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Special Squares in Tribonacci and Tribonacci Lucas numbers

Hayder R. Hashim and Anwar N. Jasim

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Corresponding Author: Hayder R. Hashim

Abstract *This paper mainly focuses on determining all the squares in the sequences of Tribonacci numbers $\{T_n\}$ and Tribonacci Lucas numbers $\{S_n\}$ that are also squares of some numbers in the sequences of Fibonacci numbers $\{F_n\}$ and Lucas numbers $\{L_n\}$, respectively. In other words, we completely solve the Diophantine equations $T_n = F_m^2$ and $S_i = L_j^2$ for the nonnegative integers n, m and i, j, respectively. As auxiliary results, we also determined all the Tribonacci numbers that are also presenting Fibonacci numbers and all the Tribonacci Lucas numbers that are presenting Lucas numbers. Namely, we obtained all the nonnegative values of the integers n_1, m_1 and n_2, m_2 with which the equations $T_{n_1} = F_{m_1}$ and $S_{n_2} = L_{m_2}$ are respectively satisfied.*

1 Introduction and auxiliary results

A linear recurrence sequence is a sequence that could be be denoted by $\{d_n\}$ and is defined by the relation

$$d_{n+k} = b_1 d_{n+k-1} + b_2 d_{n+k-2} + \ldots + b_k d_n + t(n),$$

with $n \geq 0$, $b_1, b_2, \ldots, (b_k \neq 0) \in \mathbb{C}$ and $t(n)$ is a function with respect to n. If $t(n)$ is not zero, then the sequence is called nonhomogeneous, otherwise it is homogeneous. Note that the parameter k in the indices of the recurrence relation denotes the order of the recurrence sequence. Hence, if we consider in particular $k = 2$ or 3, then the homogeneous recurrence relation respectively gives what is so called a binary recurrence sequence or a ternary recurrence sequence. Let's for instance consider the well known binary recurrence sequences named by the Lucas sequence of the first or second kind that is respectively defined by the relations

$$u_0 = 0, u_1 = 1, \quad u_{n+2} = Pu_{n+1} - Qu_n \tag{1.1}$$

and

$$v_0 = 2, v_1 = P, \quad v_{n+2} = Pv_{n+1} - Qv_n, \tag{1.2}$$

where $n \geq 0$ and the parameters $P \neq 0$ and $Q \neq 0$ are coprime integers. In fact, if $P = 1$ and $Q = -1$, we get some well known such sequences that are called by the Fibonacci sequence $\{F_n\} = \{u_n(1, -1)\}$ or Lucas sequence $\{L_n\} = \{v_n(1, -1)\}$ which are given by the formulas

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \tag{1.3}$$

$$L_0 = 2, L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \tag{1.4}$$

for $n \geq 0$. On the other hand, there are many ternary recurrence sequences that have been studied by several authors such as the sequences of Tribonacci numbers $\{T_n\}$ and Tribonacci

Lucas numbers $\{S_n\}$ whose terms are derived from the following relations:

$$T_0 = 0, T_1 = 0, T_2 = 1, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n, \tag{1.5}$$

$$S_0 = 3, S_1 = 1, S_2 = 3, \quad S_{n+3} = S_{n+2} + S_{n+1} + S_n, \tag{1.6}$$

with $n \geq 0$. Note that the sequences $\{T_n\}$ and $\{S_n\}$ are also known as generalizations of the sequences $\{F_n\}$ and $\{L_n\}$, respectively. For more details regarding these sequences, thier connections to other fields and their identities or properties, one can see e.g. [9], [10], [11], [13] and [17]. In order to see how the sequence $\{T_n\}$ is a generalization of $\{F_n\}$ (similarly, $\{S_n\}$ is a generalization of $\{L_n\}$), in the following table we list a few terms of these sequences:

Table 1: Some terms of F_n, T_n, L_n, and S_n.

n	F_n	T_n	L_n	S_n
0	0	0	2	3
1	1	0	1	1
2	1	1	3	3
3	2	1	4	7
4	3	2	7	11
5	5	4	11	21
6	8	7	18	39
7	13	13	29	71
8	21	24	47	131
9	34	44	76	241
10	55	81	123	443
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

These sequences have been in interest to many authors for their own sake or for their applications in many areas in mathematics or other fields of sciences. For instance, the terms of such sequences have been investigated as solutions to different Diophantine equations, see e.g. [2],[6],[7],[8], [12] and the references given there.

