvious. □

Lemma 2.2. *[?] Let G be a graph of order n, and let $Y_1 = (y_{1_1}, y_{1_2}, y_{1_3}, \cdots, y_{1_n})^T$ be the principal eigenvector of G corresponding to the spectral radius $\tau_1(G)$. Then G is a d-regular graph if and only if*

(i) $\tau_1(G) = d$*, and*

(ii) $y_{1_i} = \frac{1}{\sqrt{n}}, \quad \forall i = 1, 2, \cdots, n.$

Lemma 2.3. *[?] Let $Y_i = (y_{i_1}, y_{i_2}, \cdots, y_{i_n})^T$ for $i = 1, 2, \cdots, n$, where Y_i are the orthonormal eigenvectors corresponding to the eigenvalues $\tau_i(G)$ $(i = 1, 2, \cdots, n)$ of a graph G. If $C_i =$*

$\left(\sum_{j=1}^{n} y_{i_j}\right)^2$, *then*

$$W_k(G) = \tau_1(G)^k C_1 + \tau_2(G)^k C_2 + \cdots + \tau_n(G)^k C_n, \tag{2.2}$$

where $W_k(G)$ is the total number of k-length walks in G, and $k \in \mathbb{Z}^+ \cup \{0\}$.

Lemma 2.4. *[?] A graph G contains exactly one main eigenvalue if and only if G is regular.*

Lemma 2.5. *[?] Let G be a graph of order n. If y_{max} is the maximal component of the principal eigenvector Y_1 of G, then*

$$y_{max} \leq \frac{1}{\sqrt{2}}. \tag{2.3}$$

Equality holds in (2.3) if and only if G is a star graph.

Lemma 2.6. *[?] A graph G contains only one positive eigenvalue if and only if G is a complete multipartite graph.*

Lemma 2.7. *[?] Let G be a complete multipartite graph of the form $K_{m_1,m_2,\cdots,m_p}$. The graph $K_{m_1,m_2,\cdots,m_p}$ consists of p partite sets P_l such that $|P_l| = m_l$ and $\sum_{l=1}^{p} m_l = n$. For $l \in \{1, 2, \cdots, p\}$, let $P_l = \{u_{l_1}, u_{l_2}, \cdots, u_{l_{m_l}}\}$, and $Y_i = (y_{i_1}, y_{i_2}, \cdots, y_{i_n})^T$ be the eigenvector corresponding to the eigenvalue $\tau_i(G)$ of G, where $i = 1, 2, \cdots, n$. Then, the following holds:*

(i) *For some $i = 1, 2, \cdots, n$, if $\tau_i(G) = 0$, then $\sum_{j=1}^{n} y_{i_j} = 0$.*

(ii) *If $\tau_i(G) \neq 0$ for some $i = 1, 2, \cdots, n$, and ${y_{i_{l_j}}}^*$ are the components of the eigenvector Y_i corresponding to the vertices u_{l_j} $(j = 1, 2, \cdots, m_l)$, respectively, then ${y_{i_{l_1}}}^* = {y_{i_{l_2}}}^* = \cdots = {y_{i_{l_{m_l}}}}^*$.*

3 Bounds on the Spectral Radius of a Complete Multipartite Graph

Theorem 3.1. *Let G be a complete multipartite graph of order n, let and ρ be its minimum vertex cover number. Let $A = \{u_{a_1}, u_{a_2}, \cdots, u_{a_\rho}\}$ be a minimum vertex cover of G and $d_A = \sum_{i=1}^{\rho} d(u_{a_i})$. Then, for any odd positive integer $k > 2$, the following inequality holds:*

$$\tau_1(G) > \left(\frac{W_{k+2}(G)}{4{d^*}^2}\right)^{1/k}, \tag{3.1}$$

where $d^* = min\{d_A|$ A is a minimum vertex cover of $G\}$.

Proof. Let $Y_i = (y_{i_1}, y_{i_2}, \cdots, y_{i_n})^T$ be the orthonormal eigenvector corresponding to the eigenvalue $\tau_i(G)$ of the graph G, where $i \in \{1, 2, \cdots, n\}$. From the equation (2.2) we get $W_k(G) = \sum_{i=1}^{n} C_i \tau_i(G)^k$, where $C_i = \left(\sum_{j=1}^{n} y_{i_j}\right)^2$. Since k is an odd positive integer, thus from Lemma 2.6, we obtain

$$W_k(G) \leq \left(\sum_{i=1}^{n} y_{1_i}\right)^2 \tau_1(G)^k. \tag{3.2}$$

Let $A^* = \{{u_{a_1}}^*, {u_{a_2}}^*, \cdots, {u_{a_\rho}}^*\}$ be the minimum vertex cover of G, such that $d_{A^*} = d^*$. The inequalities $d^* \geq e$ and $2e = \sum_{i=1}^{n} d(u_i)$ imply

$$2d^* \geq \sum_{i=1}^{n} d(u_i). \tag{3.3}$$

Again, equation $A(G)Y_1 = \tau_1(G)Y_1$ implies $\tau_1(G)y_{1_i} = \sum_{u_i u_j \in E(G)} y_{1_j}$, $\forall i \in \{1, 2, \cdots, n\}$, leading to $\tau_1(G)\sum_{i=1}^{n} y_{1_i} = \sum_{i=1}^{n} d(u_i) y_{1_i}$. Since $y_{1_i} < 1$, $\forall\, i \in \{1, 2, \cdots, n\}$ (Lemma 2.5), we get

$$\sum_{i=1}^{n} d(u_i) > \tau_1(G) \sum_{i=1}^{n} y_{1_i}. \tag{3.4}$$

Thus, inequalities (3.3) and (3.4) give $2d^* > \tau_1(G)\sum_{i=1}^n y_{1_i}$ which implies

$$\left(\frac{2d^*}{\tau_1(G)}\right)^2 > \left(\sum_{i=1}^n y_{1_i}\right)^2. \tag{3.5}$$

