DCC ON ESSENTIAL M-CYCLIC SUBMODULES AND R-CYCLIC IDEALS

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Abstract: In this paper, we investigate the properties of essential $\mathcal{M}$-Artinian modules and essential $\mathcal{R}$-Artinian rings. A left $\mathcal{R}$-module $\mathcal{M}$ is said to be essential $\mathcal{M}$-Artinian if every descending chain of $\mathcal{R}$-cyclic submodules of $\mathcal{M}$ terminates. We prove that if $\mathcal{M}$ be an e-$\mathcal{M}$-Artinian $\mathcal{R}$-module with $Z(\mathcal{M}) \subseteq Z(\mathcal{R})$, then $\mathcal{M}$ is divisible. Additionally, we show that for an essential $\mathcal{R}$-Artinian ring $\mathcal{R}$ with unity, every essential $\mathcal{R}$-cyclic prime ideal is maximal, and there exist only finitely many essential $\mathcal{R}$-cyclic maximal ideals.

1 Introduction

In 1927, Artin introduced the concept of the descending chain condition on commutative rings, which later came to be known as Artinian rings in his honour. In [4], Osofsky studied chain conditions on essential submodules and demonstrated that, for any infinite cardinal $\aleph$, an $\aleph$-chain condition on the essential submodules of a module $\mathcal{M}$ closely resembles the $\aleph$-chain condition on all submodules of $\mathcal{M}$. In [19], Gera et al. introduced chain conditions on $\mathcal{M}$-cyclic submodules and defined notions such as $\mathcal{M}$-Artinian, $\mathcal{M}$-Noetherian, iso-$\mathcal{M}$-Artinian, and iso-$\mathcal{M}$-Noetherian modules. They also established a version of the Hilbert Basis Theorem for the iso-$\mathcal{M}$-Artinian and iso-$\mathcal{M}$-Noetherian cases.

A submodule E of $\mathcal{M}$ is said to be essential if it has a non-trivial intersection with every non-trivial submodule of $\mathcal{M}$; that is, $E \cap L = 0$ implies $L = 0$. In [7], Armendariz introduced the concept of a min-E module, also known as essential Artinian (or e-Artinian), which satisfies the descending chain condition on its essential submodules. Equivalently, a module $\mathcal{M}$ is e-Artinian if every descending chain of essential submodules $\mathcal{N}_1 \supseteq_e \mathcal{N}_2 \supseteq_e \mathcal{N}_3 \supseteq_e \cdots$ eventually becomes stationary; that is, there exists $n \in \mathbb{N}$ such that $\mathcal{N}_n = \mathcal{N}_j$ for all $j \geq n$. For example, every simple module and every finitely generated semisimple module is e-Artinian. Armendariz [7]also introduced the notion of a min-E ring, or e-Artinian ring, which is defined as a ring that is e-Artinian when considered as an e-Artinian module over itself. Equivalently, a ring $\mathcal{R}$ is e-Artinian if every descending chain of essential left ideals $\mathcal{I}_1 \supseteq_e \mathcal{I}_2 \supseteq_e \mathcal{I}_3 \supseteq_e \cdots$ eventually terminates; that is, there exists $n \in \mathbb{N}$ such that $\mathcal{I}_n = \mathcal{I}_j$ for all $j \geq n$. For example, division rings, finite rings, and principal ideal rings are all e-Artinian rings. Armendariz observed that a ring $\mathcal{R}$ is a min-E ring if and only if $\mathcal{R}$, considered as a left module over itself, is Artinian. Moreover, the socle and homomorphic images of min-E rings are themselves min-E rings. In [10], Chakraborty et al. introduced the chain conditions on essential $\mathcal{M}$-cyclic submodules and essential $\mathcal{R}$-cyclic ideals on modules and rings, respectively. Motivated by these discussions, we introduce chain conditions on essential $\mathcal{M}$-cyclic submodules and essential $\mathcal{R}$-cyclic ideals. Based on these, we define the notions of essential $\mathcal{M}$-Artinian modules and essential $\mathcal{R}$-Artinian rings as natural generalizations of e-Artinian modules and e-Artinian rings, respectively, as studied by Osofsky

[4] and Armendariz [7]. We also establish several important results concerning the structure and properties of these modules and rings This paper itself is split into three sections first section is devoted to the introduction, which includes motivation and preliminary ideas related to the article. In the second section, we discuss several fundamental properties of e-$\mathcal{M}$-Artinian modules, where we generalized a couple of results of Artinian and e-Artinian modules, and we succeeded in establishing some new statements. We proved that if every factor module of $\mathcal{M}$ is finitely embedded, then $\mathcal{M}$ is an e-$\mathcal{M}$-Artinian [Theorem 2.2].We also proved that if $\mathcal{M}$ be an e-$\mathcal{M}$-Artinian $\mathcal{R}$-module with $Z(\mathcal{M}) \subseteq Z(\mathcal{R})$, then $\mathcal{M}$ is divisible [Proposition 2.5].