On the other hand, finding the perfect powers of terms in binary and ternary recurrence sequences have been such a historical and important topic in number theory. Let's consider such related studies, starting with the results of finding the squares of Fibonacci numbers (that are $F_0 = 0, F_1 = F_2 = 1$, and $F_{12} = 144$) which was determined by Cohn [5] and independently by Wyler [19]. Furthermore, Alfred [3] obtained that the only squares in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$. Also, Pethő [14, 15] determined the third and fifth power Fibonacci numbers. For more results related to the perfect powers of binary recurrence sequences, see e.g.[4].

On the other hand, finding squares in ternary recurrence sequences such as Tribonacci sequence or Tribonacci Lucas sequence has not been achieved completely. In 1998, Pethő [16] proposed (at the 7^{th} International Research Conference on Fibonacci numbers and Their Applications) the following problem concerning the squares in the Tribonacci sequence:

Pethő's Problem 1 **:** *Are there any square terms in the Tribonacci sequence $\{T_n\}$ other than $T_0 = 0, T_1 = 0, T_2 = 1, T_3 = 1, T_5 = 2^2, T_{10} = 9^2, T_{16} = 56^2$ and $T_{18} = 103^2$?*

This problem remains unsolved till now. In fact, by the motivation of this problem in 2021 Irmak [9] determined the squares in the Tribonacci Lucas sequence $\{S_n\}$ under the conditions that $n \not\equiv 1 \pmod{32}$ and $n \not\equiv 17 \pmod{96}$. More precisely, he showed that the only square is $S_1 = 1$. Furthermore, he gave the following conjecture regarding the square terms in the Tribonacci Lucas sequence dropping out the above conditions:

Irmak's Conjecture 1 : *The only square in the Tribonacci Lucas sequence $\{S_n\}$ is given by $S_1 = 1$.*

Similarly, this conjecture has not been proved completely. By the motivation of the Pethő's Problem and Irmak's Conjecture, in this paper we investigate the terms, in the Tribonacci se-

quence $\{T_n\}$ and Tribonacci Lucas sequence $\{S_n\}$, that are presenting squares of terms in the sequences $\{F_n\}$ and $\{L_n\}$, respectively. In other words, we completely solve the following Diophantine equations:

$$T_n = F_m^2 \tag{1.7}$$

and

$$S_i = L_j^2, \tag{1.8}$$

with n, m, i, j are nonnegative integers.

In order to attack the latter equations, in the following we give and prove the following lemma that will be used later as an auxiliary result in the proofs of our main results.

Lemma 1.1. *Suppose that n_1 and n_2 are integers with $n_1, n_2 \geq 0$, then $T_{n_1} = F_{n_1}$ and $S_{n_2} = L_{n_2}$ are satisfied only if $n_1 \in \{0, 2, 7\}$ and $n_2 \in \{1, 2\}$, respectively.*

Proof. Indeed, to prove this lemma and obtain the desired values of n_1 and n_2 in which we have $T_{n_1} = F_{n_1}$ and $S_{n_2} = L_{n_2}$, it enough to show that $T_{n_1} > F_{n_1}$ and $S_{n_2} > L_{n_2}$ for all $n_1, n_2 \geq 8$. If that will be achieved, then it remains to examine the equations $T_{n_1} = F_{n_1}$ and $S_{n_2} = L_{n_2}$ with $n_1, n_2 \leq 7$. Clearly, that can be followed easily from the recurrence relations of these sequences given in (1.3)-(1.6) as we see the Tribonacci sequence grows faster than the Fibonacci sequence as $n_1 \geq 8$ (Similar idea goes to Lucas and Tribonacci Lucas sequences). Another way to achieve it easily is using mathematical induction by showing that $T_{n_1} - F_{n_1} > 0$ and $S_{n_2} - L_{n_2} > 0$ as $n_1, n_2 \geq 8$, and we omit the detail of this proof for its simpleness. Now, we seek the values of n_1 and n_2 with $0 \leq n_1, n_2 \leq 7$ in which the equations $T_{n_1} = F_{n_1}$ and $S_{n_2} = L_{n_2}$ are satisfied. In fact, from Table 1, we see that $T_{n_1} = F_{n_1}$ only if $n_1 = 0, 2$ and 7 and similarly $S_{n_2} = L_{n_2}$ in case of $n_2 = 1, 2$. Hence, the desired values of n_1 and n_2 are achieved. □

In the following, we give a generalization to the result of Lemma 1.1 in case of the indices are different:

Lemma 1.2. *Let n_1, n_2, m_1 and m_2 be nonnegative integers such that $n_1 \neq m_1$ and $n_2 \neq m_2$, then $T_{n_1} = F_{m_1}$ and $S_{n_2} = L_{m_2}$ are satisfied only if $(n_1, m_1) \in \{(1, 0), (2, 1), (3, 1), (3, 2), (4, 3)\}$ and $(n_2, m_2) \in \{(0, 2), (3, 4), (4, 5)\}$, respectively.*

Proof. Following the proof of Lemma 1.1, in order to prove this lemma it is enough to firstly show that $T_{n_1} \neq F_{m_1}$ as $n_1 \geq 8$ or $m_1 \geq 8$ and $S_{n_2} \neq L_{m_2}$ as $n_2 \geq 8$ or $m_2 \geq 8$. By showing that, it only remains to examine the values of $n_1, m_1 \leq 7$ satisfying $T_{n_1} = F_{m_1}$ and the values of $n_2, m_2 \leq 7$ satisfying $S_{n_2} = L_{m_2}$.

Let's now consider the first case in details, and the other one will be achieved similarly. Namely, we show that $T_{n_1} \neq F_{m_1}$ as $n_1 \geq 8$ or $m_1 \geq 8$, and this is equivalent to obtain $|T_{n_1} - F_{m_1}| > 0$ (equivalently, we show that $T_{n_1} - F_{m_1} > 0$ or $T_{n_1} - F_{m_1} < 0$) in case of $n_1 \geq 8$ or $m_1 \geq 8$. Here, the proof is divided to four cases:

- □ `Case` 1 : If $n_1 \geq 8$ and $n_1 > m_1$. Then we have $n_1 > m_1 \geq 0$ (with $n_1 \geq 8$ and $n_1 \neq m_1$). From Lemma 1.1, we have $T_{n_1} > F_{n_1} > F_{m_1}$ for all $n_1 \geq 8$, which implies that

$$T_{n_1} - F_{m_1} > F_{n_1} - F_{m_1} > 0$$

  as $n_1 \geq 8$ and $n_1 > m_1 \geq 0$.

- □ `Case` 2 : If $n_1 \geq 8$ and $m_1 > n_1$. This implies that $-F_{m_1} < -F_{n_1}$. Therefore,

$$T_{n_1} - F_{m_1} < T_{n_1} - F_{n_1},$$

  which is definitely greater than zero as $n_1 > 8$.

- □ `Case` 3 : If $m_1 \geq 8$ and $n_1 > m_1$. Hence, we obtain that $n_1 > m_1 \geq 8$. Similarly, we have that $T_{n_1} > F_{n_1} > F_{m_1}$, which implies that $T_{n_1} - F_{m_1} > F_{m_1} - F_{m_1} > 0$.

- □ `Case` 4 : If $m_1 \geq 8$ and $m_1 > n_1$. Thus, $m_1 > n_1 \geq 0$ (with $m_1 \geq 8$ and $n_1 \neq m_1$). This gives that

$$T_{n_1} - F_{m_1} < T_{n_1} - F_{n_1},$$

and for $n \leq 7$ the right hand side of the latter inequality is either equal to zero if $n = 0, 2, 7$ or less than zero otherwise (from the result of Lemma 1.1). On the other hand, for $n > 8$ it can be easily shown that $T_{n_1} > F_{n_1}$ (as indicated in the proof of Lemma 1.1) which leads to have the right hand side of the latter inequality is greater than zero.

From the above four cases, we have shown that $|T_{n_1} - F_{m_1}| > 0$ in case of $n_1 > 8$ or $m_1 \geq 8$. Finally, it remains to examine the distict values of n_1 and m_1 with $n_1, m_1 \leq 7$ in which $T_{n_1} = F_{m_1}$ is satisfied. In fact, these values are only given in the pairs

$$(n_1, m_1) \in \{(1, 0), (2, 1), (3, 1), (3, 2), (4, 3)\}.$$

By following the same above approach, we can also prove that $|S_{n_2} - L_{m_2}| > 0$ with $n_2 \geq 8$ or $m_2 \geq 8$. Therefore, we omit the detail of computations. Similarly, by checking the values of n_2 and m_2 (with $n_2, m_2 \leq 7$) satisfying $S_{n_2} = L_{m_2}$, we only get the pairs

$$(n_2, m_2) \in \{(0, 2), (3, 4), (4, 5)\}.$$

Hence, Lemma 1.2 is achieved. □

Remark 1.3. From Lemma 1.1, we obtained that $T_n > F_m$ and $S_n > L_m$ for all $n = m \geq 8$. Moreover, from Lemma 1.2 we got that $T_n \neq F_m$ and $S_n \neq L_m$ with $n \neq m$ and $n \geq 8$ or $m \geq 8$.