From (3.5) and (3.2) we conclude that $\tau_1(G) > \left(\frac{W_{k+2}(G)}{4d^{*2}}\right)^{1/k}$, thus completing the proof. □

Remark 3.2. It is to be noted that since $y_{1_i} < 1$, for all $i \in \{1, 2, \cdots, n\}$, equality cannot hold in (3.1).

Proposition 3.3. *Let G be a complete multipartite graph of order n, and let $\tau_1(G)$ be its spectral radius. If $W_k(G)$ is the count of k-length walks in G, where k is an odd positive integer, then*

$$\tau_1(G) \geq \left(\frac{W_k(G)}{n}\right)^{1/k}. \tag{3.6}$$

Equality holds in (3.6), if G is a complete graph K_n.

Proof. We consider $Y_i = (y_{i_1}, y_{i_2}, \cdots, y_{i_n})^T$ to be an orthonormal eigenvector corresponding to the eigenvalue $\tau_i(G)$ of G, where $i \in \{1, 2, \cdots, n\}$. Since k is an odd positive integer, thus from Lemma 2.6 and Lemma 2.3, we get $W_k(G) \leq \left(\sum_{i=1}^n y_{1_i}\right)^2 \tau_1(G)^k$. From the Rayleigh quotient, we have $\tau_1(K_n) = n-1 \geq \frac{Y_1^T A(K_n) Y_1}{Y_1^T Y_1} = \sum_{\substack{i,j=1\\ i\neq j}}^n y_{1_i} y_{1_j}$, which implies $n \geq 1 + \sum_{\substack{i,j=1\\ i\neq j}}^n y_{1_i} y_{1_j} = \left(\sum_{i=1}^n y_{1_i}\right)^2$. Substituting $\left(\sum_{i=1}^n y_{1_i}\right)^2$ by n in the inequality $W_k(G) \leq \left(\sum_{i=1}^n y_{1_i}\right)^2 \tau_1(G)^k$, we obtain $W_k(G) \leq n\tau_1(G)^k$. Therefore, $\tau_1(G) \geq \left(\frac{W_k(G)}{n}\right)^{1/k}$.

Now, let us assume that G is isomorphic to K_n. Then, clearly, $n-1 = \tau_1(K_n) = \sum_{\substack{i,j=1\\ i\neq j}} y_{1_i} y_{1_j}$, which gives $n = 1 + \sum_{\substack{i,j=1\\ i\neq j}} y_{1_i} y_{1_j}$. Thus, $n = \left(\sum_{i=1}^n y_{1_i}\right)^2$. Since G is a regular graph, from Lemma 2.4 we deduce $W_k(G) = \left(\sum_{i=1}^n y_{1_i}\right)^2 \tau_1(G)^k$, which leads to the equality in (3.6). This completes the proof. □

Remark 3.4. From llll 3.1 and propositionosition 3.3, if

$$\frac{1}{2}\left(\frac{nW_{k+2}(G)}{W_k(G)}\right)^{1/2} > d^*,$$

then for an odd positive integer k, the lower bound (3.1) on the spectral radius $\tau_1(G)$ of G outperforms the bound (3.6).

Remark 3.5. Let $D_1(\tau_1) = \tau_1(G) - \left(\frac{W_{k+2}(G)}{4d^{*2}}\right)^{1/k}$ and $D_2(\tau_1) = \tau_1(G) - \left(\frac{W_k(G)}{n}\right)^{1/k}$ be the deviations. Then, Table 1 shows the values of k, d^*, $W_k(G), W_{k+2}(G), \tau_1(G)$, $D_1(\tau_1), D_2(\tau_1)$, and $\frac{1}{2}\left(\frac{nW_{k+2}(G)}{W_k(G)}\right)^{1/2}$ for the complete multipartite graphs $K_3, K_{1,2}$ of order 3, and $K_5, K_{2,1,1,1}$, $K_{2,2,1}, K_{3,1,1}$,$K_{3,2}$, $K_{1,4}$ of order 5, where $k = 3^m$ $(m = 1, 2, 3, 4, 5)$.