In the third section, we discuss several fundamental properties of e-$\mathcal{R}$-Artinian rings, where we extend a few results of Artinian and e-Artinian rings and establish several new statements. We proved that for an ideal $\mathcal{I}$ of a ring $\mathcal{R}$, if $\mathcal{I}$ and $\mathcal{R}/\mathcal{I}$ are e-$\mathcal{I}$-Artinian and e-$(\mathcal{R}/\mathcal{I})$-Artinian, respectively, then $\mathcal{R}$ is also an e-$\mathcal{R}$-Artinian [Theorem 3.2]. In continuation, we showed that for an e-$\mathcal{R}$-Artinian ring $\mathcal{R}$, every essential $\mathcal{R}$-cyclic prime ideal is maximal, and there are only finitely many essential $\mathcal{R}$-cyclic maximal ideal [Theorem 3.8]. Apart from this, we proved that if $\mathcal{R}$ is a local e-$\mathcal{R}$-Artinian principal ideal ring, then $\mathcal{R}/\mathcal{R}'$ is self injective for every essential principal ideal $\mathcal{R}'$ of $\mathcal{R}$ [Theorem 3.10].

Throughout this paper, we consider $\mathcal{R}$ to be an associative ring with a unity and, $\mathcal{M}$ a unitary left $\mathcal{R}$-module. We will refer to all the basic terminologies and notations from [8], [6], [14], and [17]. In this paper, we consistently denote $J(\mathcal{R})$, $Soc(\mathcal{M})$, $Z(_{\mathcal{R}}\mathcal{R})$, and $E(\mathcal{R})$ as Jacobson radical of a ring, socle of a module, singular ideal of ring, and the injective envelope of ring respectively.

2   Essential M-Artinian modules:

An $\mathcal{R}$-module $\mathcal{M}$ is said to be an essential-$\mathcal{M}$-Artinian (in short, e-$\mathcal{M}$-Artinian) if any descending chain $h_1(\mathcal{M}) \supseteq_e h_2(\mathcal{M}) \supseteq_e h_3(\mathcal{M}) \supseteq_e \cdots \supseteq_e h_n(\mathcal{M}) \supseteq_e \cdots$ of essential $\mathcal{M}$-cyclic submodules of $\mathcal{M}$ is stationary i.e., there exists $r \in \mathbb{N}$, such that $h_r(\mathcal{M}) = h_{r+1}(\mathcal{M})$ where $h_i \in End(\mathcal{M})$ for all i. Dually, A module $\mathcal{M}$ is said to be an essential-$\mathcal{M}$-Noetherian (in short, e-$\mathcal{M}$-Noetherian) if any ascending chain $h_1(\mathcal{M}) \subseteq_e h_2(\mathcal{M}) \subseteq_e h_3(\mathcal{M}) \subseteq_e \cdots \subseteq_e h_n(\mathcal{M}) \subseteq_e \cdots$ of essential $\mathcal{M}$-cyclic submodules of $\mathcal{M}$ is stationary i.e., there exists $r \in \mathbb{N}$, such that $h_r(\mathcal{M}) = h_{r+1}(\mathcal{M})$ where $h_i \in End(\mathcal{M})$ for all i. Clearly, every Artinian, simple, semisimple, uniform module is e-$\mathcal{M}$-Artinian. An example of an e-$\mathcal{M}$-Artinian module will be $\mathbb{Z}_{81}$. Consider the terminating chain $\mathbb{Z}_{81} \supseteq_e 3\mathbb{Z}_{81} \supseteq_e 9\mathbb{Z}_{81} \supseteq_e 27\mathbb{Z}_{81}$; here every submodule in this descending chain is essential and hence $\mathbb{Z}_{81}$ is an e-$\mathbb{Z}_{81}$-Artinian. Clearly, if a module is uniform and Artinian, then it is an e-$\mathcal{M}$-Artinian.

We begin with Theorem 1 as generalizations of the results of [6], which will be used in the proof of ensuing results.

Theorem 2.1. *Let $\mathcal{M}$ be a semi-simple $\mathcal{R}$-module. Then the following statements are equivalent:*

 (i) $\mathcal{M}$ is the direct sum of a finite family of simple $\mathcal{M}$-cyclic submodules;

 (ii) $\mathcal{M}$ is $\mathcal{M}$-Noetherian;

(iii) $\mathcal{M}$ is $\mathcal{M}$-Artinian;

(iv) $\mathcal{M}$ is finitely generated.

Proof. Proof is similar to [6, Proposition 3.3]. □

Theorem 2.2. *Let $\mathcal{M}$ be an $\mathcal{R}$-module. If every factor module of $\mathcal{M}$ is finitely embedded, then $\mathcal{M}$ is an e-$\mathcal{M}$-Artinian.*

Proof. Suppose that every factor module of $\mathcal{M}$ is finitely embedded, and consider a descending chain $h_1(\mathcal{M}) \supseteq_e h_2(\mathcal{M}) \supseteq_e h_3(\mathcal{M}) \supseteq_e \cdots$ (*) of essential $\mathcal{M}$-cyclic submodules of $\mathcal{M}$, where $h_i \in End(\mathcal{M})$ for all $i \in \mathbb{N}$. Put $h(\mathcal{M}) = \bigcap_{n=1}^{\infty} h_n(\mathcal{M})$, then $h(\mathcal{M})$ is an essential $\mathcal{M}$-cyclic

submodule of $\mathcal{M}$. If the above chain does not terminate, then the module $\mathcal{M}/h(\mathcal{M})$ has an inverse system $h_1(\mathcal{M})/h(\mathcal{M}), h_2(\mathcal{M})/h(\mathcal{M}), h_3(\mathcal{M})/h(\mathcal{M})\cdots$, of non-zero submodules which is not bounded below by a non-zero essential $\mathcal{M}$-cyclic submodule of $\mathcal{M}$, which is a contradiction [6, Proposition 3.19]. Thus, the chain (*) must get terminated, and hence, $\mathcal{M}$ is an e-$\mathcal{M}$-Artinian.   □

