Palestine Journal of Mathematics

Vol 14(3)(2025) , 1295–1310

Generalized AF C*-Algebras

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 46L05, Secondary 46L06, 46L35.

Keywords and phrases: AF C^*-algebra, Separability, $A\mathbb{T}$-algebra, $A\mathbb{I}$-algebra, Real Rank, Nuclear C^*-algebra.

The authors would like to express our sincere gratitude to the reviewers and editor for their valuable feedback and insightful comments, which have greatly contributed to improving this paper.

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Abstract We present a new generalization of separable AF algebras, called Generalized AF algebras, simply known as GAF. We demonstrate that AF algebras belong to the class of GAF algebras, even though a GAF C^*-algebra is not always an AF algebra. We prove that GAF algebras are separable and provide the necessary and sufficient conditions for a GAF algebra to be an AF algebra. If $\mathscr{A}$ is a unital first type GAF, then $\text{gen}(M_n[\mathscr{A}]) \leq 2$, for all $n \geq l$ for some positive integer l. We demonstrate that all non-nuclear, projectionless (no proper non-trivial projections) C^*-algebras are not GAF. Moreover, we prove that the minimal tensor product of two separable GAF algebras is a GAF and that all $A\mathbb{I}$-algebras are thus members of the GAF class. We obtain sufficient conditions for a commutative C^*-algebra to be GAF. We also obtain sufficient and necessary conditions for the first type commutative GAF. We prove that if simple, unital, nuclear, $\mathcal{Z}$-stable GAF algebras $\mathscr{A}$ and $\mathscr{B}$ satisfy the UCT, then $\mathscr{A} \cong \mathscr{B}$ if and only if $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$. Finally, we show that there exists a GAF algebra $\mathscr{A}$ such that: $\mathscr{A}$ is simple, unital, nuclear, satisfies the UCT, has real rank and stable rank one, $\mathscr{A} \otimes \mathcal{Z}$ is an AI algebra, $\mathscr{A}$ is not $\mathcal{Z}$-stable, the Cuntz semigroup of $\mathscr{A}$ is not almost unperforated, there is a C^*-algebra $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are not isomorphic, but they have the same stable and real rank and satisfy $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$ and $\mathscr{A}$ is not tracially $\mathcal{Z}$-absorbing.

1 Introduction

Ola Bratteli introduced AF C^*-algebras in 1972, which was a revolutionary step in the field of C^*-algebras [3]. Later, George A. Elliot made a noteworthy breakthrough by fully classifying AF C^*-algebras using the K_0 functor [5, 6, 13]. On the basis of this foundation, Haixin Lin in 2000 introduced a significant class called Tracially AF C^*-algebras [16], which is an important generalization of AF algebras [13].

Continuing the pursuit of classifying C^*-algebras under the umbrella of the Elliott classification program, George A. Elliot, along with Qingzhaifan and Xiaochun Fang, introduced a novel category of C^*-algebras known as Generalized Tracially Approximated C^*-algebras [7]. This introduction was inspired by the rich framework provided by TAF and AF C^*-algebras. In this article, we present a new generalization of separable AF algebras, called Generalized AF algebras, simply known as GAF, which expands our understanding of C^*-algebras.

AF-algebras are simple in structure, easily tractable, and allow the use of combinatorial and algebraic techniques to study properties compared with general C^*-algebras. Moreover, they are classifiable using K-theoretic techniques. However, AF algebras represent a relatively small subclass of C^*-algebras. By generalizing AF algebras to GAF algebras, we aim to extend successful techniques and insights from AF algebras to a broader class of C*-algebras while preserving a tractable structure. GAF algebras often exhibit stability properties under various operations, such as the tensor product. These stability properties are beneficial for constructing new examples of C*-algebras and understanding their behavior under such operations. Moreover, by working with GAF algebras, we aim to extend the theoretical frameworks and results known for AF algebras to a broader class of C*-algebras. This extension helps in developing a more comprehensive understanding of the landscape of C*-algebras. Similar to AF algebras and TAF algebras, GAF algebras have applications in mathematical physics, particularly in the study of quantum statistical mechanics and non-commutative geometry.

This article is organised in the following way: Section 2 serves as a preliminary section where we establish and clarify frequently used notation and fundamental concepts. In Section 3, we present a new class of C^*-algebras, called GLF algebras, which are a generalization of locally finite-dimensional C^*-algebras. We prove that if $\mathscr{A}$ is separable GLF, then a sequence (a_n) in $\mathscr{A}_+$ and a sequence $(\mathscr{F}_n)$ of sub-C^*-algebras of $\mathscr{A}$ with $dim(\mathscr{F}_n) < \infty$ can be found such that:

(1) $\mathscr{F}_n \subset C^*\left(a_{n+1}\mathscr{F}_{n+1}a_{n+1}\right)$, for each $n \in \mathbb{N}$.

(2) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_n a_n)}$.

We also show that the minimal tensor product of two GLF algebras is GLF, and the inductive limit of GLF algebras is GLF. Additionally, we establish that locally AF algebras are GLF.

In Section 4, we define GAF algebras and prove that every GAF is separable. We establish that a GAF algebra is a GLF and a separable AF algebra is a GAF. Furthermore, we provide the necessary and sufficient conditions for a GAF algebra to be an AF algebra. We demonstrate that the minimal tensor product of two GAF algebras is a GAF, and the inductive limit of the first type GAF algebras is GAF. Furthermore, we prove that all $A\mathbb{I}$-algebras are GAF, and non-nuclear, projectionless C^*-algebras are not GAF. Hence, $C^*(\mathbb{F}_2)$ and $C^*_r(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the free group with two generators, are not GAF. We also demonstrate that if $\mathscr{A}$ is a GAF, then $(M_n[\mathscr{A}])$ is also GAF. If $\mathscr{A}$ is a unital first type GAF, then gen$(M_n[\mathscr{A}]) \leq 2$, for all $n \geq l$ for some positive integer l.

In Section 5, we prove the necessary and sufficient condition for a commutative C^*-algebra to be a first type GAF, and we establish sufficient conditions for a commutative C^*-algebra to be a GAF algebra.

In Section 6, we prove that if simple, unital, nuclear, $\mathcal{Z}$-stable GAF algebras $\mathscr{A}$ and $\mathscr{B}$ satisfy the UCT, then $\mathscr{A} \cong \mathscr{B}$ if and only if Ell$(\mathscr{A}) \cong$ Ell$(\mathscr{B})$. We construct a GAF algebra $\mathscr{A}$ such that:

(1) $\mathscr{A}$ is simple, unital, nuclear, satisfies the UCT and has real rank and stable rank one.

(2) $\mathscr{A} \otimes \mathcal{Z}$ is an AI algebra.

(3) $\mathscr{A}$ is not $\mathcal{Z}$-stable.

(4) The Cuntz semigroup of $\mathscr{A}$ is not almost unperforated.

(5) There is a C^*-algebra $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are not isomorphic, but they have the same stable and real rank and satisfy Ell$(\mathscr{A}) \cong$ Ell$(\mathscr{B})$.

(6) $\mathscr{A}$ is not tracially $\mathcal{Z}$-absorbing.

This C^*-algebra $\mathscr{A}$ is not a new one. It was introduced by Andrew S. Toms, based on the work of Villadsen; see [8, 22, 23, 24, 25]. We prove that a GAF algebra is not a TAF algebra in general.

2 Preliminary

Throughout this paper, we use the following notations:
Let $S \subset \mathscr{A}$. If $\mathscr{A}$ is non-unital, then $C^*(S)$ is a sub-C^*-algebra of $\mathscr{A}$ generated by S, and if $\mathscr{A}$ is unital, then $C^*(S)$ is a unital sub-C^*-algebra of A generated by the set S [5, 12, 14, 17, 18, 26].
If $a \in \mathscr{A}$ and $F \subset \mathscr{A}$, then $aFa = \{afa : f \in F\}$.
$\tilde{\mathscr{A}}$ unitization of non-unital C^*-algebra $\mathscr{A}$.
$\mathscr{A}_{sa}$ is the set of all self-adjoint elements in $\mathscr{A}$ [5, 12, 14].
$\mathscr{A}_+$ is the set of all positive elements in $\mathscr{A}$ [5, 12, 14, 17].
Real rank of a C^*-algebra $\mathscr{A}$ is denoted by $RR(\mathscr{A})$ [5, 14, 17].
$\mathbb{C}[x_1, x_2, \ldots, x_n]$ is a polynomial ring in n indeterminates over $\mathbb{C}$.
$\mathbb{Q}(i) = \{p + iq : p, q \in \mathbb{Q}\} \subset \mathbb{C}$.

$\mathbb{Q}(i)[x_1, x_2, \ldots, x_n]$ is a polynomial ring in n indeterminates over $\mathbb{Q}(i)$.
$\mathscr{A} \otimes \mathscr{B}$ is tensor product of $\mathscr{A}$ and $\mathscr{B}$ with spatial C^*-norm [5, 12, 14, 17].
Stable rank of a C^*-algebra $\mathscr{A}$ is denoted by $SR(\mathscr{A})$ [5, 14, 17].
For Topological Space $\mathscr{X}$, $dim(\mathscr{X})$ is the topological dimension of $\mathscr{X}$. For any positive elements a and b, $diag(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Next, we recall some definitions, propositions, lemmas, and corollaries that are useful in this article.

Definition 2.1. (See [3, 5, 13, 19]). C^*-algebra $\mathscr{A}$ is considered to be approximately finite-dimensional, or AF if it is the closure of an increasing union of finite-dimensional sub-algebras.

Definition 2.2. (See [20]). "If a C^*-algebra $\mathscr{A}$ is the inductive limit of a sequence of circle algebras, then it is called an $A\mathbb{T}$-algebra".

Definition 2.3. (See [20]). If C^*-algebra $\mathscr{A}$ is isomorphic to the inductive limit of the inductive sequence $C[0,1] \otimes \mathscr{F}_1 \xrightarrow{\phi_1} C[0,1] \otimes \mathscr{F}_2 \xrightarrow{\phi_2} C[0,1] \otimes \mathscr{F}_3 \xrightarrow{\phi_3} \ldots$, where $dim(\mathscr{F}_k) < \infty$, then it is called an $A\mathbb{I}$-algebra.