As k increases, both $D_1(\tau_1)$ and $D_2(\tau_1)$ decrease for all the considered graphs (such as $K_3, K_{1,2}$, K_5, $K_{2,1,1,1}$, $K_{2,2,1}$, and others). This is because, for each of the complete multipartite graphs mentioned above, given d^* and n (i.e., the order of the graph), when k increases from 3 to 243, $W_k(G)$ and $W_{k+2}(G)$ begin to increase rapidly. The graphs $K_{1,2}, K_3$, and $K_{1,4}$, $K_{3,2}$, $K_{3,1,1}$, $K_{2,2,1}$, $K_{2,1,1,1}$, K_5 are the arrangements of the complete multipartite graphs of order 3 and 5, respectively, listed in descending order of the performance of the bound (3.1). Here, it is evident that the values obtained from the bound (3.1) are closer to $\tau_1(G)$ than those from (3.6) for complete bipartite graphs, particularly in the case of the star graph. In contrast, (3.6) yields the best result for complete graphs other than the star graph. Furthermore, based on the increasing values of $D_2(\tau_1)$, the graphs can be arranged as follows: $K_3, K_{1,2}$ (of order 3) and $K_5, K_{2,2,1}, K_{2,1,1,1}, K_{3,2}, K_{3,1,1}, K_{1,4}$ (of order 5). When comparing complete multipartite graphs of the same order (3 or 5) for each k, graphs with more evenly distributed vertices

Table 1. Walk lengths and the deviations $D_1(\tau_1)$ and $D_2(\tau_1)$

k	G	d^*	$W_k(G)$	$W_{k+2}(G)$	$\tau_1(G)$	$D_1(\tau_1)$	$D_2(\tau_1)$	$\frac{1}{2}\left(\frac{nW_{k+2}(G)}{W_k(G)}\right)^{\frac{1}{2}}$
3	K_3	4	24	96	2.0000	0.8553	0.0000	1.7321
	$K_{1,2}$	2	8	16	1.4142	0.4142	0.0275	1.2247
	K_5	16	320	5120	4.0000	2.2900	0.0000	4.4721
	$K_{2,1,1,1}$	12	240	3192	3.6458	1.8762	0.0116	4.0774
	$K_{2,2,1}$	10	168	1760	3.2361	1.5975	0.0092	3.6187
	$K_{3,1,1}$	8	128	1160	3.0000	1.3452	0.0528	**3.3657**
	$K_{3,2}$	6	72	432	2.4495	0.8905	0.0166	2.7386
	$K_{1,4}$	4	32	128	2.0000	0.7401	0.1434	2.2361
9	K_3	4	1536	6144	2.0000	0.3394	0.0000	1.7321
	$K_{1,2}$	2	64	128	1.4142	0.1543	0.0092	1.2247
	K_5	16	1310720	20971520	4.0000	0.9867	0.0000	4.4721
	$K_{2,1,1,1}$	12	564000	7496448	3.6458	0.7806	0.0036	4.0761
	$K_{2,2,1}$	10	193024	2021376	3.2361	0.7592	0.0029	3.6180
	$K_{3,1,1}$	8	94376	849896	3.0000	0.5383	0.0139	**3.3551**
	$K_{3,2}$	6	15552	93312	2.4495	0.3965	0.0056	2.7386
	$K_{1,4}$	4	2048	8192	2.0000	0.2855	0.0490	2.2361
	$K_{1,4}$	4	2048	8192	2.0000	0.2855	0.0490	2.2361
27	K_3	4	402653184	1.6106e+09	2.0000	0.1202	0.0000	1.7321
	$K_{1,2}$	2	32768	65536	1.4142	0.0534	0.0031	1.2247
	K_5	16	9.0072e+16	20971520	4.0000	0.3604	0.0000	4.4721
	$K_{2,1,1,1}$	12	7.3021e+15	9.7056e+16	3.6458	0.2814	0.0012	4.0761
	$K_{2,2,1}$	10	2.9237e+14	3.0617e+15	3.2361	0.2357	9.6798e-04	3.6180
	$K_{3,1,1}$	8	3.6603e+13	3.2943e+14	3.0000	0.1913	0.0045	**3.3541**
	$K_{3,2}$	6	1.5673e+11	9.4037e+11	2.4495	0.1400	0.0019	2.7386
	$K_{1,4}$	4	536870912	2.1475e+09	2.0000	0.1001	0.0165	2.2361
81	K_3	4	7.2536e+24	2.9014e+25	2.0000	0.0409	0.0000	1.7321
	$K_{1,2}$	2	4.3980e+12	8.7961e+12	1.4142	0.0180	0.0070	1.2247
	K_5	16	2.9230e+49	4.6768e+50	4.0000	0.1239	0.0000	4.4721
	$K_{2,1,1,1}$	12	1.5847e+46	2.1063e+47	3.6458	0.0963	3.9083e-04	4.0761
	$K_{2,2,1}$	10	1.0160e+42	1.0639e+43	3.2361	0.0806	3.2350e-04	3.6180
	$K_{3,1,1}$	8	2.1284e+39	1.9156e+40	3.0000	0.0652	0.0015	**3.3541**
	$K_{3,2}$	6	1.6041e+32	9.6246e+32	2.4495	0.0476	5.8974e-04	2.7386
	$K_{1,4}$	4	9.6714e+24	3.8686e+25	2.0000	0.0339	0.0055	2.2361
	K_3	4	4.2404e+73	1.6962e+74	2.0000	0.0137	0.0000	1.7321
	$K_{1,2}$	2	1.0634e+37	2.1268e+37	1.4142	0.0060	3.0000e-04	1.2247
	K_5	16	9.9896e+146	1.5983e+148	4.0000	0.0417	0.0000	4.4721
	$K_{2,1,1,1}$	12	1.6198e+137	2.1529e+138	3.6458	0.0324	1.3028e-04	4.0761
243	$K_{2,2,1}$	10	4.2632e+124	4.4645e+125	3.2361	0.0271	1.0784e-04	3.6180
	$K_{3,1,1}$	8	4.1851e+116	3.7666e+117	3.0000	0.0219	5.0000e-04	**3.3541**
	$K_{3,2}$	6	1.7198e+95	1.0319e+96	2.4495	0.0160	1.8974e-04	2.7386
	$K_{1,4}$	4	5.6539e+73	2.2616e+74	2.0000	0.0114	0.0018	2.2361