Proposition 2.3. *Let $\mathcal{M}$ be a simple $\mathcal{R}$-module, and $\mathcal{L}$ be an e-$\mathcal{L}$-Artinian submodule of $\mathcal{M}$, then $\mathcal{M} \oplus \mathcal{L}$ is an e-($\mathcal{M} \oplus \mathcal{L}$)-Artinian.*

Proof. Proof is similar to [18, Theorem 3.1].   □

Recall that an $\mathcal{R}$-module $\mathcal{M}$ is nonsingular if $Z(\mathcal{M})= \{m \in \mathcal{M}| Ann_R(m)$ is an essential right ideal of $\mathcal{R}\} = \{0\}$. i.e., $\mathcal{M}$ has no non-zero singular elements.

Proposition 2.4. *Every epi-$\mathcal{M}$-Noetherian uniform nonsingular module is iso-$\mathcal{M}$-Noetherian.*

Proof. Let $\mathcal{M}$ be an epi-$\mathcal{M}$-Noetherian uniform nonsingular module. Consider an ascending chain $f_1(\mathcal{M}) \subseteq_e f_2(\mathcal{M}) \subseteq e \cdots$ of $\mathcal{M}$-cyclic submodules of $\mathcal{M}$, then exists a positive integer k such that $f_k(\mathcal{M})$ is a homomorphic image of $f_{k+1}(\mathcal{M})$. Let $\phi : f_{k+1}(\mathcal{M}) \longrightarrow f_k(\mathcal{M})$ be a surjective homomorphism. Then $f_{k+1}(\mathcal{M})/ker(\phi) \cong f_k(\mathcal{M})$. Since $f_{k+1}(\mathcal{M})$ is uniform and $f_{k+1}(\mathcal{M})/ker(\phi)$ is nonsingular, we have that $Ker(\phi) = 0$. Thus, ϕ is an isomorphism. This implies that $\mathcal{M}$ is iso-$\mathcal{M}$-Noetherian.   □

Recall that an $\mathcal{R}$-module $\mathcal{M}$ is divisible if for any nonzero zero divisor r in $\mathcal{R}$, $\mathcal{M} = r\mathcal{M}$, and that $Z(\mathcal{M})$ will denote the set of zero divisors of $\mathcal{M}$.

Proposition 2.5. *If $\mathcal{M}$ be an e-$\mathcal{M}$-Artinian $\mathcal{R}$-module with $Z(\mathcal{M}) \subseteq Z(\mathcal{R})$, then $\mathcal{M}$ is divisible.*

Proof. Let $r \in \mathcal{R} - Z(\mathcal{R}) \subseteq \mathcal{R} - Z(\mathcal{M})$. Now consider a descending chain $rf(\mathcal{M}) \supseteq_e r^2 f(\mathcal{M}) \supseteq_e \cdots$ of essential $\mathcal{M}$-cyclic submodules of $\mathcal{M}$. Since $\mathcal{M}$ is e-$\mathcal{M}$-Artinian, then it must be stablized i.e. $r^n f(\mathcal{M}) = r^{n+1} f(\mathcal{M})$ for some positive integer n. Let $a \in \mathcal{M}$, then $r^n a = r^{n+1}b$ for some $b \in \mathcal{M}$. Hence, $r^n(a - rb) = 0$, and since $r^n \in \mathcal{R} - Z(\mathcal{M})$, $a - rb = 0$ implies $a = rb$. Therefore, $\mathcal{M} = r\mathcal{M}$ for all $r \in \mathcal{R} - z(\mathcal{R})$, and so $\mathcal{M}$ is divisible.   □

Recall from [13] that a left $\mathcal{R}$-module $\mathcal{M}$ is called $\mathcal{M}$-compressible if, for each nonzero $\mathcal{M}$-cyclic submodule $f(\mathcal{M})$ of $\mathcal{M}$, there exists a monomorphism $\alpha : \mathcal{M} \longrightarrow f(\mathcal{M})$. Clearly, every essentially compressible module is essentially $\mathcal{M}$-compressible. We said to be an $\mathcal{R}$-module $\mathcal{M}$ is subisomorphic to an $\mathcal{R}$-module $\mathcal{M}$ if there exist $\mathcal{R}$-monomorphisms $\alpha : \mathcal{M} \longrightarrow \mathcal{M}'$ and $\beta : \mathcal{M}' \longrightarrow \mathcal{M}$, and in this case we call the modules $\mathcal{M}$ and $\mathcal{M}'$ subisomorphic.

Proposition 2.6. *The following statements are equivalent for a module $\mathcal{M}$:*

(i) $\mathcal{M}$ is essentially $\mathcal{M}$-compressible;

(ii) $\mathcal{M}$ is subisomorphic to an essentially $\mathcal{M}$-compressible module;

(iii) $\mathcal{M}$ contains compressible essential $\mathcal{M}$-cyclic submodule $f(\mathcal{M}')$ such that there exists a monomorphism $\phi : \mathcal{M} \longrightarrow f(\mathcal{M}')$;

(iv) There is an essentially monomorphism $\psi : \mathcal{M} \longrightarrow \mathcal{M}'$ for some essential $\mathcal{M}$-compressible module $\mathcal{M}$.