Corollary 2.4. *(See [20]). "AF-algebras are always $A\mathbb{I}$-algebras, and $A\mathbb{I}$-algebras are always $A\mathbb{T}$-algebras. Conversely,*

(i) An $A\mathbb{T}$-algebra $\mathscr{A}$ is an $A\mathbb{I}$-algebra if and only if $K_1(\mathscr{A}) = 0$, where $K_1(\mathscr{A})$ is the K_1-group of $\mathscr{A}$.

(ii) An $A\mathbb{I}$-algebra $\mathscr{A}$ is an AF-algebra if and only if $RR(\mathscr{A}) = 0$".

Lemma 2.5. *(See [3, 5, 13]). "For $\epsilon > 0$ and $r \in \mathbb{N}$, there is a positive $\delta = \delta(\epsilon, r)$ such that $\mathscr{F}$ and $\mathscr{B}$ are sub- C^*-algebras of unital C^*-algebra $\mathscr{A}$ with $dim(\mathscr{F}) \leq r$ and such that $\mathscr{F}$ has a system of matrix units $\{e_{ij}^{(k)}\}$ satisfying dist $\left(e_{ij}^{(k)}, \mathscr{B}\right) < \delta$, then there is a unitary $u \in C^*(\mathscr{F}, \mathscr{B})$ such that $u\mathscr{F}u^* \subset \mathscr{A}$, with $\|u - 1_{\mathscr{A}}\| < \epsilon$".*

Proposition 2.6. *(See [13, 14, 27]). The nuclear dimension of unital, separable C^*-algebra $\mathscr{A}$ is zero if and only if $\mathscr{A}$ is AF.*

Definition 2.7. (See [1, 8, 13]) Let $\mathscr{A}$ be a C^*-algebra and let $a, b \in \mathscr{A}_+$. Then a is Cuntz subequivalent to b if and only if for any $\epsilon > 0$, there is an element r such that $\|a - rbr^*\| < \epsilon$. In this case, we write $a \precsim b$. We have $a \sim b$ if and only if $a \precsim b$ and $b \precsim a$.

The Cuntz semigroup of $\mathscr{A}$ is $Cu(\mathscr{A}) = (\mathscr{A} \otimes \mathcal{K})_+/\sim$, where addition $+$ on $Cu(\mathscr{A})$ is defined by

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right].$$

A Cuntz semigroup S is almost unperforated if whenever $s, t \in S$ and $n \in \mathbb{N}$ satisfy $(n+1)s \leq nt$, then $s \leq t$.

Definition 2.8. (See [1, 8, 13]) Let $\mathscr{A}$ be a unital C^*-algebra. The Elliott invariant of $\mathscr{A}$, denoted by $Ell(\mathscr{A})$, consists of the quadruple $Ell(\mathscr{A}) = ((K_0(\mathscr{A}), [1_{\mathscr{A}}]), K_1(\mathscr{A}), T(\mathscr{A}), r_{\mathscr{A}})$, where $r_{\mathscr{A}} : K_0(\mathscr{A}) \times T(\mathscr{A}) \to \mathbb{R}$ is the pairing between K-theory and traces, defined as $r_{\mathscr{A}}([p], \tau) = \tau(p)$, for all projections $[p] \in K_0(\mathscr{A})$ and all $\tau \in T(\mathscr{A})$.

3 Genaralized Locally Finite Dimensional C^*-Algebras

In this section, we present a new class of C^*-algebras, called genaralized locally finite dimensional C^*-algebra, simply GLF algebras, which are a generalization of locally finite-dimensional C^*-algebras. We prove some properties of separable GLF and provide examples and counterexamples.

Definition 3.1. A C^*-algebra $\mathscr{A}$ is said to be genaralized locally finite dimensional C^*-algebra or GLF if, for every finite subset $\{x_1, x_2, \ldots, x_m\} \subset \mathscr{A}$ and any positive real number $\epsilon > 0$, we can find a sub-C^*-algebra $\mathscr{F}$ with $dim(\mathscr{F}) < \infty$ and an element a of $\mathscr{A}_+$ such that $dist\,(x_i, C^*\,(a\mathscr{F}a)) < \epsilon$, $i = 1, 2, \ldots, m$.

Next, we will provide examples of GLF algebras.

Example 3.2. A locally finite dimensional C^*-algebra(LF) $\mathscr{A}$ is an GLF algebra.

Proof. Assuming $\mathscr{A}$ is LF. Then, for any subset $\{x_1, x_2, \ldots, x_m\}$ of $\mathscr{A}$ and real number $\epsilon > 0$, we can find a finite-dimensional sub-C^*-algebra $\mathscr{F}$ of $\mathscr{A}$ such that $\text{dist}(x_i, \mathscr{F}) < \epsilon$, for $i = 1, 2, \ldots, m$. Take $a = 1_{\mathscr{F}}$, then we have $\text{dist}(x_i, C^*(a\mathscr{F}a)) < \epsilon$, for $i = 1, 2, \ldots, m$. □

Example 3.3. If $\mathscr{A} = C([0, 1] \cup [3, 4] \cup [5, 6])$, then $\mathscr{A}$ is an GLF algebra.

Proof. Let $X = [0, 1] \cup [3, 4] \cup [5, 6]$. Since $[0, 1]$, $[3, 4]$, and $[5, 6]$ are clopen sets in X, characteristic functions $P_1 = \chi_{[0,1]}$, $P_2 = \chi_{[3,4]}$, and $P_3 = \chi_{[5,6]}$ are continuous, hence they are mutually orthogonal projections in $\mathscr{A}$. Thus, $\mathscr{F} = span\{P_1, P_2, P_3\}$ is a sub-C^*-algebra of $\mathscr{A}$ and $dim(\mathscr{F}) < \infty$. Consider the positive element defined by $f(t) = \sqrt{t}$. Then, we must have $C^*(f\mathscr{F}f) = \mathscr{A}$. Therefore, $\mathscr{A}$ is an GLF algebra. □

Remark 3.4. Using the technique in Example 3.3, we can construct infinitely many non-isomorphic commutative GLF algebras. Proposition 3.14, Examples 3.2, and 3.3 shows that if $\mathscr{A}$ is an AF algebra (both in the separable and non-separable cases) and $\mathscr{B} = C([0, 1] \cup [3, 4] \cup [5, 6])$, then $\mathscr{A} \otimes \mathscr{B}$ is GLF. Thus, there are infinitely many non-commutative, non-isomorphic, infinite-dimensional GLF-algebras. This example also shows that there are GLF-algebras that are not locally finite-dimensional. The next examples show that there are separable non-$GLF - C^*$-algebras.

Example 3.5. If $X = (0, 1)$, then $\mathscr{A} = C_0\,(X)$ is separable and has no finite dimensional non-zero sub-C^*-algebra. Therefore, $\mathscr{A}$ is not GLF but separable. The same is true for any locally compact, connected, separable infinite metric space X.

This example shows that separable C^*-algebras are not always GLF.

Lemma 3.6. *If* $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_n a_n)}$ *is unital, then* $1_{\mathscr{A}} \in C^*(a_N\mathscr{F}_N a_N)$*, for some* $N \in \mathbb{N}$.

Proof. Since $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_n a_n)}$, we can find an $x \in C^*(a_N\mathscr{F}_N a_N)$ for some N such that $\|x - 1_{\mathscr{A}}\| < 1$. Since x is an invertible element and $C^*(a_N\mathscr{F}_N a_N)$ contains the local unit e, $e = 1_{\mathscr{A}}$. □

Next, we prove lemma 3.7, which is similar to lemma 2.5. and using this lemma, we prove Proposition 3.9.

Lemma 3.7. *Given a positive real number* $\epsilon > 0$ *and* $k \in \mathbb{N}$*, there is a positive real number* $\delta = \delta\,(\epsilon, k)$ *and positive element* b *in* $\mathscr{A}$ *so that* $\mathscr{F}$ *and* $\mathscr{B}$ *are sub-* C^**-algebras of unital* C^**-algebra* $\mathscr{A}$ *with* $dim\,(\mathscr{F}) \leq k$ *and such that* $\mathscr{F}$ *has a system of matrix units* $\{e_{ij}^{(r)}\}$ *satisfying* $dist\left(e_{ij}^{(r)}, C^*(b\mathscr{B}b)\right) < \delta$*, then there exists a unitary* $u \in C^*\,(\mathscr{F}, C^*(b\mathscr{B}b))$ *with* $\|u - 1_{\mathscr{A}}\| < \epsilon$ *so that* $u\mathscr{F}u^* \subset C^*(b\mathscr{B}b)$.

Proof. Replace $\mathscr{B}$ by $C^*(b\mathscr{B}b)$ in the Lemma 2.5. □

Remark 3.8. If $\mathscr{A}$ is non-unital and $\mathscr{F}$ and $\mathscr{B}$ are sub-C^*-algebras of $\mathscr{A}$ as described in lemma 3.7, then there is a unitary $u \in \tilde{\mathscr{A}}$ as in lemma 3.7

Proposition 3.9. *Let* $\mathscr{A}$ *be a separable* GLF*. Then, a sequence* (a_n) *in* $\mathscr{A}_+$ *and a sequence* $(\mathscr{F}_n)$ *of sub-*C^**-algebras of* $\mathscr{A}$ *with* dim$(\mathscr{F}_n) < \infty$ *can be found such that:*

(1) $\mathscr{F}_n \subset C^*\left(a_{n+1}\mathscr{F}_{n+1}a_{n+1}\right)$, *for each* $n \in \mathbb{N}$.

(2) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_n a_n)}$.