across their partite sets, such as $K_5(\simeq K_{1,1,1,1,1})$ and $K_3(\simeq K_{1,1,1})$ tend to yield better results for (3.6). The performances tend to degrade as the even distribution of the vertices among the graph's partite sets becomes less even. Table 1 also shows that bound (3.6) outperforms (3.1) for all of the $K_3, K_{1,2}, K_5, K_{2,1,1,1}, K_{2,2,1}, K_{3,1,1}, K_{3,2}$, and $K_{1,4}$, because in these graphs, d^* cannot exceed $\frac{1}{2}\left(\frac{nW_{k+2}(G)}{W_k(G)}\right)^{1/2}$. Moreover, the values of $\frac{1}{2}\left(\frac{nW_{k+2}(G)}{W_k(G)}\right)^{1/2}$ change slowly compared to the increasing values of k, and the only graph where the values decrease is $K_{3,1,1}$. For a fixed value of k, it is observed that $D_1(\tau_1)$ decreases faster than $D_2(\tau_1)$ for $K_{1,2}$ and $K_{1,4}$ while the decrease is slower for K_3 and K_5.

Theorem 3.6. *Let G be a complete multipartite graph of the form $K_{m_1,m_2,\cdots,m_p}$, with n vertices,* *and let the graph contain p partite sets P_l, where $|P_l| = m_l$ for $l = 1, 2, \cdots, p$. If $Y_1 = \{y_{1_1}, y_{1_2}, \cdots, y_{1_n}\}^T$ is the principal eigenvector corresponding to the spectral radius $\tau_1(G)$ of G, where, y_{max} and y_{min} are the maximal and minimal components of Y_1, respectively, then*

$$\prod_{i=1}^{p} m_i p(p-1) {y_{max}}^2 - \left(\prod_{i=1}^{p} m_i p(p-1) - \sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j \right) {y_{min}}^2 \geq \tau_1(G). \tag{3.7}$$

Equality in (3.7) holds if and only if G is a regular complete multipartite graph $K_{m,m,\cdots,m}$ with p partite sets.

Proof. From Lemma 2.7, let us assume that ${y_{1_l}}'$ is the component of the vector $Y_1 = \{y_{1_1}, y_{1_2}, \cdots, y_{1_n}\}^T$ corresponding to the vertices contained in the partite set $P_l = \{u_{l_1}, u_{l_2}, \cdots, u_{l_{m_l}}\}$ for $l = 1, 2, \cdots, p$. It is clear that the complete multipartite graph G $(\simeq K_{m_1,m_2,\cdots,m_p})$ contains $\prod_{i=1}^{p} m_i$ distinct p-cliques. Now, consider two distinct partite sets of G, P_i $(i = 1, 2, \cdots, p)$ and P_j $(j = 1, 2, \cdots, p)$, such that for some $k \in \{1, 2, \cdots, m_i\}$ and $q \in \{1, 2, \cdots, m_j\}$, $u_{i_k} \in P_i$ and $u_{j_q} \in P_j$. Then, the edge $u_{i_k} u_{j_q}$ occurs $\prod_{\substack{l=1 \\ l \neq i,j}}^{p} m_l$ times in the $\prod_{i=1}^{p} m_i$ distinct p-cliques present in G. However, there are $m_i m_j$ edges between the partite sets P_i and P_j of G. Thus,

$$\tau_1(G) \leq \prod_{i=1}^{p} m_i p(p-1) {y_{max}}^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j \left(\frac{\prod_{i=1}^{p} m_i}{m_i m_j} - 1 \right) {y_{min}}^2, \tag{3.8}$$

which implies (3.7).

Let G be a regular complete multipartite graph $K_{m,m,\cdots,m}$, containing p partite sets. Then $pm = n$. From Lemma 2.2, we get $\tau_1(G) = (p-1)m$, and $y_{max} = y_{min} = \frac{1}{\sqrt{pm}} (= \frac{1}{\sqrt{n}})$. Thus, substituting the values of y_{max}, y_{min} as $\frac{1}{\sqrt{pm}}$, and m_i, m_j as m on the left side of (3.7), we obtain the value $(p-1)m$, which implies equality in (3.7).