Proof. $(a) \implies (b)$ is clear.
$(b) \implies (c)$ Suppose that there exists an essentially $\mathcal{M}$-compressible module $\mathcal{M}'$ and monomorphism $\alpha : \mathcal{M} \longrightarrow \mathcal{M}'$, $f : \mathcal{M}' \longrightarrow \mathcal{M}$. Then $\mathcal{N} = f(\mathcal{M}')$. Then $f(\mathcal{M}')$ is an essentially compressible $\mathcal{M}$-cyclic submodule of $\mathcal{M}$, and $f\alpha : \mathcal{M} \longrightarrow f(\mathcal{M}')$ is a monomorphism.
$(c) \implies (a)$ Let $f_1(\mathcal{M})$ be any essential $\mathcal{M}$-cyclic submodule of $\mathcal{M}$. Then $f(\mathcal{M}') \cap f_1(\mathcal{M})$ is an essentially $f(\mathcal{M}')$-cyclic submodule of $f(\mathcal{M}')$ and so there is monomorphism $\theta : f(\mathcal{M}') \longrightarrow (f_1(\mathcal{M}) \cap f(\mathcal{M}))$. If $\psi : (f_1(\mathcal{M}) \cap f(\mathcal{M})) \longrightarrow f_1(\mathcal{M})$ is the inclusion mapping then $\psi\theta\phi : \mathcal{M} \longrightarrow f_1(\mathcal{M})$ is a monomorphism. It follows that $\mathcal{M}$ is essentially $\mathcal{M}$-compressible module.
$(a) \implies (d)$ it is clear.   □

Proposition 2.7. *Every direct sum of essentially $\mathcal{M}$-compressible modules is essentially $\mathcal{M}$-compressible.*

Proof. Let a module $\mathcal{M} = \oplus_{i \in \mathbb{N}} \mathcal{M}_i$ be a direct sum of essentially $\mathcal{M}$-compressible modules $\mathcal{M}_i (i \in I)$ for some (non-empty) index set I. Let $f(\mathcal{M}) = \oplus_{i \in \mathbb{N}}(f(\mathcal{M}) \cap \mathcal{M}_i)$ be an essential $\mathcal{M}$-cyclic submodule of $\mathcal{M}_i$. Then, for each $i \in I$, $f(\mathcal{M}) \cap \mathcal{M}_i$ is an essential $\mathcal{M}$-cyclic submodule of $\mathcal{M}_i$ and hence, there exists a monomorphism $\theta_i : \mathcal{M}_i \longrightarrow f(\mathcal{M}) \cap \mathcal{M}_i$. Clearly, the mapping $\theta = \sum \theta_i : \mathcal{M} \longrightarrow f(\mathcal{M})$ is a monomorphism. Hence $\mathcal{M}$ is essentially an $\mathcal{M}$-compressible module. $\qquad\square$

3 Essential $\mathcal{M}$-Artinian Rings:

We call a ring $\mathcal{R}$ an essential $\mathcal{R}$-Artinian ring (in short, e-$\mathcal{R}$-Artinian), if it is an e-$\mathcal{M}$-Artinian module, where $\mathcal{M} = {}_{\mathcal{R}}\mathcal{R}$, for example, every residue class of integers modulo n for $n \geq 0$ and semisimple rings. It is observed that $\mathcal{R}$-cyclic ideals are exactly the principal ideal, and verified that the ideal $\mathcal{I}$ of an e-$\mathcal{R}$-Artinian ring is essential $\mathcal{I}$-Artinian.

Lemma 3.1. *Let $\mathcal{R}$ be an essential $\mathcal{R}$-Artinian domain, then it is field.*

Proof. Let $\mathcal{R}$ be an essential $\mathcal{R}$-Artinian domain and $0 \neq a \in \mathcal{R}$. Then for descending chain of principal ideals: $< a > \supseteq < a^2 > \subseteq < a^3 > \supseteq \cdots$, there exists an index n such that, $< a^n > = < a^{n+1} > = \cdots$. Hence $< a^n > = < a^{n+1} >$ implies that $a^n = r a^{n+1}$ for some $r \in \mathcal{R}$, implies that $1 = ra$. Since $\mathcal{R}$ is a field. $\qquad\square$

Theorem 3.2. *Let $\mathcal{I}$ be an ideal of a ring $\mathcal{R}$. If $\mathcal{I}$ and $\mathcal{R}/\mathcal{I}$ are e-$\mathcal{I}$-Artinian and e-$(\mathcal{R}/\mathcal{I})$-Artinian respectively, then $\mathcal{R}$ will be an e-$\mathcal{R}$-Artinian.*