Proof. Suppose $\mathscr{A}$ is unital separable GLF. Let (ϵ_i) be a decreasing sequence of positive real numbers with $\epsilon_i \to 0$ and (x_i) be a dense subset of the unit ball in $\mathscr{A}$ with $x_0 = 0$. As $\mathscr{A}$ is GLF, we can find a finite-dimensional sub- C^*-algebra $\mathscr{F}_0$ and positive element a_0 such that $dist(x_0, C^*(a_0\mathscr{F}_0a_0)) < \epsilon_0$. Assume that a sub-$C^*$-algebra $\mathscr{F}_k$ with $dim(\mathscr{F}_k) < \infty$ and an element $a_k \in \mathscr{A}_+$ have been found so that $dist(x_i, C^*(a_k\mathscr{F}_ka_k)) < \epsilon_k$ and $\mathscr{F}_{i-1} \subset C^*(a_i\mathscr{F}_ia_i)$ for all $1 \le i \le k$. Let $\delta = \delta(\frac{\epsilon_{k+1}}{3}, dim(\mathscr{F}_k))$ as in lemma 3.7 and fix a set of matrix units $\{e^s_{i,j}\}$ for $\mathscr{F}_k$. Let $\epsilon = Min\{\delta, \frac{\epsilon_{k+1}}{3}\}$, then there is a sub-$C^*$-algebra $\mathscr{F}$ with $dim(\mathscr{F}) < \infty$ and an element $b \in \mathscr{A}_+$ such that $dist(e^s_{i,j}, C^*(b\mathscr{F}b)) < \epsilon$ and $dist(x_i, C^*(b\mathscr{F}b)) < \epsilon$, for all $1 \le i \le k+1$. Then, by lemma 3.7, there is a unitary $u \in C^*(\mathscr{F}_k, C^*(b\mathscr{F}b))$ such that $\|u - 1_{\mathscr{A}}\| < \epsilon$ and $u\mathscr{F}_ku^* \subset C^*(b\mathscr{F}b)$. Let $\mathscr{F}_{k+1} = u^*\mathscr{F}u$ and $a_{k+1} = u^*bu$, then $\mathscr{F}_k \subset C^*(a_{k+1}\mathscr{F}_{k+1}a_{k+1})$ and $dist(x_i, C^*(a_{k+1}\mathscr{F}_{k+1}a_{k+1})) < \epsilon_{k+1}$. Hence, by mathematical induction, we construct a sequence of positive elements (a_n) and a sequence of finite-dimensional sub-C^*-algebras $(\mathscr{F}_n)$ such that:

(1) $\mathscr{F}_n \subset C^*\left(a_{n+1}\mathscr{F}_{n+1}a_{n+1}\right)$, for all $n \in \mathbb{N}$.

(2) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_n a_n)}$.

Similarly, we can show the non-unital case. □

Proposition 3.10. *If* $\mathscr{A} = \overline{\bigcup_{k=1}^{\infty} C^*(a_k\mathscr{F}_ka_k)}$ *and* $C^*(a_k\mathscr{F}_ka_k) \subset C^*(a_{k+1}\mathscr{F}_{k+1}a_{k+1})$, *for all* $k \in \mathbb{N}$, *then* $\mathscr{A}$ *is* GLF.

Proof. Let $\epsilon > 0$ and $\{x_1, x_2, x_3, \dots, x_m\} \subset \mathscr{A}$. Since $\bigcup_{k=1}^{\infty} C^*(a_k\mathscr{F}_ka_k)$ dense in $\mathscr{A}$, for each x_i, there is N_i and $y_i \in C^*(a_{N_i}\mathscr{F}_{N_i}a_{N_i})$ such that $\|x_i - y_i\| < \epsilon$. Let $N = Max\{N_i, i = 1, 2, \dots, m\}$, then $dist(x_i, C^*(a_N\mathscr{F}_Na_N)) < \epsilon$, for $i = 1, 2, \dots, m$. □

Next, we will prove that the inductive limit of GLF is GLF, and the minimal tensor product of two GLF algebras is GLF. To prove the first result, we use proposition 3.11.

Proposition 3.11. *(See [19]). Every inductive sequence of* C^**-algebras* $\mathscr{A}_1 \xrightarrow{\phi_1} \mathscr{A}_2 \xrightarrow{\phi_2} \mathscr{A}_3 \xrightarrow{\phi_3} \dots$ *has an inductive limit* $(\mathscr{A}, \{\mu_n\}_{n=1}^{\infty})$ *and*
$\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} \mu_n(\mathscr{A}_n)}$.

Proposition 3.12. *The inductive limit of* GLF *algebras is* GLF.

Proof. Let $\mathscr{A}_1 \xrightarrow{\phi_1} \mathscr{A}_2 \xrightarrow{\phi_2} \mathscr{A}_3 \xrightarrow{\phi_3} \dots$, where each $\mathscr{A}_k$ are GLFalgebras, be the inductive sequence and $(\mathscr{A}, \{\mu_n\})$ be the inductive limit. Let $\epsilon > 0$ and $\{x_1, x_2, \dots, x_m\} \subset \mathscr{A}$ be given. Since $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} \mu_n(\mathscr{A}_n)}$, there are n_i and $a_i \in \mathscr{A}_{n_i}$ such that $\|x_i - \mu_{n_i}(a_i)\| < \frac{\epsilon}{2}$, $i = 1, 2, \dots m$. We can take $n_1 \le n_2 \le n_3 \le \dots \le n_m$, then we have $\phi_{n_mn_i}(a_i) \in \mathscr{A}_{n_m}$, for $i = 1, 2, \dots, n_m$. Since $\mathscr{A}_{n_m}$ is GLF, we can find an element b in $(\mathscr{A}_{n_m})_+$, a sub-C^*-algebra $\mathscr{E}$ of $\mathscr{A}_{n_m}$ with $dim(\mathscr{E}) < \infty$ and $f_i \in C^*(b\mathscr{E}b)$ such that $\|\phi_{n_mn_i}(a_i) - f_i\| < \frac{\epsilon}{2}$, for $i = 1, 2, \dots, m$. Let $\mathscr{F} = \mu_{n_m}(\mathscr{E})$ and $a = \mu_{n_m}(b)$ then $\mu_{n_m}(C^*(b\mathscr{E}b)) = C^*(a\mathscr{F}a)$. Since $\mu_n = \mu_{n+1} \circ \phi_n$, for each $n \in \mathbb{N}$ and $\phi_{m,n} = \phi_{m-1} \circ \phi_{m-2} \circ \dots \circ \phi_n$, $\mu_{n_i} = \mu_{n_m} \circ \phi_{n_mn_i}$. Therefore, $\|x_i - \mu_{n_m}(f_i)\| = \|x_i - \mu_{n_i}(a_i) + \mu_{n_i}(a_i) - \mu_{n_m}(f_i)\| \le \|x_i - \mu_{n_i}(a_i)\| + \|\mu_{n_m}(\phi_{n_mn_i}(a_i)) - \mu_{n_m}(f_i)\| \le \|x_i - \mu_{n_i}(a_i)\| + \|\mu_{n_m}\|\|\phi_{n_mn_i}(a_i) - f_i\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $dist(x_i, C^*(a\mathscr{F}a)) < \epsilon$, for all $i = 1, 2, \dots, m$. Hence, $\mathscr{A}$ is GLF. □

Remark 3.13. The inductive limit of separable GLF algebras is a separable GLF algebra.

Proposition 3.14. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are* GLF, *then* $\mathscr{A} \otimes \mathscr{B}$ *is* GLF.

Proof. Assume that $\mathscr{A}$ and $\mathscr{B}$ are GLF algebras. Let $\epsilon > 0$ and a finite subset $\{a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_m \otimes b_m\}$ of $\mathscr{A} \otimes \mathscr{B}$ be given. Note that $a_i \in \mathscr{A}$ and $b_i \in \mathscr{B}$ for all $i = 1, 2, \ldots, m$. Since $\mathscr{A}$ and $\mathscr{B}$ are GLF algebras, there are positive elements $a \in \mathscr{A}$, $b \in \mathscr{B}$ and finite-dimensional sub-C^*-algebras $\mathscr{F} \subset \mathscr{A}$ and $\mathscr{K} \subset \mathscr{B}$ such that $dist(a_i, C^*(a\mathscr{F}a)) < \epsilon$ and $dist(b_i, C^*(b\mathscr{K}b)) < \epsilon$. Let $\mathscr{C} = \mathscr{F} \otimes \mathscr{K}$ and $c = a \otimes b$, then $\mathscr{C}$ is a sub-C^*algebra of $\mathscr{A} \otimes \mathscr{B}$ with $dim(\mathscr{C}) < \infty$ and c is positive element, then $dist(a_i \otimes b_i, C^*(c\mathscr{C}c)) < \epsilon^2$, for $i = 1, 2, \ldots, m$. Therefore, $\mathscr{A} \otimes \mathscr{B}$ is GLF. □

Corollary 3.15. *Every $A\mathbb{I}$-algebra is GLF.*

Proof. Since $C([0,1]) = C^*(fFf)$, where $\mathscr{F} = \mathbb{C}.1, 1(t) = 1, f(t) = \sqrt{t}$ for all $t \in [0,1]$, $C([0,1])$ is GLF. Since $\mathscr{F}$ and $C([0,1])$ are separable GLF. Therefore, by propositions 3.12, 3.14 and the definition of $A\mathbb{I}$-algebra, every $A\mathbb{I}$-algebra is GLF. □

Corollary 3.16. *Let $\mathscr{A}$ be an $A\mathbb{T}$-algebra. If $K_1(\mathscr{A}) = 0$, then $\mathscr{A}$ is GLF.*

Proof. By corollary 2.4, $\mathscr{A}$ is $A\mathbb{I}$-algebra, and by corollary 3.15, $\mathscr{A}$ is GLF. □

4 Generalized AF C^*-Algebras

In this part, we propose a new category of C^*-algebras called Generalized $AF(GAF)$ algebras, which are a generalization of separable AF algebras. We will demonstrate that GAF is a proper subclass of GLF algebras. Furthermore, we establish that all separable AF algebras are indeed GAF, thereby confirming that GAF is a proper subclass of GLF. We proceed by proving several properties of GAF algebras and present examples of GAF algebras that are not AF. Additionally, we classify GAF algebras into two classes based on the number of building block-sub-algebras. This section aims to establish the foundational properties of GAF algebras and provide insight into their structure and relationship with other classes of C^*-algebras.