Conversely, assuming that equality holds in (3.7), we obtain equality in (3.8). Now, from Lemma 2.7 and the Rayleigh quotient, we get $\tau_1(G) = \sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j {y_{1_i}}' {y_{1_j}}'$, which, together with (3.8), implies

$$\prod_{i=1}^{p} m_i p(p-1) \left({y_{max}}^2 - {y_{min}}^2 \right) = \sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j \left({y_{1_i}}' {y_{1_j}}' - {y_{min}}^2 \right). \tag{3.9}$$

Now, if we assume G to be non-regular, then the components of the vector Y_1 are not equal (Lemma 2.2). Thus, from Lemma 2.7, we infer that there exist some partite sets P_i and P_j in G such that the vertices contained in P_i and P_j correspond to y_{max} and y_{min}, respectively. Hence, for some $i, j \in \{1, 2, \cdots, p\}$ with $i \neq j$, ${y_{max}}^2 - {y_{min}}^2 > {y_{1_i}}' {y_{1_j}}' - {y_{min}}^2$. Since $\prod_{i=1}^{p} m_i p(p-1) \geq \sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j$, equality does not hold in (3.9), leading to a contradiction. □

Remark 3.7. Let $D(\tau_1)$ be the deviation of the bound (3.7) from $\tau_1(G)$. Table 2 enlists the values of n, p, y_{max}, y_{min}, $\prod_{i=1}^{p} m_i p(p-1)$, $\sum_{\substack{i,j=1 \\ i \neq j}}^{p} m_i m_j$, and the deviation $D(\tau_1)$ for the complete multipartite graphs considered in the table.

In Table 2, we categorize a set of 24 graphs into 6 groups, each containing 4 graphs of consecutive orders. The cardinalities of one of the partite sets in the graphs within each group are increased, while the cardinalities of the other partite sets remarkain constant. We observe that $D(\tau_1)$ increases as the order of the graph increases. Notably, for $p = 2$, the graphs with partite sets that have the same cardinality difference show a decrease in $D(\tau_1)$ as the orders (n) of the corresponding graphs increase. For instance, $\{K_{1,2}, K_{2,3}, K_{3,4}\}$, $\{K_{1,3}, K_{2,4}, K_{3,5}\}$, $\{K_{1,4}, K_{2,5}, K_{3,6}\}$, and $\{K_{1,5}, K_{2,6}, K_{3,7}\}$ are 4 groups of 2-partite graphs with partite sets that have the same cardinality differences. In the graphs from $\{K_{1,2}, K_{2,3}, K_{3,4}\}$, $D(\tau_1)$ decreases from $K_{1,2}$ to $K_{2,3}$ and then to $K_{3,4}$. The same trend is observed in the other 3 groups: $\{K_{1,3}, K_{2,4}, K_{3,5}\}$, $\{K_{1,4}, K_{2,5}, K_{3,6}\}$ and $\{K_{1,5}, K_{2,6}, K_{3,7}\}$. However, this pattern does not hold for $p = 3$.

Table 2. Extremarkal components of the principal eigenvector of complete multipartite graph G and the deviation $D(\tau_1)$

G	n	p	y_{max}	y_{min}	$\prod_{i=1}^{p} m_i p(p-1)$	$\sum_{\substack{i,j=1\\i\neq j}}^{p} m_i m_j$	$D(\tau_1)$
$K_{1,2}$	3	2	0.7071	0.5000	4	4	0.2422
$K_{1,3}$	4	2	0.7071	0.4082	6	6	0.4608
$K_{1,4}$	5	2	0.7071	0.3536	8	8	0.6584
$K_{1,5}$	6	2	0.7071	0.3162	10	10	0.8392
$K_{2,3}$	5	2	0.5000	0.4082	12	12	0.1236
$K_{2,4}$	6	2	0.5000	0.3536	16	16	0.2422
$K_{2,5}$	7	2	0.5000	0.3162	20	20	0.3544
$K_{2,6}$	8	2	0.5000	0.2887	24	24	0.4609
$K_{3,4}$	7	2	0.4082	0.3536	24	24	0.0829
$K_{3,5}$	8	2	0.4082	0.3162	30	30	0.1638
$K_{3,6}$	9	2	0.4082	0.2887	36	36	0.2422
$K_{3,7}$	10	2	0.4082	0.2673	42	42	0.3176
$K_{1,1,1}$	3	3	0.5774	0.5774	6	6	0.0000
$K_{1,1,2}$	4	3	0.5573	0.4352	12	10	0.1506
$K_{1,1,3}$	4	3	0.5477	0.3651	18	14	0.3074
$K_{1,1,4}$	6	3	0.5418	0.3213	24	18	0.4521
$K_{1,2,2}$	5	3	0.5257	0.4253	24	16	0.2004
$K_{1,2,3}$	6	3	0.5257	0.4253	24	16	0.2004
$K_{1,2,4}$	7	3	0.4987	0.3166	48	28	0.4832
$K_{1,2,5}$	8	3	0.4910	0.2868	60	34	0.5919
$K_{1,3,3}$	7	3	0.4888	0.3562	54	30	0.3364
$K_{1,3,4}$	8	3	0.4752	0.3149	72	38	0.4655
$K_{1,3,5}$	9	3	0.4654	0.2855	90	46	0.5699
$K_{1,3,6}$	10	3	0.4578	0.2633	108	54	0.6613

Remark 3.8. We know that all of the diagonal elements in the adjacency matrix $A(G)$ of a graph G are zero. Therefore, $tr\,(A(G)) = \sum_{i=1}^{n} \tau_i(G) = 0$. Since G is a complete multipartite graph, it contains exactly one positive eigenvalue, $\tau_1(G)$ (Lemma 2.6). Hence, $\tau_1(G) = \sum_{i=2}^{n} |\tau_i(G)|$. Thus, the energy E_G of G is given by $E_G = \sum_{i=1}^{n} |\tau_i(G)| = \tau_1(G) + \sum_{i=2}^{n} |\tau_i(G)| = 2\tau_1(G)$. Using (3.2), (3.6), and (3.7), we obtain lower bounds as well as an upper bound on the energy E_G of a complete multipartite graph G in the following lllls.