Proof. Let $\mathcal{R}_1 \supseteq_e \mathcal{R}_2 \supseteq_e \cdots \supseteq_e \mathcal{R}_n) \supseteq_e \mathcal{R}_{n+1} \supseteq_e \cdots$ be a descending chain of essential principal ideals of $\mathcal{R}$. Let $p : \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{I}$ be the natural homomorphism of $\mathcal{R}$ onto $\mathcal{R}/\mathcal{I}$. Then $p(\mathcal{R}_1) \supseteq_e p(\mathcal{R}_2) \supseteq_e \cdots \supseteq_e p(\mathcal{R}_n) \supseteq_e p(\mathcal{R}_{n+1}) \supseteq_e \cdots$ is a descending chain of essential principal ideals in $\mathcal{R}/\mathcal{I}$. Since $\mathcal{R}/\mathcal{I}$ is e-$(\mathcal{R}/\mathcal{I})$-Artinian, there exists a positive integer n such that $p(\mathcal{R}_n) = p(\mathcal{R}_{n+1})$ for all $i \geq 1$. Also, $\mathcal{R}_1 \cap \mathcal{I} \supseteq_e \mathcal{R}_2 \cap \mathcal{I} \supseteq_e \cdots \supseteq_e \mathcal{R}_n \cap \mathcal{I} \supseteq_e \mathcal{R}_{n+1} \cap \mathcal{I} \supseteq_e \cdots$ is a descending chain of essential principal ideals in $\mathcal{I}$. Since $\mathcal{I}$ is an e-$\mathcal{I}$-Artinian, there exists a natural number m such that $\mathcal{R}_m \cap \mathcal{I} = \mathcal{R}_{m+i} \cap \mathcal{I}$ for all $i \geq 1$. Let $r = max(m,n)$, then $p(\mathcal{R}_r) = p(\mathcal{R}_{r+i})$ and $\mathcal{R}_r \cap \mathcal{I} = \mathcal{R}_{r+i} \cap \mathcal{I}$ for all $i \geq 1$. Let $a \in \mathcal{R}_r$, then there exists an element $x \in \mathcal{R}_{r+i}$ such that $p(a) = p(x)$ i.e., $a + \mathcal{I} = x + \mathcal{I}$, hence $a - x \in \mathcal{I}$ and als $a - hx \in \mathcal{R}_r$. This shows that $a - x \in \mathcal{R}_r \cap \mathcal{I} = \mathcal{R}_{r+i} \cap \mathcal{I}$ for all $i \geq 1$. Again, as $x \in \mathcal{R}_{r+i}$. Then $a \in \mathcal{R}_{r+i}$. Consequently, $\mathcal{R}_r = \mathcal{R}_{r+i}$ for all $i \geq 1$, this shows that $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian. $\qquad\square$

Proposition 3.3. *Let $\mathcal{R} = \prod_{i=1}^{n} \mathcal{R}_i$. Then $\mathcal{R}$ is left essentially iso $\mathcal{R}$-Artinian ring if and only if each $\mathcal{R}_i$ is a left essentially iso $\mathcal{R}_i$-Artinian.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \cdots, \mathcal{R}_n$ be left essentially iso $\mathcal{R}_i$-Artinian rings and $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \cdots \times \mathcal{R}_n$. Let $\mathcal{I}_1 \supseteq_e \mathcal{I}_2 \supseteq_e \mathcal{I}_3 \supseteq_e \cdots \supseteq_e \mathcal{I}_n \supseteq_e \cdots$ be a descending chain of essential left principal ideals of $\mathcal{R}$. For each $j \in \mathbb{N}$, $\mathcal{I}_j$ is of the form $\mathcal{I}_j = \mathcal{B}_{j1} \times \mathcal{B}_{j2} \times \cdots \times \mathcal{B}_{jn}$, where each $\mathcal{I}_{jk}$ is an essential left principal ideal of $\mathcal{R}_j$ for each $k \in \{1,2,3,\cdots,n\}$, $\mathcal{R}_k$ is left essentially iso $\mathcal{R}_k$-Artinian, there exists $m \in \mathbb{N}$ such that for each $j \geq m$, there is a iso-morphism, say $\psi_{jk} : \mathcal{I}_{jk} \longrightarrow \mathcal{I}_{(j+1)k}$ for each $j \geq m$, we define a map $\psi_j : \mathcal{I}_j \longrightarrow \mathcal{I}_{j+1}$ by $\psi_j(i_{j1}, i_{j2}, i_{j3}, \cdots, i_{jn}) = (\psi_{j1}(i_{j1}), \psi_{j2}(i_{j2}), \cdots, \psi_{jn}(i_{jn}))$, for every $i_{j1}, i_{j2}, \cdots, i_{jn}) \in \mathcal{I}_j$. Since each ψ_{jk} is an isomorphism, ψ_j is an isomorphism. Hence, $\mathcal{R}$ is left essentially iso $\mathcal{R}$-Artinian. Reverse part is obvious. $\qquad\square$

Recall that for any element a of ring $\mathcal{R}$, we get $\mathbf{l(a)} = \{\mathbf{r} \in \mathcal{R} : \mathbf{ra} = \mathbf{0}\}$, i.e., $\mathbf{l(a)}$ is the left annihilator of a in $\mathcal{R}$.

Lemma 3.4. *Let $\mathcal{R}$ be a ring which has many essential left principal ideals, which satisfies the ascending chain condition on left principal ideals of the form $l(a)$, where $a \in \mathcal{R}$. Then $l(b)$ is nilpotent, where $b \in \mathcal{R}$.*

Proof. Suppose that $l(b)$ is not nilpotent. By hypothesis, the ascending chain $l(b) \subseteq l(b^2) \subseteq l(b^3) \subseteq \cdots$ terminates. Let k be a positive integer such that $l(b^k) = l(b^{k+1}) = \cdots$. Let $0 \neq c = b^k$ and $l(c) = l(c^2)$. If $x \in cR \cap l(c)$, then $x = cs$ for some $s \in R$ and $0 = cx = c^2 s$, so that $x = cs = 0$. Hence, $Rc \cap l(c) = 0$. This implies that $l(c)$ is not essential, so that Rc is essential and hence $l(c) = 0$. Therefore, $l(b^k) = 0$. $\qquad \square$