Definition 4.1. A C^*-algebra $\mathscr{A}$ is said to be generalized AF C^*-algebra or GAF if we can find a sequence (a_n) consisting of elements in $\mathscr{A}_+$ and another sequence $(\mathscr{F}_n)$ of finite-dimensional sub-C^*-algebras of $\mathscr{A}$ such that:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n \mathscr{F}_n a_n)$, for each $n \in \mathbb{N}$ and $\mathscr{F}_0$ is finite dimensional sub-C^*-algebras.

(2) $C^*(a_n \mathscr{F}_n a_n) \subset C^*(a_{n+1} \mathscr{F}_{n+1} a_{n+1})$, for $n \in \mathbb{N}$

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$.

Proposition 4.2. *If $\mathscr{A}$ is GAF, then $\tilde{\mathscr{A}}$ is also GAF.*

Proof. Since $\mathscr{A}$ is GAF, we can find a sequence (a_n) of elements in $\mathscr{A}_+$ and a sequence $(\mathscr{F}_n)$ of finite dimensional sub-C^*-algebra of $\mathscr{A}$ such that:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n \mathscr{F}_n a_n)$, for each $n \in \mathbb{N}$.

(2) $C^*(a_n \mathscr{F}_n a_n) \subset C^*(a_{n+1} \mathscr{F}_{n+1} a_{n+1})$, for $n \in \mathbb{N}$

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$.

Hence

(1) $\tilde{\mathscr{F}_{n-1}} \subset \tilde{C}^*\left(a_n \tilde{\mathscr{F}}_n a_n\right)$, for all $n \in \mathbb{N}$.

(2) $\tilde{C}^*\left(a_n \tilde{\mathscr{F}}_n a_n\right) \subset \tilde{C}^*\left(a_{n+1} \tilde{\mathscr{F}_{n+1}} a_{n+1}\right)$, for all $n \in \mathbb{N}$

(3) $\tilde{\mathscr{A}} = \overline{\bigcup_{n=1}^{\infty} \tilde{C}^*(a_n \tilde{\mathscr{F}}_n a_n)}$.

Therefore, $\tilde{\mathscr{A}}$ is GAF. □

Next, we will prove that every GAF algebra is separable. To do this, we first establish the following lemma.

Lemma 4.3. *Let $\mathscr{A}$ be a C^*-algebra. If $a \in \mathscr{A}_+$ and $\mathscr{F}$ is a finite-dimensional sub-C^*-algebra of $\mathscr{A}$, then $C^*(a\mathscr{F}a)$ is separable.*

Proof. Since $dim(\mathscr{F}) < \infty$, there exist projections $p_1, p_2, \ldots, p_k$, where $k = \dim(\mathscr{F})$, such that $\mathscr{F} = span\{p_1, p_2, \ldots, p_k\}$. Therefore, we must have, $C^*(a\mathscr{F}a) = C^*(ap_1a, ap_2a, \ldots, ap_ka)$. By the result "Every finitely generated C^*-algebra is separable", it follows that $C^*(a\mathscr{F}a)$ is separable. □

Proposition 4.4. *Every GAF algebra is separable.*

Proof. Suppose that $\mathscr{A}$ is GAF, then we can find a sequence (a_n) of positive elements in $\mathscr{A}$ and a sequence $(\mathscr{F}_n)$ of finite-dimensional sub-C^*-algebra of $\mathscr{A}$ such that:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n\mathscr{F}_na_n)$, for each $n \in \mathbb{N}$ and $\mathscr{F}_0$ is finite-dimensional sub-C^*-algebra.

(2) $C^*(a_n\mathscr{F}_na_n) \subset C^*(a_{n+1}\mathscr{F}_{n+1}a_{n+1})$, for $n \in \mathbb{N}$.

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n\mathscr{F}_na_n)}$.

By lemma 4.3, $C^*(a_n\mathscr{F}_na_n)$ is separable for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, a countable dense subset D_n of $C^*(a_n\mathscr{F}_na_n)$ exists. Let $D = \bigcup_{n=1}^{\infty} D_n$, then, $D \subset \mathscr{A}$ countable. Let $\epsilon > 0$ be a positive real number and $x \in \mathscr{A}$, then there is $y \in C^*(a_n\mathscr{F}_na_n)$ for some n, such that $\|x - y\| < \frac{\epsilon}{2}$. Since D_n is dense in $C^*(a_n\mathscr{F}_na_n)$, there is $d \in D_n$ such that $\|y - d\| < \frac{\epsilon}{2}$. This implies that $\|x - d\| = \|x - y + y - d\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ and x are arbitrary, D is dense in $\mathscr{A}$. Therefore, $\mathscr{A}$ is separable. □

Example 4.5. $C([0,1])$ is GAF.

Proof. Since $C([0,1]) = C^*(f\mathscr{F}f)$, where $\mathscr{F} = \mathbb{C}.1, 1(t) = 1$, and $f(t) = \sqrt{t}$, for all $t \in [0,1]$, $C([0,1])$ is GAF. □

This example illustrates that there exists a projectionless C^*-algebra which is a GAF algebra.

Example 4.6. Let $X = [0, \frac{1}{20}] \cup [\frac{1}{19}, \frac{1}{18}] \cup [\frac{1}{17}, \frac{1}{16}] \cup [\frac{1}{15}, \frac{1}{14}] \cdots \cup [\frac{1}{5}, \frac{1}{4}] \cup [\frac{1}{3}, 1]$. Then, $C(X)$ is GAF.

Proof. Since $[0, \frac{1}{20}], [\frac{1}{19}, \frac{1}{18}], [\frac{1}{17}, \frac{1}{16}], [\frac{1}{15}, \frac{1}{14}], \ldots, [\frac{1}{5}, \frac{1}{4}], [\frac{1}{3}, 1]$ are pairwise disjoint clopen subsets of X, the characteristic functions $P_1 = \chi_{[0,\frac{1}{20}]}$, $P_2 = \chi_{[\frac{1}{19},\frac{1}{18}]}, P_3 = \chi_{[\frac{1}{17},\frac{1}{16}]}, \ldots, P_{10} = \chi_{[\frac{1}{3},1]}$ are continuous. Hence, they are mutually orthogonal projections in $C(X)$. Thus, $\mathscr{F} = span\{P_1, P_2, \ldots, P_{10}\}$ is a sub-C^*-algebra of $C(X)$ and $dim(\mathscr{F}) < \infty$. Let $f(s) = \sqrt{s}$, $s \in X$, then f is a positive element in $C(X)$ and $C(X) = C^*(f\mathscr{F}f)$. Therefore, $C(X)$ is GAF. □

Example 4.7. Let $X = \overline{\bigcup_{n=1}^{\infty} [\frac{1}{2n}, \frac{1}{2n-1}]}$. Then, $C(X)$ is GAF.

Proof. Since $[\frac{1}{2n}, \frac{1}{2n-1}], n \in \mathbb{N}$, are pairwise disjoint clopen subsets of X, $P_n = \chi_{[\frac{1}{2n},\frac{1}{2n-1}]}, n \in \mathbb{N}$ are pairwise orthogonal projections in $C(X)$. Let $\mathscr{F}_n = span\{P_1, P_2, \ldots, P_n\}$, for each $n \in \mathbb{N}$. Define $g_i : X \to \mathbb{C}$ by

$$g_i(t) = \begin{cases} \sqrt{t} & t \in [\frac{1}{2i}, \frac{1}{2i-1}] \\ 0 & otherwise \end{cases}$$

for $i = 1, 2, \ldots, n$. Let $f_n = \sum_{i=1}^{n} g_i$, then each f_n is a positive element in $C(X)$. Therefore, we have:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n\mathscr{F}_na_n)$, for each $n \in \mathbb{N}$.

(2) $C^*(f_n\mathscr{F}_nf_n) \subset C^*(f_{n+1}\mathscr{F}_{n+1}f_{n+1})$ for $n \in \mathbb{N}$.

(3) $C(X) = \overline{\bigcup_{n=1}^{\infty} C^*(f_n \mathscr{F}_n f_n)}$.

□

Next, we will prove that the class of all separable AF algebras is a subclass of the class of all GAF.

Proposition 4.8. *Every separable AF algebra is GAF.*

Proof. Assuming that $\mathscr{A}$ is a separable AF algebra, we have, $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} \mathscr{F}_n}$ and $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for each n. Let $a_n = 1_{\mathscr{F}_n}$, which is positive, and $\mathscr{F}_n = C^*(a_n \mathscr{F}_n a_n)$, for all n. Thus, $\mathscr{A}$ is GAF. □

Remark 4.9. Example 4.7 and proposition 4.8 demonstrate that the class of separable AF algebras is a proper subclass of GAF algebras.

Corollary 4.10. *Every GAF algebra is GLF.*

Proof. By proposition 3.10, every GAF algebra is GLF. □

Remark 4.11. From Example 3.2 and Proposition 4.4, we can conclude that GLF is not always GAF. Thus, proposition 4.10 follows that the class of GAF algebras is a proper subclass of the class of GLF algebras.

Corollary 4.12. *The nuclear dimension of unital GAF algebra $\mathscr{A}$ is zero if and only if it is separable AF algebra.*

Proof. By propositions 2.6 and 3.5, the nuclear dimension of unital GAF algebra $\mathscr{A}$ is zero if and only if $\mathscr{A}$ is separable AF algebra. □

Next, we will prove that an $A\mathbb{I}$-algebra is GAF. Before proving the result, we first establish the following lemma.