Theorem 3.9. *Let G be a complete multipartite graph of order n. Let $A = \{u_{a_1}, u_{a_2}, \cdots, u_{a_\rho}\}$ be a minimum vertex cover of G, and $d_A = \sum_{i=1}^{\rho} d(u_{a_i})$. If E_G is the energy of G, and $k > 2$ is an odd positive integer, then*

$$E_G > 2\left(\frac{W_{k+2}(G)}{4{d^*}^2}\right)^{1/k}, \tag{3.10}$$

where $d^* = min\{d_A|\ A$ is a minimum vertex cover of $G\}$.

Theorem 3.10. *Let G be a complete multipartite graph of order n with energy E_G. If $W_k(G)$ is the total number of k-length walks in G, then*

$$E_G \geq 2\left(\frac{W_k(G)}{n}\right)^{1/k}, \tag{3.11}$$

where k is an odd positive integer. Equality exists in (3.11) if G is a complete graph K_n.

Theorem 3.11. *Let G be a complete multipartite graph $K_{m_1,m_2,m_3,\cdots,m_p}$ of order $n > 2$, containing p partite sets P_l such that $|P_l| = m_l$ for $l = 1, 2, \cdots, p$. Denote, y_{max}, y_{min}, and E_G are the extremarkal components of the principal eigenvector Y_1 and the energy of G, respectively. Then the following inequality holds:*

$$2\prod_{i=1}^{p} m_i p(p-1){y_{max}}^2 - 2\left(\prod_{i=1}^{p} m_i p(p-1) - \sum_{\substack{i,j=1\\ i\neq j}}^{p} m_i m_j\right){y_{min}}^2 \geq E_G. \tag{3.12}$$

Equality in (3.12) holds if and only if G is a regular complete multipartite graph $K_{m,m,\cdots,m}$ with p partite sets.

4 Bounds on the Extremarke components of the Principal Eigenvector of a Complete Multipartite Graph

Theorem 4.1. *Let G be a non-regular complete multipartite graph of order n and size e. Let $Y_1 = (y_{1_1}, y_{1_2}, \cdots, y_{1_n})^T$ be the principal eigenvector corresponding to the spectral radius $\tau_1(G)$ of G. If δ is the minimum degree of G and k is an odd positive integer, then*

$$y_{max} > \frac{(\tau_1(G) - \delta)\left(\frac{W_K(G)}{{\tau_1}^k}\right)^{1/2}}{2e - n\delta}, \tag{4.1}$$

where $y_{max} = max\{y_{1_1}, y_{1_2}, \cdots, y_{1_n}\}$.

Proof. Since $Y_1 = (y_{1_1}, y_{1_2}, \cdots, y_{1_n})^T$ is the principal eigenvector of G, from the Rayleigh quotient, we get $\tau_1(G) = \frac{{y_1}^T A(G) y_1}{{y_1}^T y_1} = \sum_{u_i u_j \in E(G)} y_{1_i} y_{1_j}$. For all $i, j \in \{1, 2, \cdots, n\}$, we have $\tau_1(G) y_{1_i} = \sum_{u_i u_j \in E(G)} y_{1_i} y_{1_j}$. Therefore, $\sum_{i=1}^{n} d(u_i) y_{1_i} = \tau_1(G) \sum_{i=1}^{n} y_{1_i}$. Given that $2e = \sum_{i=1}^{n} d(u_i)$, we can deduce $(2e - n\delta) y_{max} > (\tau_1(G) - \delta) \sum_{i=1}^{n} y_{1_i}$. Therefore, the inequality becomes

$$y_{max} > \frac{(\tau_1(G) - \delta)\sum_{i=1}^{n} y_{1_i}}{2e - n\delta}. \tag{4.2}$$

From (2.2), we know that $W_k(G) = \sum_{i=1}^{n} \tau_i(G)^k C_i$. According to Lemma 2.6, G has only one positive eigenvalue $\tau_1(G)$, since it is a complete multipartite graph. Moreover, k is an odd positive integer. Therefore, G being a non-regular graph implies $\tau_1(G)^k C_1 = \tau_1(G)^k (\sum_{i=1}^{n} y_{1_i})^2 > W_k(G)$, which gives

$$\sum_{i=1}^{n} y_{1_i} > \left(\frac{W_k(G)}{\tau_1(G)^k}\right)^{1/2}. \tag{4.3}$$

Combining the inequalities from (4.2) and (4.3), we conclude the proof of equation (4.1). □