Theorem 3.5. *Let a ring $\mathcal{R}$ be an e-$\mathcal{R}$-Artinian, then every finitely generated $\mathcal{R}$-module $\mathcal{M}$ is an e-$\mathcal{M}$-Artinian.*

Proof. Consider $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian ring and $\mathcal{M}$ is a finitely generated $\mathcal{R}$-module. Let $\mathcal{F}$ be any free module, so there exists a surjective homomorphism $\psi : \mathcal{F} \longrightarrow \mathcal{M}$. Because $\mathcal{M}$ is finitely generated so, $\mathcal{F} \cong \bigoplus_{i=1}^{n} \mathcal{R}$ where $n \in \mathbb{N}$. Consider $\phi : \bigoplus_{i=1}^{n} \mathcal{R} \longrightarrow \mathcal{M}$ is an onto $\mathcal{R}$-homomorphism. Clearly, $\bigoplus_{i=1}^{n} \mathcal{R}$ is an e-$\mathcal{R}$-Artinian module. Hence we have a homomorphic image of ϕ, i.e., $\mathcal{M}$ is also an e-$\mathcal{M}$-Artinian. $\qquad \square$

Recall that Let $\mathcal{R}$ be a ring, an element $r \in \mathcal{R}$ is called a regular element if it is not a zero divisor.

Proposition 3.6. *Let $\mathcal{R}$ be a left iso-e-$\mathcal{R}$-Artinian ring. If every non-zero left ideal of $\mathcal{R}$ contains a regular element, $\mathcal{R}$ is left e-$\mathcal{R}$-Noetherian.*

Proof. Let us consider $\mathcal{I}$ is a left ideal of $\mathcal{R}$ which is not finitely generated, let $x \in \mathcal{I}$ be a left regular element in $\mathcal{R}$. Then $x\mathcal{R}$ is isomorphic to $\mathcal{R}$ and contains the left ideal $x\mathcal{I}$, which is isomorphic to $\mathcal{I}$ as a left $\mathcal{R}$-module. Thus, we construct a descending chain $\mathcal{R} \supseteq_e x\mathcal{I} \supseteq_e x^2\mathcal{R} \supseteq_e x^3\mathcal{I} \supseteq_e \cdots$, where $x^n\mathcal{I} \cong \mathcal{I}$ is not finitely generated and $x^n\mathcal{R} \cong \mathcal{R}$. This gives a contradiction. Hence, $\mathcal{R}$ is e-$\mathcal{R}$-Northerian. $\qquad \square$

Recall that let $\mathcal{R}$ be a commutative ring and $\mathcal{A}$ be any subset of $\mathcal{R}$. Then the annihilator of $\mathcal{A}$ is denoted and defined as, $ann(\mathcal{A}) = \{r \in R | ra = 0 \text{ for all } a \in \mathcal{R}\}$. A ring R is called self injective if it is injective as module over itself. In the following lemma, we relate e-R-Artinian ring with the ascending chain condition on essential $\mathcal{R}$-cyclic annihilators and self-injective rings.

Lemma 3.7. *For a commutative ring $\mathcal{R}$, we have:*

 (i) *If $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian, then it holds ascending chain condition on essential principal annihilators.*

 (ii) *For a self injective ring, the converse of part (1) holds, i.e., if $\mathcal{R}$ satisfies the ascending chain condition on essential principal annihilators, then $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian.*

Proof. (i) Consider the ascending chain of essential principal ideals which are annihilators of $\mathcal{R}$, i.e., $\mathcal{R}_1 \subseteq_e \mathcal{R}_2 \subseteq_e \cdots$. If we take the annihilator on the above chain, we obtain descending chain i.e., $ann(\mathcal{R}_1) \supseteq_e ann(\mathcal{R}_2) \supseteq_e \cdots$. Because $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian, we have an index r, such that $ann(\mathcal{R}_i) = ann(\mathcal{R}_r)$ for all $i \geq r$. Again, considering the annihilator of these annihilators ideals, we obtain $\mathcal{R}_i = \mathcal{R}_r$ for all $i \geq r$, hence the result.

 (ii) Assume that $\mathcal{R}$ is self injective ring that satisfies the ascending chain condition on essential principal annihilators. By [11, Theorem 2], the ring $\mathcal{R}$ is Artinian and hence e-$\mathcal{R}$-Artinian. $\qquad \square$

Let $\mathcal{R}$ be any ring with unity, an ideal $\mathcal{R}_1$ is called the principal maximal ideal of $\mathcal{R}$, if $\mathcal{R}_1 \neq \mathcal{R}$ and for any ideal $\mathcal{R}_2 \subseteq \mathcal{R}_1$, then either $\mathcal{R}_2 = \mathcal{R}_1$ or $\mathcal{R}_2 = \mathcal{R}$. An ideal $\mathcal{R}_1$ is said to be principal prime ideal in $\mathcal{R}$, if $\mathcal{R}_2$ and $\mathcal{R}_3$ are principal ideals of $\mathcal{R}$ such that $\mathcal{R}_2\mathcal{R}_3 \subseteq \mathcal{R}_1$, then either $\mathcal{R}_2 \subseteq \mathcal{R}_1$ or $\mathcal{R}_3 \subseteq \mathcal{R}_1$. The Jacobson radical of $\mathcal{R}$ is defined as the intersection of all maximal left ideals of $\mathcal{R}$ which is denoted by $J(\mathcal{R})$. A ring is said to be left semi simple if it is semi simple as a left module over itself. In the following results, we find the relationship between e-R-Artinian ring with their essential principal prime ideal, essential principal maximal ideal and Jacobson radical.