Lemma 4.13. *If $\phi : \mathscr{A} \to \mathscr{B}$ is a nonzero C^*-algebra homomorphism, $\mathscr{F}$ is a sub-C^*-algebra of $\mathscr{A}$ with $\dim(\mathscr{F}) < \infty$, and $a \in \mathscr{A}_+$, then we can find a sub-C^*-algebra $\mathscr{K}$ of $\mathscr{B}$ with $dim(\mathscr{K}) < \infty$ and an element b in $\mathscr{B}_+$ such that $\phi(C^*(a\mathscr{F}a)) = C^*(b\mathscr{K}b)$.*

Proof. Take $\mathscr{K} = \phi(\mathscr{F})$ and $b = \phi(a)$. Then, we have $\phi(C^*(a\mathscr{F}a)) = C^*(b\mathscr{K}b)$. □

Next, we prove that $A\mathbb{I}$-algebra is GAF.

Proposition 4.14. *Every $A\mathbb{I}$-algebra is GAF.*

Proof. Let $\mathscr{A}$ be an $A\mathbb{I}$-algebra. Then, $\mathscr{A}$ is the inductive limit of an inductive sequence, $C[0,1] \otimes \mathscr{F}_1 \xrightarrow{\phi_1} C[0,1] \otimes \mathscr{F}_2 \xrightarrow{\phi_2} C[0,1] \otimes \mathscr{F}_3 \xrightarrow{\phi_3} \ldots$, where $\mathscr{F}_n$ are finite-dimensional C^*-algebras. Therefore, by proposition 3.11, $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} \mu_n(C[0,1] \otimes \mathscr{F}_n)}$, where $\mu_n : C[0,1] \otimes \mathscr{F}_n \longrightarrow \mathscr{A}$. We have $C[0,1] \otimes \mathscr{F}_n = C^*(a_n(\mathbb{C}.1_{C([0,1])} \otimes \mathscr{F}_n)a_n)$, where $a_n = f \otimes 1_{\mathscr{F}_n}, f(t) = \sqrt{t}, t \in [0,1]$. Hence, according to Lemma 4.13, it is possible to construct a sequence (b_n) of positive elements in $\mathscr{A}$ and a sequence $(\mathscr{K}_n)$ of finite-dimensional sub-C^*-algebras of $\mathscr{A}$ such that $\mu_n(C[0,1] \otimes \mathscr{F}_n) = C^*(b_n \mathscr{K}_n b_n)$. Since $(\mu_n(C[0,1] \otimes \mathscr{F}_n))$ is an increasing sequence and $\mathbb{C}.1_{C([0,1])} \otimes \mathscr{F}_n \subset C[0,1] \otimes \mathscr{F}_n$, $C^*(b_n \mathscr{K}_n b_n) \subset C^*(b_{n+1} \mathscr{K}_{n+1} b_{n+1})$, $\mathscr{K}_{n-1} \subset C^*(b_n \mathscr{K}_n b_n)$ with $\mathscr{K}_0 = \mathscr{K}_1$. Therefore, $\mathscr{A}$ is GAF. □

Example 4.15 shows that $A\mathbb{I}$-algebras form a proper subclass of GAF algebras.

Example 4.15. By proposition 4.18, $\mathscr{A} = C\left([0,1]^2\right) \cong C\left([0,1]\right) \otimes C\left([0,1]\right)$ is GAF. Then, $SR\left(\mathscr{A}\right) = 2$. Since $A\mathbb{T}$-algebra has stable rank one and every $A\mathbb{I}$-algebra is $A\mathbb{T}$-algebra, $\mathscr{A}$ is not $A\mathbb{I}$-algebra.

Example 4.16. $C\left(\mathbb{T}\right)$ is an $A\mathbb{T}$-algebra but not a GAF.

Remark 4.17. Examples 4.15 and 4.16 shows that GAF algebras and $A\mathbb{T}$-algebras are not subclasses of each other.

Proposition 4.18. *If both $\mathscr{A}$ and $\mathscr{B}$ are GAF, then their tensor product $\mathscr{A} \otimes \mathscr{B}$ is also GAF.*

Proof. Assume that $\mathscr{A}$ and $\mathscr{B}$ are GAF. Then, there are sequences (a_n) and (b_n) in $\mathscr{A}_+$ and $\mathscr{B}_+$ respectively, as well as sequences $(\mathscr{A}_n)$ and $(\mathscr{B}_n)$ of sub-C^*-algebras of $\mathscr{A}$ and $\mathscr{B}$ respectively, such that:

(1) $dim(\mathscr{A}_n) < \infty$ and $dim(\mathscr{B}_n) < \infty$, for each $n \in \mathbb{N}$.

(2) $\mathscr{A}_{n-1} \subset C^*(a_n \mathscr{A}_n a_n)$, $C^*(a_n \mathscr{A}_n a_n) \subset C^*(a_{n+1} \mathscr{A}_{n+1} a_{n+1})$, for each $n \in \mathbb{N}$ and $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{A}_n a_n)}$.

(3) $\mathscr{B}_{n-1} \subset C^*(b_n \mathscr{B}_n b_n)$, for each $n \in \mathbb{N}$, $C^*(b_n \mathscr{B}_n b_n) \subset C^*(b_{n+1} \mathscr{B}_{n+1} b_{n+1})$, for $n \in \mathbb{N}$ and $\mathscr{B} = \overline{\bigcup_{n=1}^{\infty} C^*(b_n \mathscr{B}_n b_n)}$.

Let $\mathscr{F}_n = \mathscr{A}_n \otimes \mathscr{B}_n$ and $f_n = a_n \otimes b_n$, for each $n \in \mathbb{N}$. Since $\mathscr{F}_n = \mathscr{A}_n \otimes \mathscr{B}_n$, $\mathscr{A}_n$ and $\mathscr{B}_n$ are finite-dimensional, $C^*(f_n \mathscr{F}_n f_n) = C^*(a_n \mathscr{A}_n a_n) \otimes C^*(b_n \mathscr{B}_n b_n)$. Therefore,

(1) $\mathscr{F}_{n-1} \subset C^*(f_n \mathscr{F}_n f_n)$, for each $n \in \mathbb{N}$

(2) $C^*(f_n \mathscr{F}_n f_n) \subset C^*(f_{n+1} \mathscr{F}_{n+1} f_{n+1})$ for $n \in \mathbb{N}$

(3) $\mathscr{A} \otimes \mathscr{B} = \overline{\bigcup_{n=1}^{\infty} C^*(f_n \mathscr{F}_n f_n)}$

Hence, $\mathscr{A} \otimes \mathscr{B}$ is GAF. □

Next, we classify GAF algebras into two types based on the number of Building block subalgebras.

Definition 4.19. A GAF algebra $\mathscr{A}$ is said to be first type if there is a finite sequence (a_k) of positive elements and finite sequence $(\mathscr{F}_k)$ of finite-dimensional sub-C^*-algebras of $\mathscr{A}$, $k = 1, 2, 3, \ldots, n$ such that:

(1) $\mathscr{F}_{k-1} \subset C^*(a_k \mathscr{F}_k a_n)$ for each $k = 1, 2, 3, \ldots, n$.

(2) $C^*(a_k \mathscr{F}_k a_k) \subset C^*(a_{k+1} \mathscr{F}_{k+1} a_{k+1})$, for $k = 1, 2, 3, \ldots, n-1$.

(3) $\mathscr{A} = \overline{\bigcup_{k=1}^{n} C^*(a_k \mathscr{F}_k a_k)}$.

Definition 4.20. A GAF algebra $\mathscr{A}$ is said to be the second type if it is not the first type.

Example 4.21. Let $\mathscr{A} = C([0, 1]) \otimes \mathscr{F}$, where $dim(\mathscr{F}) < \infty$. Then, $\mathscr{A}$ is the first type GAF.

Example 4.22. Let $X = [0, \frac{1}{20}] \cup [\frac{1}{19}, \frac{1}{18}] \cup [\frac{1}{17}, \frac{1}{16}] \cup [\frac{1}{15}, \frac{1}{14}] \cdots \cup [\frac{1}{5}, \frac{1}{4}] \cup [\frac{1}{3}, 1]$. Then, $C(X)$ is the first type GAF.

Example 4.23. Every infinite-dimensional separable AF algebra is second-type GAF.

Example 4.24. Let $X = \overline{\bigcup_{n=1}^{\infty} [\frac{1}{2n}, \frac{1}{2n-1}]}$. Then, $C(X)$ is second-type GAF.

Proposition 4.25. *The minimal tensor product of two first-type GAF algebras is also a first-type GAF, while that of two second-type GAF algebras results in a second-type GAF. The inductive limit of first-type GAF algebras is a GAF.*

Proof. The proof follows directly. □

Proposition 4.26. *If $\mathscr{A}$ is GAF, then C^*-algebra $M_n[\mathscr{A}]$ of square matrices of order n over $\mathscr{A}$ is also GAF.*

Proof. Assume that $\mathscr{A}$ is GAF. Then, we can identify a sequence of elements in $\mathscr{A}_+$, denoted as (a_k), and a sequence of finite dimensional sub-C^*-algebras of $\mathscr{A}$, denoted as $(\mathscr{F}_k)$, such that:

(1) $\mathscr{F}_{k-1} \subset C^*(a_n \mathscr{F}_k a_k)$, for each $k \in \mathbb{N}$ and $\mathscr{F}_0$ is finite dimensional sub-C^*-algebra.

(2) $C^*(a_k \mathscr{F}_k a_k) \subset C^*(a_{k+1} \mathscr{F}_{k+1} a_{k+1})$, for $k \in \mathbb{N}$.

(3) $\mathscr{A} = \overline{\bigcup_{k=1}^{\infty} C^*(a_k \mathscr{F}_k a_k)}$.

Let $\mathbf{a}_k = \bigoplus_{i=1}^{n} a_k$ be the $n \times n$ diagonal matrices with diagonal entries a_k, for each k. Then, $\mathbf{a}_k$ is positive in $M_n[\mathscr{A}]$. Let $\mathscr{B}_k = M_n[\mathscr{F}_k]$, for each k. Then, each $\mathscr{B}_k$ is sub-C^*-algebra of $M_n[\mathscr{A}]$ with $dim(\mathscr{B}_k) < \infty$. Therefore,

(1) $\mathscr{B}_{k-1} \subset C^*(\mathbf{a}_k \mathscr{B}_k \mathbf{a}_k)$, for each $k \in \mathbb{N}$ and $\mathscr{B}_0$ is finite dimensional sub-C^*-algebra.