Remark 4.2. Suppose that,

$$y_{max}\,(2e - n\delta) = (\tau_1(G) - \delta)\sum_{i=1}^{n} y_{1_i}, \tag{4.4}$$

and

$$\tau_1(G)^k \left(\sum_{i=1}^{n} y_{1_i}\right)^2 = \tau_1(G)^k C_1 = W_k(G). \tag{4.5}$$

From these, we deduce that $(2e - n\delta) y_{max} = (\tau_1(G) - \delta)\left(\frac{W_k G}{{\tau_1}^k}\right)^{1/2}$. If we assume that $y_{max} = y_{1_i}$, for all $i \in \{1, 2, \cdots, n\}$, then (4.4) holds. From Lemma 2.2, it follows that G must be a regular graph. Similarly, from Lemma 2.4, we know that to hold (4.5), G must be regular. For a regular graph with order n and size e, we know that $2e = n\delta$. hence we get the strict inequality (4.1).

Theorem 4.3. *Let G be a complete multipartite graph of the form $K_{m_1,m_2,\cdots,m_p}$ with order n. Let G contain partite sets P_l such that $|P_l| = m_l$, for $l = 1, 2, \cdots, p$. For $i = 1, 2, \cdots, n$, let $Y_i = (y_{i_1}, y_{i_2}, y_{i_3}, \cdots, y_{i_n})^T$ be the orthonormal eigenvectors corresponding to the eigenvalues $\tau_i(G)$ of G, respectively, and let $C_i = \left(\sum_{j=1}^n y_{i_j}\right)^2$. Let y_{1_l}' denote a component of the principal eigenvector $Y_1 = (y_{1_1}, y_{1_2}, y_{1_3}, \cdots, y_{1_n})^T$ of G corresponding to the vertices of P_l. If ψ ($\psi > 1$), and χ are the independence number and chromatic number of G, respectively, and if the following condition holds: $\tau_1(G)^k \sum_{l=1}^k m_l(m_l-1){y_{1_l}'}^2 \geq |\sum_{i=2}^n C_i \tau_i(G)^k|$, where k is an odd positive integer, then*

$$y_{max} \geq \left(\frac{W_k(G) - \left(\tau_1(G)^k + \tau_1(G)^{k+1}\right)}{\tau_1(G)^k \chi\psi(\psi-1)}\right)^{1/2}. \tag{4.6}$$

Equality holds in (4.6) if and only if G is a regular complete multipartite graph $K_{\psi,\psi,\cdots,\psi}$ of order n.

Proof. Let $P_m = \{u_{m_1}, u_{m_2}, \cdots, u_{m_{|P_m|}}\}$ ($m \in \{1, 2, \cdots, \chi\}$), be a partite set of the complete multipartite graph G. Thus, each P_m forms an independent set in G and contains vertices of the same color. From Lemma 2.7, we assume that y_{1_m}' is the component of the principal eigenvector Y_1 corresponding to the vertices $u_{m_i} \in P_m$, where $i \in \{1, 2, \cdots, |P_m|\}$. Suppose, $P_{m^*} = \{{u_{m_1}}^*, {u_{m_2}}^*, \cdots, {u_{m_\psi}}^*\}$ is a maximum independent set in G. Then, for all $m \in \{1, 2, \cdots, \leq \chi\}$, we have

$$\psi = |P_{m^*}| \geq |P_m|. \tag{4.7}$$

Again, $y_{max} \geq y_{1_m}'$, $\forall m \in \{1, 2, \cdots, \chi\}$. Thus from (4.7), we obtain

$$\psi(\psi-1){y_{max}}^2 \geq \sum_{\substack{\left({u_{m_i}}^*, {u_{m_j}}^*\right) \in P_{m^*} \times P_{m^*} \\ i \neq j}} {y_{1_{m^*}}'}^2, \tag{4.8}$$

and

$$\psi(\psi-1){y_{max}}^2 \geq \sum_{m=1}^{\chi} \left(\sum_{\substack{\left(u_{m_i}, u_{m_j}\right) \in P_m \times P_m \\ i \neq j}} {y_{1_m}'}^2\right) \quad (i, j \in \{1, 2, \cdots, |P_m|\}). \tag{4.9}$$

Hence, for the complete multipartite graph G, we obtain

$$\begin{aligned} \chi\psi(\psi-1){y_{max}}^2 &\geq \sum_{m=1}^{\chi} \left(\sum_{\substack{\left(u_{m_i}, u_{m_j}\right) \in P_m \times P_m \\ i \neq j}} {y_{1_m}'}^2\right) \\ &= \sum_{u_k u_s \notin E(G)} y_{1_k} y_{1_s}, \quad (i, j \in \{1, 2, \cdots, |P_m|\}). \end{aligned} \tag{4.10}$$

We know that

$$\begin{aligned} \left(\sum_{i=1}^n y_{1_i}\right)^2 &= \sum_{i=1}^n {y_{1_i}}^2 + \sum_{u_i u_j \in E(G)} y_{1_i} y_{1_j} + \sum_{u_i u_j \notin E(G)} y_{1_i} y_{1_j} \\ &= 1 + \tau_1(G) + \sum_{u_i u_j \notin E(G)} y_{1_i} y_{1_j}. \end{aligned} \tag{4.11}$$

Thus, from (4.10) and (4.11), we obtain

$$1 + \tau_1(G) + \chi\psi(\psi-1){y_{max}}^2 \geq \left(\sum_{i=1}^n y_{1_i}\right)^2. \tag{4.12}$$

Applying the lower bound of $(\sum_{i=1}^{n} y_{1_i})^2$ obtained from (3.2) into (4.12), we get

$$1 + \tau_1(G) + \chi\psi(\psi - 1){y_{max}}^2 \geq \frac{W_k(G)}{{\tau_1(G)}^k}.$$

This gives us the desired result in (4.6).