Theorem 3.8. *Let $\mathcal{R}$ be an e-$\mathcal{R}$-Artinian ring with unity. Then, every essential principal prime ideal is maximal. Also, there are only finitely many essential principal maximal ideals.*

Proof. Let us consider an essential principal prime ideal $\mathcal{R}'$ of $\mathcal{R}$. Let $0 \neq r \notin \mathcal{R}'$ and $r \in \mathcal{R}$, then the descending chain of essential principal ideals $\mathcal{R}r \supseteq_e \mathcal{R}r^2 \supseteq_e \cdots \supseteq_e \mathcal{R}r^n \supseteq_e \mathcal{R}r^{n+1} \supseteq_e \cdots$ of $\mathcal{R}$, then there exists $n \in \mathbb{N}$, such that $\mathcal{R}r^n = \mathcal{R}r^{n+1}$. Then $r^n = r'r^{n+1}$ for some $r' \in \mathcal{R}$. So, $r^n(1 - r'r) = 0$. But $r^n \notin \mathcal{R}'$, hence $1 - r'r \in \mathcal{R}'$, which implies that $\mathcal{R} = \mathcal{R}' + \mathcal{R}r$. Thus, $\mathcal{R}_1$ is essential principal maximal ideal. For the second statement, consider $\mathcal{R}$ to be an e-$\mathcal{R}$-Artinian ring and Δ to be the set of all essential principal ideals which are the intersection of a finite number of essential principal maximal ideals of $\mathcal{R}$. Then Δ is non empty (as $\mathcal{R}$ itself is a member of Δ) and so has a minimal member, say $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n$ where $\mathcal{R}_i$ are essential principal maximals ideal of $\mathcal{R}$. Let $\mathcal{R}'$ be an principal maximal ideal of $\mathcal{R}$, then $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n \cap \mathcal{R}' \subseteq \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n$. So that $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n \cap \mathcal{R}' = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n$ and $\mathcal{R}' \supseteq \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \cdots \cap \mathcal{R}_n \supseteq \mathcal{R}_1 \mathcal{R}_2 \cdots \mathcal{R}_n$. Since $\mathcal{R}_i$ are essential principal prime, it follows that $\mathcal{R}' \supseteq \mathcal{R}_i$ for some i. Hence $\mathcal{R}' = \mathcal{R}_i$ (all principal prime are maximal). Therefore $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \cdots, \mathcal{R}_n$ are all the essential principal maximal ideals of $\mathcal{R}$. □

Proposition 3.9. *Let $\mathcal{R}$ be an e-$\mathcal{R}$-Artinian ring and $J(\mathcal{R})$ be the Jacobson radical of $\mathcal{R}$. Then $\mathcal{R}/J(\mathcal{R})$ is a semi simple ring.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n$ are maximal principal ideals of $\mathcal{R}$. By [6, Proposition 3.22], $J(\mathcal{R}) = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_n$. We define a mapping $\phi : \mathcal{R}/J(\mathcal{R}) \longrightarrow (\mathcal{R}/\mathcal{R}_1) \oplus \cdots \oplus (\mathcal{R}/\mathcal{R}_n)$ by $\phi(r + J(\mathcal{R}) = (r + \mathcal{R}_1, \cdots, r + \mathcal{R}_n)$ for all $r \in \mathcal{R}$, this is a well defined monomorphism. Now $(\mathcal{R}/\mathcal{R}_1) \oplus \cdots \oplus (\mathcal{R}/\mathcal{R}_n)$ is a semi simple ring. As $\mathcal{R}/J(\mathcal{R})$ is isomorphic to some ideal of the semi simple ring $(\mathcal{R}/\mathcal{R}_1) \oplus \cdots \oplus (\mathcal{R}/\mathcal{R}_n)$. Thus $\mathcal{R}/J(\mathcal{R})$ is semi simple ring. □

A ring $\mathcal{R}$ with unity is called a local ring if the set of all non-units in $\mathcal{R}$ is an ideal. Finally, we prove that, quotient of e-$\mathcal{R}$-Artinian ring is self injective.

Theorem 3.10. *Let $\mathcal{R}$ be a local e-$\mathcal{R}$-Artinian principal ideal ring. Then $\mathcal{R}/\mathcal{R}'$ is self injective, for every essential principal ideal $\mathcal{R}'$ of R.*