2 Main results

Theorem 2.1. *If the integers $n, m \geq 0$, then the set of solutions (n, m) of equation (1.7) is given by*

$$(n, m) \in \{(0, 0), (1, 0), (2, 1), (2, 2), (3, 1), (3, 2), (5, 3)\}.$$

Proof. We prove this theorem by two cases. Starting with the case when $n, m \leq 7$, and the other is when n or $m \geq 8$. Namely,

- `Case 1` : If $n, m \leq 7$. Clearly, we are here seeking the values of n and m with $0 \leq n, m \leq 7$, that satisfy equation (1.7). In fact, with the help of any mathematical software such as SageMath Software [18] we determine the integers n and m with $0 \leq n, m \leq 7$ satisfying the equation $T_n = F_m^2$, which are given by the pairs

$$(n, m) \in \{(0, 0), (1, 0), (2, 1), (2, 2), (3, 1), (3, 2), (5, 3)\}.$$

- `Case 2` : If $n \geq 8$ or $m \geq 8$. Here, we divide this case into five subcases regarding the relation between the integers n and m. Indeed, in all of these sucases we will show that $T_n \neq F_m^2$

  - If $n = m$. In either of $n \geq 8$ or $m \geq 8$, we have $n = m \geq 8$. Then equation (1.7) becomes $T_n = F_n^2$. Since $n \geq 8$, we claim that $F_n^2 > T_n$. In order to prove this claim, we may assume for a contradiction that

$$T_n - F_n^2 \geq 0 \tag{2.1}$$

for all $n \geq 8$. From Remark 1.3 (or Lemma 1.1), we have the fact $T_n > F_n$ for all $n \geq 8$. Now, by using this fact in the left hand side of inequality (2.1), we get that

$$T_n - F_n^2 > F_n - F_n^2 = F_n(1 - F_n), \tag{2.2}$$

which is clearly less than zero as $F_n > 0$ and $(1 - F_n) < 0$ for all $n \geq 8$. Hence, that contradicts inequality (2.1) and implies that $T_n - F_n^2 < 0$ or $F_n^2 > T_n$ for all $n \geq 8$. Therefore, equation (1.7) is not solvable in the positive integers n and m such that $n = m \geq 8$.

- If $m \geq 8$ and $n > m$. Here, we are looking for the values of n and m such that $n > m \geq 8$ with which the equation (1.7) (i.e. $T_n = F_m^2$) is satisfied. More precisely, we again claim this equation has no solutions in the integers n and m with this range. To prove this claim, it is enough to show that $|T_n - F_m^2| > 0$ (i.e. $|T_n - F_m^2| \neq 0$ or $T_n \neq F_m^2$) with all $n > m \geq 8$. Indeed, we can prove the truthiness of this inequality (namely, $|T_n - F_m^2| > 0$ with all $n > m \geq 8$) using a mathematical induction (what is so called the "Two-Dimensional Induction") by showing the following three statements are true: $S(n_0, m_0) = S(9, 8)$, $S(k+1, m_0) = S(k+1, 8)$ (if $S(k, m_0) = S(k, 8)$ is true) and $S(h, k+1)$ (if $S(h, k)$ is true), where

$$S(n, m) = |T_n - F_m^2| > 0.$$

For more details about this type of induction, see e.g. [1]. In other words, we show that

  - $S(9, 8) = |T_9 - F_8^2| = 397$, which is greater than zero.
  - Now, we assume that $S(k, 8) = |T_k - F_8^2| = |T_k - 441| > 0$ for all $k \geq 9$. Next, we have to show the statement at $(n, m) = (k+1, 8)$ is true. Namely, we must prove that $S(k+1, 8) > 0$ for all $k \geq 9$, and that is

$$\begin{aligned} S(k+1, 8) = |T_{k+1} - 441| &= |T_k + T_{k-1} + T_{k-2} - 441| \\ &= |T_k - 441 + T_{k-1} + T_{k-2}| \\ &\leq |T_k - 441| + |T_{k-1} + T_{k-2}| \end{aligned}$$