The necessary and sufficient condition for equality in (4.6) is that equality must hold in both (3.2) and (4.12). Let G be a regular complete multipartite graph $K_{\psi,\psi,\cdots,\psi}$ of order n such that $\psi > 1$. This implies $\psi = |P_m|$, $\forall m \in \{1, 2, \cdots, \chi\}$, and $U(G) = \cup_{m=1}^{\chi} P_m$. From Lemma 2.2, we assert that $y_{max} = {y_{1_m}}'$, $\forall m \in \{1, 2, \cdots, \chi\}$. Therefore (4.10) and (4.11) lead us to the equality in (4.12). Furthermore, from Lemma 2.4, we can state that if $G \simeq K_{\psi,\psi,\cdots,\psi}$, which is of order n, then equality holds in (3.2).

Conversely, let equality hold in (4.6), and hence in both (3.2) and (4.12). The equality in (3.2) implies that G is regular (Lemma 2.4), whereas equality in (4.12) implies $\chi\psi(\psi - 1){y_{max}}^2 = \sum_{u_i u_j \notin E(G)} y_{1_i} y_{1_j} = \sum_{l=1}^{k} m_l(m_l - 1){y_{1_l}}^2$. Since G is a complete multipartite graph with chromatic number χ and independence number ψ, we get $\chi = p$, and $m_l = \psi$, $\forall l \in \{1, 2, \cdots, p\}$. This implies that G must be $K_{\psi,\psi,\cdots,\psi}$ of order n, and thus the llll follows. □

Proposition 4.4. *Let G be a complete multipartite graph of the form $K_{m_1,m_2,\cdots,m_p}$. Let G be of order n and consist of p partite sets P_l $(l = 1, 2, \cdots, p)$, such that $|P_l| = m_l$. If $\tau_1(G)$ is the spectral radius of G, then*

$$y_{max} \geq \left(\frac{\tau_1(G)}{\prod_{i=1}^{p} m_i p(p-1)}\right)^{1/2}. \tag{4.13}$$

Equality holds in (4.13) if and only if G is K_2.

Proof. For all $l \in \{1, 2, \cdots, p\}$, let $P_l = \{u_{l_1}, u_{l_2}, \cdots, u_{l_{m_l}}\}$. From Lemma 2.7, we assume that ${y_{1_l}}^*$ $(l \in \{1, 2, \cdots, p\})$ is a component of the principal eigenvector $Y_1 = (y_{1_1}, y_{1_2}, \cdots, y_{1_n})^T$ of G, corresponding to the vertices u_{l_i} $(i \in \{1, 2, \cdots, m_l\})$ of P_l. Since G is a complete multipartite graph with p partite sets, there are $\prod_{i=1}^{p} m_l$ distinct p-cliques in G, where $\omega(G) = p$. By the Rayleigh quotient, we obtain $\tau_1(G) = \sum_{u_{i_s} u_{j_q} \in E(G)} {y_{1_{i_s}}}^* {y_{1_{j_q}}}^*$, where $i, j \in \{1, 2, \cdots, p\}$ $(i \neq j)$, $s \in \{1, 2, \cdots, m_i\}$, and $q \in \{1, 2, \cdots, m_j\}$. On the other hand, $y_{max} \geq y_{1_i}$, $\forall i \in \{1, 2, \cdots, n\}$. Thus, we have

$$\prod_{l=1}^{p} m_l p(p-1){y_{max}}^2 \geq \tau_1(G). \tag{4.14}$$

From this inequality, we get the upper bound (4.13) on y_{max}.

Equality holds in (4.13) if and only if the same holds in (4.14). For a complete multipartite graph G, the necessary and sufficient condition for the existence of equality in (4.14) is that G must be regular (Lemma 2.2), and $\omega(G) = p = 2$. Thus, equality exists in (4.13) if and only if G is K_2. □

Proposition 4.5. *Let G be a complete multipartite graph $K_{m_1,m_2,\cdots,m_p}$, consisting of $p > 1$ partite sets P_l such that $|P_l| = m_l$. If y_{min} is the minimal component of the principal eigenvector corresponding to the spectral radius $\tau_1(G)$ of G, then*

$$\left(\frac{p(p-1)\prod_{i=1}^{p} m_i - 2\tau_1(G)}{2\left(p(p-1)\prod_{i=1}^{p} m_i - \sum_{\substack{i,j=1 \\ i\neq j}}^{p} m_i m_j\right)}\right)^{1/2} \geq y_{min}. \tag{4.15}$$

Equality in (4.15) holds if and only G is K_2.

Proof. From (2.3) and (3.7), we obtain the upper bound (4.15) on y_{min}. □

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Received: 2024-08-27
Accepted: 2025-05-02