Proof. We first show that the ring $\mathcal{R}$ is self injective. Consider

$$0 \longrightarrow \mathcal{R}' \xrightarrow{inc} \mathcal{R}$$

$$f \downarrow \quad \swarrow \phi$$

$$\mathcal{R}$$

where $\mathcal{R}'$ is an essential principal ideal of $\mathcal{R}$ and $f : \mathcal{R}' \longrightarrow \mathcal{R}$ is an $\mathcal{R}$-homomorphism. By [6, Lemma 6.3], there is an element $p \in \mathcal{R}$ for which the non-zero ideals of $\mathcal{R}$ are $\mathcal{R}, \mathcal{R}p, \mathcal{R}p^2, \cdots$ and $\bigcap_{n=1}^{\infty} \mathcal{R}p^n = 0$. But $\mathcal{R}$ is an e-$\mathcal{R}$-Artinian. There is an integer k for which $p^k = 0$ but $p^{(k-1)} \neq 0$. Then the ideals of $\mathcal{R}$ are $\mathcal{R}, \mathcal{R}p, \mathcal{R}p^2, \cdots, \mathcal{R}p^k = 0$ and all these are distinct. We take $\mathcal{A} = \mathcal{R}p^t$, where $t \geq k$. Assume that $\mathcal{R}p^m$ is the smallest ideal of $\mathcal{R}$ to which $f(p^t)$ belongs. Then we can write $f(p^t) = up^m$ for some unit $u \in \mathcal{R}$. Now $p^{k-t}h(p^t) = 0$, so that $p^{k-t+m} = 0$ for all $m \geq t$, therefore the above diagram is commutes and hence $\mathcal{R}$ is self-injective. Now consider an essential principal ideal $\mathcal{R}'$ of $\mathcal{R}$. If $\mathcal{R}' = \mathcal{R}$, then the ring $\mathcal{R}/\mathcal{R}$, is self-injective. If $\mathcal{R}' \neq \mathcal{R}$, then $\mathcal{R}/\mathcal{R}'$ is also a local e-$(\mathcal{R}/\mathcal{R}')$-Artinian principal ideal ring. Thus, $\mathcal{R}/\mathcal{R}'$ is self-injective. □

Lemma 3.11. *Let $\mathcal{R}$ be a serial ring with Jacobson radical $\mathcal{J}$ if $n\mathcal{J}^n = 0$, then $\mathcal{R}$ is a $\mathcal{R}$-Noetherian, if for some n, $\mathcal{J}^n = 0$, $\mathcal{R}$-Artinian.*

Proof. Now $\mathcal{R} = e_1\mathcal{R} \oplus e_2\mathcal{R} \oplus \cdots \oplus e_k\mathcal{R}$ for some orthogonal indecomposable idempotents e_i. Consider $x = 0$ in e_iR. As $\cap e_i\mathcal{J}^n = 0$, for some n $x \notin e_i\mathcal{J}^n/e_i\mathcal{J}^{n+1}$. Then $x\mathcal{R} = e_i\mathcal{J}^n$. This immediately gives that $e_i\mathcal{R}$ is left Noetherian. Hence $\mathcal{R}_\mathcal{R}$ is $\mathcal{R}$-Noetherian. □

Recall that an ideal $\mathcal{I}$ of ring $\mathcal{R}$ is said to be a nil if every element is nilpotent, while $\mathcal{I}$ is a nilpotent if every element is nilpotent and $\mathcal{I}^n = 0$ for some natural number n. An ideal $\mathcal{I}$ is considered as left T-nilpotent if for every sequence $\{x_n\}$ of elements in $\mathcal{I}$ there is a natural number n such that $x_n x_{n-1} \cdots x_3 x_2 x_1 = 0$.

Theorem 3.12. *For a principal ideal $\mathcal{R}'$ of an essential $\mathcal{R}$-Artinian ring $\mathcal{R}$, the following are equal:*

  (i) $\mathcal{R}'$ *is nil;*

 (ii) $\mathcal{R}'$ *is T-nilpotent;*

(iii) $\mathcal{R}'$ *is nilpotent.*

Proof. (1) $\implies$ (2) let $< a_1 >, < a_2 >, < a_3 >, \cdots$ be a sequence in principal ideal $f(\mathcal{R}')$. Consider the descencing chain $< a_1 > \supseteq_e < a_1, a_2 > \supseteq_e < a_1, a_2, a_3 > \supseteq_e \cdots$. By assumption, there is an integer number k such that $< a_1, a_2, \cdots, a_k > = < a_1, a_2, \cdots, a_k, a_{k+1} > = \cdots$. Let m be the smallest positive integer with $a_{k+1}^m = 0$ and nonzero element a_{k+1}^{m-1} belongs to the annihilator of $< a_1 >, f(a_2), f(a_3 \cdots, a_{k+1} >$. Hence, $a_1 a_2 \cdots a_k a_{k+1}^{m-1} = a_1 a_2) \cdots a_k a_{k+1} a_{k+1}^{m-2} = 0$. It follows that the nonzero element a_{k+1}^{m-2} belongs to the annihilator of $< a_1 a_2 \cdots a_k >$. This process can be repeated, so we must have $a_1 a_2 \cdots a_k a_{k+1} = 0$, as desired.
(2) $\implies$ (3) Consider the descending chain $\mathcal{R}' \supseteq_e \mathcal{R}'^2 \supseteq_e \cdots$. By supposition, $\mathcal{R}$ is essential $\mathcal{R}$-Artinian and so there is a natural number k, $\mathcal{R}'^k = \mathcal{R}'^{k+1} = \cdots$. This implies ann($\mathcal{R}'^k$)=ann($\mathcal{R}'^{K+1} = \cdots$). We claim $\mathcal{R}'^{2k} = 0$. If not, then for some $x \in \mathcal{R}'^k \neq 0$. Hence, $x_1 \mathcal{R}'^{2k} \neq 0$ and for some $x_2 \in \mathcal{R}'$, we have $x_1 x_2 \mathcal{R}'^k \neq 0$. It follows $x_1 x_2 \mathcal{R}'^{2k} \neq 0$. This process can be continued unabated, so we can contradict T-nilpotency of $\mathcal{R}'$. Thus, we must have $\mathcal{R}'^{2k} = 0$, then $\mathcal{R}'$ is T-nilpotent.
(3) $\implies$ (1) This is clear. $\square$

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