    and this is definitely greater than zero as $|T_k - 441| > 0$ for all $k \geq 9$.
  - The final step is to prove that $S(h, k+1) > 0$ for all $h > k \geq 8$ by assuming that $S(h, k) = |T_h - F_k^2| > 0$ is true with $h > k \geq 8$. Therefore,

$$\begin{aligned} S(h, k+1) = |T_h - F_{k+1}^2| &= |T_h - F_k^2 - 2F_kF_{k-1} - F_{k-1}^2| \\ &\geq ||T_h - F_k^2| - |2F_kF_{k-1} + F_{k-1}^2|| \\ &> |0 - |2F_kF_{k-1} + F_{k-1}^2|| \\ &> 0 \quad \text{for all} \quad k \geq 8, \end{aligned}$$

    and this proves the statement $S(n, m) = |T_n - F_m^2| > 0$ for all $n > m \geq 8$.

  Thus, we have achieved our claim of having no solutions in n and m in the case of $n > m \geq 8$ satisfying equation (1.7).

- If $m \geq 8$ and $m > n$. Similarly, we again claim that the equation (1.7) is not solvable with $m > n \geq 0$ such that $m \geq 8$. We prove this claim by showing that $|T_n - F_m^2| > 0$ for all $m > n \geq 0$ with $m \geq 8$. Thus,

$$T_n - F_m^2 < T_n - F_n^2.$$

  From Case 1 and from the first subcase of Case 2 (when $n = m \geq 8$), one can easily obtain that the right hand side of the latter inequality is either equal to zero only if $n = 0, 2$ or less than zero otherwise. Hence, our claim is achieved, and this proves that the equation (1.7) has no such solutions.

- If $n \geq 8$ and $n > m$. This means that $n > m \geq 0$ such that $n \geq 8$. Again, we can show that $|T_n - F_m^2| > 0$ for all $n > m \geq 0$ and $n \geq 8$. In other words,

$$T_n - F_m^2 \geq T_n - F_n^2$$

  and this less than zero as shown in identity (2.2). Therefore, we have concluded that the equation (1.7) does not have solutions in n and m under the condition that $n > m \geq 0$ for $n \geq 8$.

- If $n \geq 8$ and $m > n$. We again can prove that equation (1.7) is not solvable in the integers n and m with $m > n \geq 8$. As done in Case 2 (with $n > m \geq 8$), we apply mathematical induction to show that $T_n - F_m^2 > 0$ in the case of $m > n \geq 8$.

From these above subcases of Case 1, we conclude that the equation (1.7) has no solutions in the integers n and m with $n \geq 8$ or $m \geq 8$.

Hence, the results of Theorem 2.1 is obtained from Case 1, namely equation (1.7) is satisfied only with

$$(n, m) \in \{(0, 0), (1, 0), (2, 1), (2, 2), (3, 1), (3, 2), (5, 3)\}.$$

Thus, the proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let i and j be integers with $i, j \geq 0$, then the only solution to equation (1.8) is given by $(i, j) = (1, 1)$.*

Proof. By following the same the approach used in the proof of Theorem 2.1, we can prove this theorem. Hence, we summarize the detail of the proof as follows.

■ We firstly consider $i \geq 8$ or $j \geq 8$, which can be divided into five subcases, namely $i = j \geq 8$; $j \geq 8$ and $i > j$; $j \geq 8$ and $j > i$; $i \geq 8$ and $i > j$, and lastly $i \geq 8$ and $j > i$. In the following, we will show that the equation (1.8) has no solutions in the integers i and j with all of these mentioned five subcases.

As done in the proof of Theorem 2.1, in case of $i = j \geq 8$ we can show that the equation (1.8) (namely $S_i = L_i^2$) has no solutions. That can be achieved by assuming for a contradiction that $S_i - L_i^2 \geq 0$, and by using the fact of $S_i > L_i$ for all $i \geq 8$ (from the result of Lemma 1.1, see also Remark 1.3) we can easily show that $S_i - L_i^2 < 0$. Hence, we conclude that for $i = j \geq 8$ the equation (1.8) is not solvable.