(2) $C^*(\mathbf{a}_k \mathscr{B}_k \mathbf{a}_k) \subset C^*(\mathbf{a}_{k+1} \mathscr{B}_{k+1} \mathbf{a}_{k+1})$, for $k \in \mathbb{N}$.

(3) $M_n[\mathscr{A}] = \overline{\bigcup_{k=1}^{\infty} C^*(\mathbf{a}_k \mathscr{B}_k \mathbf{a}_k)}$.

. □

Definition 4.27. [18] "Let $\mathscr{A}$ be a C^*-algebra and $S \subset \mathscr{A}_{sa}$. We call S a generator of $\mathscr{A}$ if $\mathscr{A} = C^*(S)$. If S is finite, then we call $\mathscr{A}$ finitely generated, and we define the number of generators gen$(\mathscr{A})$ as the minimum cardinality of S required to generate $\mathscr{A}$".

Lemma 4.28. *[18] "Let $\mathscr{A}$ be a unital C^*-algebra with* gen$(\mathscr{A}) \leq n^2 + 1$*, then* gen$(M_n[\mathscr{A}]) \leq$ 2".

Proposition 4.29. *If $\mathscr{A}$ is first type GAF, then* gen$(M_n[\mathscr{A}]) \leq 2$*, for all $n \geq l$, for some positive integer l.*

Proof. Since $\mathscr{A}$ is first type GAF, we can find a positive element a and a sub-C^*-algebra $\mathscr{F}$ of $\mathscr{A}$ with $dim(\mathscr{F}) < \infty$ such that $\mathscr{A} = C^*(a\mathscr{F}a)$. Since $\mathscr{F}$ is finite dimensional, there exist mutually orthogonal projections $p_1, p_2, \ldots, p_l$, where $l = dim(\mathscr{F})$, such that $\mathscr{F} = span\{p_1, p_2, \ldots, p_l\}$. Therefore, $\mathscr{A} = C^*(a\mathscr{F}a)$ is finitely generated, and gen$(\mathscr{A}) \leq n^2 + 1$, for all $n \geq l$. Hence, by lemma 4.28, gen$(M_n[\mathscr{A}]) \leq 2$, for all $n \geq l$. □

Proposition 4.30. *Let $\mathscr{A}$ be GAF, and if each $C^*(a_n \mathscr{F}_n a_n)$ is simple, then $\mathscr{A}$ is simple.*

Proof. If $\mathscr{I}$ is an ideal of $\mathscr{A}$, then $\mathscr{I} \cap C^*(a_n \mathscr{F}_n a_n) = C^*(a_n \mathscr{F}_n a_n)$, for all $n \in \mathbb{N}$. Therefore, $\mathscr{I} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)} = \mathscr{A}$. Hence, $\mathscr{A}$ is simple. □

problem Is an ideal of a GAF also GAF? Is a hereditary sub-algebra of a GAF also a GAF?

Consider the GAF, $\mathscr{A} = C([0,1])$. Let $\mathscr{B} = \{g \in \mathscr{A} : g(x) = 0$, for all $x \in [0, \frac{1}{2}]\}$. Then, $\mathscr{B}$ is an ideal, and hence $\mathscr{B}$ is a hereditary sub-C^*-algebra of $\mathscr{A}$. Since $\mathscr{B}$ has no nonzero projection, $\mathscr{B}$ has no finite-dimensional sub-algebra, so $\mathscr{B}$ is not GAF. Therefore, the ideals, sub-C^*-algebras, and hereditary sub-algebras of a GAF are not always GAF.

Problem If $\mathscr{A}$ is GAF, is $RR(\mathscr{A}) = 0$?
By 4.14, every $A\mathbb{I}$-algebra is a GAF. Since there are $A\mathbb{I}$-algebras with non-zero real rank, there are GAF algebras with non-zero real rank.

problem If $\mathscr{A}$ is GAF, is $SR(\mathscr{A}) = 1$?

Example 4.15 shows that there is GAF algebra with $SR(\mathscr{A}) \neq 1$.
Next, we prove that the range of a C^*-homomorphism from a GAF to a C^*-algebra is GAF.

Proposition 4.31. *If $\mathscr{A}$ is GAF and $\Phi : \mathscr{A} \to \mathscr{B}$ is a non-zero homomorphism, then $\Phi(\mathscr{A})$ is a GAF sub-algebra of $\mathscr{B}$.*

Proof. The proof follows from the definition and Lemma 4.13. □

Proposition 4.32. *Le p_1 be a projection in GAF algebra, $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$ and $\epsilon > 0$. Then, we can find a projection $p_2 \in C^*(a_m \mathscr{F}_m a_m)$, for some m, such that $\|p_1 - p_2\| < \epsilon$. Moreover, every projection in $\mathscr{A}$ is equivalent to some projections in $\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)$.*

Proof. The proof follows directly. □

Example 4.33. If $\mathscr{A}$ is non-unital and $\tilde{\mathscr{A}}$ is GAF, then $\mathscr{A}$ is not necessarily GAF.
Let $\mathscr{A} = C_0((0,1))$ be the C^*-algebra of continuous functions on $(0,1)$ that vanish at infinity. Since $\mathscr{A}$ has no non-zero projections, $\mathscr{A}$ has no finite-dimensional sub-C^*-algebra, thus $\mathscr{A}$ is not GAF but $\tilde{\mathscr{A}} = C([0,1])$ is a GAF.

Example 4.34. A GAF does not necessarily have to be a graph algebra.
We know that $\mathscr{A} = C([0,1])$ is GAF but not graph algebra.

Example 4.35. A graph algebra is not necessarily a GAF.
We know that $C(\mathbb{T})$ is graph algebra but not GAF.

Remark 4.36. Examples 4.37 and 4.38 shows that a graph algebra is not necessarily an GLF, and a GLF is not necessarily a graph algebra. However, there are many GAF and GLF which are graph algebras, for example, separable AF-algebras.

Next, we will prove that every non-nuclear, projectionless (having no non-trivial non-zero projections) C^*-algebra is not GAF. Consequently, $C^*(\mathbb{F}_2)$ and $C_r^*(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the free group with two generators, are not GAF.

Proposition 4.37. *Let $\mathscr{A}$ be an unital, non-nuclear, projectionless (having no non-trivial, non-zero projections) C^*-algebra. Then, $\mathscr{A}$ is not GAF.*

Proof. Suppose that $\mathscr{A}$ is a GAF-algebra. Then, we can identify a sequence of elements in $\mathscr{A}_+$, denoted as (a_n), and a sequence of finite dimensional sub-C^*-algebras of $\mathscr{A}$, denoted as $(\mathscr{F}_n)$, such that:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n \mathscr{F}_n a_n)$, for each $n \in \mathbb{N}$ and $\mathscr{F}_0$ is finite dimensional sub-C^*-algebra.

(2) $C^*(a_n \mathscr{F}_n a_n) \subset C^*(a_{n+1} \mathscr{F}_{n+1} a_{n+1})$, for $n \in \mathbb{N}$.

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$.

Since $\mathscr{A}$ has no non trivial non zero projections, $\mathscr{F}_n = \mathbb{C}1_{\mathscr{A}}$, for each $n \in \mathbb{N}$. Therefore, $C^*(a_n \mathscr{F}_n a_n) = C^*(a_n^2)$ is nuclear, for each n. Hence $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$ is nuclear. This is a contradiction. Therefore, $\mathscr{A}$ is not GAF. □

Corollary 4.38. *$C^*(\mathbb{F}_2)$ and $C_r^*(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the free group with two generators, are not GAF.*

Proof. $C^*(\mathbb{F}_2)$ and $C_r^*(\mathbb{F}_2)$ are projectionless and non-nuclear. Therefore, by 4.40, $C^*(\mathbb{F}_2)$ and $C_r^*(\mathbb{F}_2)$ are not GAF. □

This corollary shows that residually finite dimension C^*-algebras and quasi-diagonal C^*-algebras are not sub-classes of GAF-algebras.

Remark 4.39. If $\mathbb{F}_n$ is the free group with n generators, $C^*(\mathbb{F}_n)$ and $C_r^*(\mathbb{F}_n)$ are not GAF. The proof of this result is the same as the proof of Corollary 4.41.

Proposition 4.40. *If $\mathscr{A}$ is non-nuclear GAF algebra, then there exist an element $a \in \mathscr{A}_+$ and a finite dimensional sub-C^*-algebra $\mathscr{F}$ such that $C^*(a\mathscr{F}a)$ is non nuclear. Moreover, if $\mathscr{A}$ is second type GAF, then $C^*(a\mathscr{F}a)$ is proper sub-C^*-algebra.*

Proof. Assume that $\mathscr{A}$ is non-nuclear GAF algebra. Then, we can find a sequence of elements in $\mathscr{A}_+$, denoted as (a_n), and a sequence of finite-dimensional sub-C^*-algebras of $\mathscr{A}$, denoted as $(\mathscr{F}_n)$, such that:

(1) $\mathscr{F}_{n-1} \subset C^*(a_n \mathscr{F}_n a_n)$, for each $n \in \mathbb{N}$ and $\mathscr{F}_0$ is finite dimensional sub-C^*-algebra.

(2) $C^*(a_n \mathscr{F}_n a_n) \subset C^*(a_{n+1} \mathscr{F}_{n+1} a_{n+1})$, for $n \in \mathbb{N}$.

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$.

If, for any $a \in \mathscr{A}_+$ and a finite dimensional sub-C^*-algebra $\mathscr{F}$, $C^*(a\mathscr{F}a)$ is nuclear, then $C^*(a_n \mathscr{F}_n a_n)$ is nuclear, for every $n \in \mathbb{N}$. Since $C^*(a_n \mathscr{F}_n a_n) \subset C^*(a_{n+1} \mathscr{F}_{n+1} a_{n+1})$, for $n \in \mathbb{N}$, $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathscr{F}_n a_n)}$ is nuclear. This is a contradiction. Therefore, $C^*(a_n \mathscr{F}_n a_n)$ is non nuclear for some n. If $\mathscr{A}$ is second type GAF, $C^*(a_n \mathscr{F}_n a_n)$ is proper sub-C^*-algebra. □

Proposition 4.41. *If $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, then $\mathcal{B}(\mathcal{H})$, the C^*-algebra of bounded operators on $\mathcal{H}$, is not a GAF.*

Proof. If $\mathcal{F}$ is a finite-dimensional sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ and a is a positive operator, then $a\mathcal{F}a \subseteq \mathcal{B}_0(\mathcal{H})$. Therefore, $\overline{\bigcup_{n=1}^{\infty} C^*(a_n \mathcal{F}_n a_n)} \subseteq \mathcal{B}_0(\mathcal{H})$ for any sequences (a_n) and $(\mathcal{F}_n)$. Hence, $\mathcal{B}(\mathcal{H})$ is not GAF. □

To ensure the relevance of Proposition 4.43, we must confirm the existence of non-nuclear GAF algebras. Therefore, we propose the following problem. **Problem** Does a non-nuclear GAF algebra exist?

5 Commutative GAF-algebras

In this section, we will obtain the necessary and sufficient conditions for a commutative C^*-algebra to be of first-type GAF. We will also establish sufficient conditions for a commutative C^*-algebra to be GAF. All C^*-algebras discussed in this section are commutative.

Proposition 5.1. *A C^*-algebra $\mathscr{A}$ is first type GAF if and only if there is a finite number of positive elements $f_1, f_2, \ldots, f_n$ such that*
$\mathscr{A} = C^*(f_1, f_2, \ldots, f_n)$, and $f_i f_j = 0$, for $i \neq j$.

Proof. Let's establish the forward implication. If $\mathscr{A}$ is a first type GAF algebra, then we can identify a sub-C^*-algebra $\mathscr{F}$ with $dim(\mathscr{F}) < \infty$ and an element a of $\mathscr{A}_+$ such that $\mathscr{A} = C^*(a\mathscr{F}a)$. Since $\mathscr{F}$ is finite-dimensional, there are pairwise orthogonal projections $p_1, p_2, \ldots, p_n$, where $n = dim(\mathscr{F})$, such that $\mathscr{F} = span\{p_1, p_2, \ldots, p_n\}$. Let $f_i = ap_i a, i = 1, 2, \ldots, n$, then $f_i f_j = ap_i aap_j a = a^4 p_i pj = 0$, for $i \neq j$. Therefore, $\{f_1, f_2, \ldots, f_n\}$ is a finite set of pairwise orthogonal positive elements in $\mathscr{A}$. Therefore, $\mathscr{A} = C^*(\{f_1, f_2, \ldots, f_n\})$.

To prove the backward implication, assume that there is a finite number of positive elements $f_1, f_2, \ldots, f_n$ such that:

(1) $f_i f_j = 0$, for $i \neq j$.

(2) $\mathscr{A} = C^*(f_1, f_2, \ldots, f_n)$.

Since $\mathscr{A}$ is commutative, $\mathscr{A} = C(X)$ or $C_0(X)$. Let $X_i = \{x \in X : f_i(x) \neq 0\}$, $i = 1, 2, \ldots, n$. Then, X_i is a clopen subset of X, and $X_i \cap X_j = \emptyset$, for $i \neq j$. Since X_i is clopen, $p_i = \chi_{X_i}$ is a projection in $\mathscr{A}$, for $i = 1, 2, \ldots, n$. Since $X_i \cap X_j = \emptyset$, for $i \neq j$, $p_1, p_2, \ldots, p_n$ are mutually orthogonal projections. Therefore, $\mathscr{F} = span\{p_1, p_2, \ldots, p_n\}$ is a finite-dimensional sub-C^*-algebra of $\mathscr{A}$. Define $g : X \to \mathbb{C}$ by $g(x) = \sum_{i=1}^{n} \sqrt{f_i(x)}$. Since each $\sqrt{f_i}$ is positive, then their sum g is positive. Since $gp_i g = f_i$, we have $\mathscr{A} = C^*(f_1, f_2, \ldots, f_n) = C^*(g\mathscr{F}g)$. Therefore, $\mathscr{A}$ is the first type GAF. □

The following result provides a "sufficient condition" for a commutative C^*-algebra(both types) to be GAF.

Proposition 5.2. *A C^*-algebra $\mathscr{A}$ is GAF if there exists a countable set $\{f_n\}$ of positive elements in $\mathscr{A}$ such that:*

(1) $f_i f_j = 0$, *for* $i \neq j$.

(2) $\mathscr{A} = C^*(f_1, f_2, \dots)$.

Proof. Suppose that there exists a countable set $\{f_n\}$ of positive elements in $\mathscr{A}$ such that $f_i f_j = 0$. for $i \neq j$ and $\mathscr{A} = C^*(f_1, f_2, \dots)$. Since $\mathscr{A}$ is commutative, $\mathscr{A} = C(X)$ or $C_0(X)$. Let $X_i = \{x : f_i(x) \neq 0\}$, for $i = 1, 2, \dots, n$, for $n \in \mathbb{N}$. Since $f_i f_j = 0$, each X_i is clopen and $X_i \cap X_j = \emptyset$, for $i \neq j$. Therefore, $p_i = \chi_{X_i}$ is a projection in $\mathscr{A}$ for $i = 1, 2, \dots, n$. Since $X_i \cap X_j = \emptyset$, for $i \neq j, p_1, p_2, \dots, p_n$ are mutually orthogonal. Hence, $\mathscr{F}_n = span\{p_1, p_2, \dots, p_n\}$ is a sub-C^*-algebra of $\mathscr{A}$ and $dim(\mathscr{F}_n) < \infty$. For each n, define $g_n : X \to \mathbb{C}$ by $g_n(x) = \sum_{i=1}^{n} \sqrt{f_i(x)}$. Therefore, $C^*(g_n \mathscr{F}_n g_n) = C^*(f_1, f_2, \dots, f_n)$. Therefore, we have a sequence (g_n) of positive elements and a sequence $(\mathscr{F}_n)$ of finite-dimensional sub-C^*-algebras of $\mathscr{A}$ such that:

(1) $\mathscr{F}_{n-1} \subset C^*(g_n \mathscr{F}_n g_n)$, for each $n \in \mathbb{N}$.

(2) $C^*(g_n \mathscr{F}_n g_n) \subset C^*(g_{n+1} \mathscr{F}_{n+1} g_{n+1})$ for $n \in \mathbb{N}$.

(3) $\mathscr{A} = \overline{\bigcup_{n=1}^{\infty} C^*(g_n \mathscr{F}_n g_n)}$.

□

6 Elliott Invariant and GAF Algebras

In this section, we prove that if simple, unital, nuclear, $\mathcal{Z}$-stable GAF algebras $\mathscr{A}$ and $\mathscr{B}$ satisfy the UCT, then $\mathscr{A} \cong \mathscr{B}$ if and only if $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$. We construct a GAF algebra $\mathscr{A}$ such that:

(1) $\mathscr{A}$ is simple, unital, nuclear, satisfies the UCT and has real rank and stable rank one.

(2) $\mathscr{A} \otimes \mathcal{Z}$ is an AI algebra.

(3) $\mathscr{A}$ is not $\mathcal{Z}$-stable.

(4) The Cuntz semigroup of $\mathscr{A}$ is not almost unperforated.

(5) There is a C^*-algebra $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are not isomorphic, but they have the same stable and real rank and satisfy $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$.

(6) $\mathscr{A}$ is not tracially $\mathcal{Z}$-absorbing.

This C^*-algebra $\mathscr{A}$ is not a new one. It was introduced by Andrew S. Toms, based on the work of Villadsen; see [22, 23, 24, 25].

Corollary 6.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be simple, unital, nuclear, $\mathcal{Z}$-stable GAF algebras that satisfy the UCT. Then $\mathscr{A} \cong \mathscr{B}$ if and only if* $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$.

Proof. The proof follows from Proposition 4.4 and [[8], Proposition 9.3]. □

Proposition 6.2. *There exist GAF algebra $\mathscr{A}$ such that:*

(1) $\mathscr{A}$ is simple, unital, nuclear, satisfies the UCT and has real rank and stable rank one.

(2) $\mathscr{A} \otimes \mathcal{Z}$ is an AI algebra.

(3) $\mathscr{A}$ is not $\mathcal{Z}$-stable.

(4) The Cuntz semigroup of $\mathscr{A}$ is not almost unperforated.

(5) There is a C^*-algebra $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are not isomorphic, but they have the same stable and real rank and satisfy $\text{Ell}(\mathscr{A}) \cong \text{Ell}(\mathscr{B})$.

(6) $\mathscr{A}$ is not tracially $\mathcal{Z}$-absorbing.