Next, we again prove the unsolvability of equation (1.8) with $j \geq 8$ and $i > j$ (equivalently $i > j \geq 8$). By having that $S(i, j) = |S_i - L_j^2| > 0$ and using the Two-Dimensional Induction, in which we have to prove the truthiness of $S(9, 8)$, $S(k + 1, 8)$ (if $S(k, 8)$ is true) and $S(h, k + 1)$ (if $S(h, k)$ is true). The first two statements can be achieved as done in Theorem 2.1, therefore in the following we only consider $S(h, k + 1)$ if $S(h, k) = |S_h - L_k^2| > 0$ is true. Namely,

$$\begin{aligned} S(h, k + 1) = |S_h - L_{k+1}^2| &= |S_h - L_k^2 - 2L_k L_{k-1} - L_{k-1}^2| \\ &\geq ||S_h - L_k^2| - |2L_k L_{k-1} + L_{k-1}^2|| \\ &> |0 - |2L_k L_{k-1} + L_{k-1}^2|| \\ &> 0 \quad \text{for all} \quad k \geq 8. \end{aligned}$$

Thus, we obtain $S(i, j) = |S_i - L_j^2| > 0$ for all $i > j \geq 8$. That shows the equation (1.8) (i.e. $S_i = L_j^2$) has no solutions in the integers i and j with $i > j \geq 8$.

Now, we consider the third subcase, i.e. $j > i$ such that $j \geq 8$ and $i \geq 0$. From the fact $S_i < S_j$ for all $j \geq 8$, we have that

$$S_i - L_j^2 < S_j - L_j^2 < 0 \tag{2.3}$$

since $S_j < L_j^2$ for all $j \geq 8$ as shown above in the case of $i = j \geq 8$. Again, this shows that the equation (1.8) has no solutions in the integers i and j with $0 \leq i < j$ and $j \geq 8$.

Here, we claim again that the equation (1.8) has no solutions in the integers i and j with $i \geq 8$ and $i > j \geq 0$. That can be shown by proving that $|S_i - L_j^2| > 0$. Since $i \geq 8$ and $j \geq 0$, then $L_i^2 > L_j^2 = 131$. Thus,

$$S_i - L_j^2 > S_i - L_i^2$$

and this less than zero as shown in identity (2.3). Therefore, the claim is proved.

Lastly, we deal with $i \geq 8$ and $j > i$ (i.e. $j > i \geq 8$). In fact, as done earlier we can easily use mathematical induction to show that $|S_i - L_j^2| > 0$ for al $j > i \geq 8$. Hence, there are no i and j (with $j > i \geq 8$) in which the equation (1.8) is satisfied.

- It only remains to consider the case of $i, j < 8$, and that can be done using the SageMath Software to search for the positive integers $i, j \leq 7$ in which equation (1.8) holds. Here, we obtain these values in the following set of solutions:

$$(n, m) \in \{(0,0), (1,0), (2,1), (2,2), (3,1), (3,2), (5,3)\}.$$

Thus, Theorem 2.2 is proved. □

3 Conclusion

In this paper, we mainly have shown the squares in the sequence of Tribonacci numbers, that are presenting squares of some Fibonacci numbers, are $T_0 = T_1 = 0 = F_0^2$, $T_2 = T_3 = 1 = F_1^2 = F_2^2$, $T_5 = 2^2 = F_3^2$. Moreover, we have also proved that the only square term in sequence of Tribonacci Lucas numbers, that is also a square of a number in the sequence of Lucas numbers, is $S_1 = 1 = L_1^2$. The former result answers the Pethő's problem positively (see, Pethő's Problem 1) in case of the squares in $\{T_n\}$ are squares of some Fibonacci numbers, and the latter one proves the Irmak's conjecture (see, Irmak's Conjecture 1) in case of the square in $\{S_n\}$ is a square of a Lucas number. As an auxiliary result that's used to prove these main results, we also determined all the Tribonacci numbers that are also presenting Fibonacci numbers, namely $T_0 = F_0 = 0$, $T_2 = F_2 = 1$, $T_7 = F_7 = 13$, $T_1 = F_0 = 0$, $T_2 = F_1 = 1$, $T_3 = F_1 = 1$, $T_3 = F_2 = 1$ and $T_4 = F_3 = 2$. As a generalization of this auxiliary result, we obtained all the Tribonacci Lucas numbers that are also presenting Lucas numbers, i.e. $S_1 = L_1 = 1$, $S_2 = L_2 = 3$, $S_0 = L_2 = 3$, $S_3 = L_4 = 7$ and $S_4 = L_5 = 11$.

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Author information

Hayder R. Hashim, Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Iraq.
E-mail: hayderr.almuswi@uokufa.edu.iq

Anwar N. Jasim, Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Iraq.
E-mail: anwern.jaseem@uokufa.edu.iq

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