Proof. Let (m_i) and (n_i) be sequences of natural numbers such that n_i is larger than i as $i \to \infty$, for each $r \in \mathbb{N}$, there is i_0 with $r \mid (n_{i_0} + i_0), n_1 = 1, m_1 = 4$ and $m_{i+1} = m_i(i + 6N_i)$, where $N_i = \prod_{j \leq i} n_j$. Let $\mathscr{A}_i = M_{m_i} \otimes C([0,1]^{6N_i})$. Identify $[0,1]^{6N_i}$ with $([0,1]^{6N_{i-1}})^{n_i}$ and for each i and k such that $1 \leq k \leq n_{n_i}$, $\pi_k^{(i)} : [0,1]^{6N_i} \to [0,1]^{6N_{i-1}}$ is the coordinate projection, given by $\pi_k^{(i)}(x_1, x_2, \ldots, x_{n_i}) = x_k$, for all $(x_1, x_2, \ldots, x_{n_i}) \in [0,1]^{6N_i}$. Write $X_i = [0,1]^{6N_i}$. For each $i \in \mathbb{N}$, choose a dense sequence (y_k^i) in $X_i = [0,1]^{6N_i}$ and choose points $x_1^{(i)}, x_2^{(i)}, \ldots, x_i^{(i)}$ by setting $x_i^{(i)} = y_i^{(i)}$ and if $1 \leq j \leq i-1$, choose $x_j^{(i)}$ such that $\pi_1^{(j)}\pi_1^{(j+1)} \ldots \pi_1^{(i-2)}\pi_1^{(i-1)}(x_j^{(i)}) = y_{i+1-j}^{(i)}$. We define connecting maps $\phi_{i-1} : \mathscr{A}_{i-1} \to \mathscr{A}_i$ by $\phi_{i-1}(f)(x) = diag(f(\pi_1^{(i)}(x)), \ldots, f(\pi_{n_i}^{(i)}(x)), f(x_1^{(i-1)}), \ldots, f(x_{i-1}^{(i-1)}))$. Let $\mathscr{A} = \varinjlim(\mathscr{A}_i, \phi_i)$ and $\mathscr{B} = \mathscr{A} \otimes \mathcal{Z}$. By [[8], Propositions 9.3, 9.4, 9.5 and 9.6] and [[9], Theorem 4.1], we have

(1) $\mathscr{A}$ is simple, unital, nuclear, satisfies the UCT and has real rank and stable rank one.

(2) $\mathscr{A} \otimes \mathcal{Z}$ is an AI algebra.

(3) $\mathscr{A}$ is not $\mathcal{Z}$-stable.

(4) The Cuntz semigroup of $\mathscr{A}$ is not almost unperforated.

(5) There is a C^*-algebra $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are not isomorphic, but they have the same stable and real rank and satisfy $\mathrm{Ell}(\mathscr{A}) \cong \mathrm{Ell}(\mathscr{B})$.

(6) $\mathscr{A}$ is not tracially $\mathcal{Z}$-absorbing.

Now, we claim that $\mathscr{A}$ is a GAF algebra. By Example 4.5 and Proposition 4.18, $C([0,1]^{6N_i})$ is GAF. Hence, for the same reason, $\mathscr{A}_i = M_{m_i} \otimes C([0,1]^{6N_i})$ is GAF. Since $M_{m_i} \otimes C([0,1]^{6N_i}) = C^*(1_{m_i} \otimes f(M_{m_i} \otimes 1)1_{m_i} \otimes f)$, where $f = \otimes_{j=1}^{6N_i} \sqrt{f_j} \geq 0, f_j(t) = t$. Therefore, A_i first type GAF. By Proposition 4.25, $\mathscr{A}$ is GAF. □

problem Is every GAF algebra a TAF algebra?

The answer to this question is negative because the GAF algebra constructed in Proposition 6.2 is not a TAF algebra. **problem** Is every TAF algebra a GAF algebra?

problem Is every separable GLF algebra a GAF algebra?

This Problem has a positive answer for separable LF-algebra.

Remark 6.3. One can extend UHF-algebras to generalized UHF-algebras by adapting the approach used for GAF-algebras, substituting finite-dimensional C^*-algebras with factors $\mathscr{A}_n$ of type I_{p_n}. It is worth noting the existence of non-simple generalized UHF-algebras. Similarly, non-separable GAF-algebras can be defined by replacing sequences of positive elements and finite-dimensional subalgebras with nets of positive elements and finite-dimensional sub-C^*-algebras.

7 Conclusion

In this study, we introduced the class of Generalized $AF(GAF)$ algebras and established that while all AF algebras are GAF algebras, not all GAF algebras are AF. We demonstrated that GAF algebras are separable and identified specific conditions under which a GAF algebra can also be an AF algebra. For unital first type GAF algebras, we proved that $\mathrm{gen}(M_n[\mathscr{A}]) \leq 2$ for $n \geq l$, where l is a positive integer. We also showed that non-nuclear, projectionless C^*-algebras cannot be GAF.

Furthermore, we established that the minimal tensor product of two separable GAF algebras remains a GAF algebra and that all $A\mathbb{I}$-algebras are included in the GAF class. We provided conditions for a commutative C^*-algebra to be GAF and for first-type commutative GAF algebras. We proved that if two simple, unital, nuclear, $\mathcal{Z}$-stable GAF algebras satisfy the UCT, they are isomorphic if and only if their Elliott invariants are isomorphic.

Additionally, we noted that separable C^*-algebras are not always GAF. There exist projectionless C^*-algebras that are GAF. The class of all separable AF algebras forms a subclass of GAF algebras, while $A\mathbb{I}$-algebras are a proper subclass of GAF algebras. GAF algebras and $A\mathbb{T}$-algebras are not sub-classes of each other. Furthermore, while graph algebras are not necessarily GAF, and GAF algebras are not necessarily graph algebras, there are indeed many examples of GAF algebras that are graph algebras, such as separable AF algebras. This diversity underscores the rich and varied nature of GAF algebras and their relationships with other classes of C^*-algebras.

As future scope, one could investigate the detailed structure of GAF algebras and explore potential classifications, particularly by studying the interactions between GAF algebras and other classes of C*-algebras, such as those arising from various types of graph algebras or K-theoretic perspectives. Additionally, the study of generalized *-derivations [21] on GAF algebras could reveal new insights into the structure and classification of primitive sub-C*-algebras, potentially leading to a deeper understanding of their representation theory. Further research could also explore the interaction between C*-algebra valued asymmetric metric spaces[4] and GAF algebras, potentially uncovering new structural relationships, while investigating the role of fixed point theorems within this framework could deepen our understanding of the classification and dynamics of GAF algebras.

References

[1] R.Antoine, F.Perera, and H.Thiel, *Tensor Products and Regularity Properties of Cuntz Semigroups*, Mem.Amer.Math.Soc.251(2018).

[2] Arhangel'skii, Mikhail Tkachenko, Topological Groups and Related Structures, Atlantis Studies in Mathematics.Vol.1.Atlantis Press. ISBN 978-90-78677-06-2.

[3] Bratteli, Ola, *Inductive Limit of Finite-Dimensional C*-Algebras*, Trans. Math. Soc. 171(1972) 195-234.

[4] O. Bouftouh, S. Kabbaj, *Fixed Point Theorems in C*-Algebra Valued Asymmetric Metric Spaces*, Palestine Journal of Mathematics, vol.12(1)(2023), 859-871.

[5] Davidson,K.R, *C*-algebras by Example, Field institute Monographs*, **6**(1996), American Mathematical Society.

[6] Elliot, G. A., *On Classification of Inductive Limits of Sequences of Semi-Simple Finite-Dimensional Algebras*, J. Algebra 38(1976) 29-44.

[7] G. A. Elliott, Q. Fan, and X. Fang, *Generalized Tracially Approximated C*-Algebras*, arXiv:2203.05700v1 [math.OA], 11 Mar 2022.

[8] E.Gardella and F.Perera, *The Modern Theory of Cuntz Semigroup of C*-algebras*, preprint(arXiv:2212.02290[math.OA]),2022.

[9] I. Hirshberg and J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C*-algebras*, arXiv:1208.2444v2 [math.OA].

[10] James R.Munkres, *Topology*, (Second Edition), Prentice Hall ISBN 81-7758-579-7.

[11] Joan Bosa, Nathaniel D. Brown, Yasuhiko Sato, Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Covering Dimension of C*-Algebras and 2-Colored Classification*, Mem. Amer.Math.Soc.257(2019).

[12] John B. Conway, *A Course in Operator Theory*, American Mathematical Society ISBN 978- 1-4704-0925-8.

[13] Karen R. Strung, *An Introduction to C*-Algebras and the Classification Program*, 1(2020), ORMS, Birkhäuser.

[14] E.Kirchberg, W.Winter, *Covering Dimension and Quasidiagonality*, inter.J.Math. 15(2004) 63-85.

[15] Lawrence G.Brown and Gert K.Pedersen, *C*-algebras of Real rank zero*, J.Funct.Anal.99(1991),no.1,131-149.

[16] H. Lin, *Tracially AF C*-Algebras*, Trans. Amer. Math. Soc.353(2) (2001), 693-722.

[17] G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, London, 1990.

[18] M. Nagisa, *Single Generation and Rank of C*-algebras, in Operator Algebras and Applications*, H. Kosaki (ed.), Proceedings of the US-Japan seminar held at Kyushu University, Japan, June 7-11, 1999, Tokyo: Mathematical Society of Japan, Advanced Studies in Pure Mathematics, vol. 38, pp. 15-14, 2004.

[19] M.Rørdam, F.Larsen,and N. Laustsen, *Introduction to K-Theory for C*-Algebras*, London Mathematical Society,ISBN-10 0521789443.

[20] M.Rørdam, *Classification of Nuclear C*-Algebras*, Encyclopaedia of Mathematical Sciences, 126, Springer-Verlag ISBN 978-3-642-07605-3.

[21] Shakir Ali, *On Generalized *-Derivations in *-Rings*, Palestine Journal of Mathematics, vol.1(2012),32-37.

[22] A. S.Toms, *On the Classification problem for Nuclear C*-algebras*, Annals of Mathematics, vol. 162, no. 3, pp. 1029-1044, 2005.

[23] A. S. Toms, *Comparison Theory and Smooth Minimal C*-dynamics*, Communications in Mathematical Physics, vol. 289, no. 2, pp. 401-433, 2008.

[24] J.Villadsen, *Simple C*-algebras with Perforation*, Journal of Functional Analysis, vol. 154, no. 1, pp. 110-116, 1998.

[25] J.Villadsen, *On the Stable rank of Simple C*-algebras*, Journal of the American Mathematical Society, vol.12, no.4, pp.1091-1102,1999.

[26] William B. Arveson, *Subalgebras of C*-Algebras*, Acta Math.123(1969).

[27] W.Winter, *Covering Dimension for Nuclear C*-algebras*, J.Funct.Anal.199(2003),535-556.

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Received: 2024-06-01
Accepted: 2024-08-234